7 Commutators, Measurement and The Uncertainty Principle

A black cat went past us, and then another that looked just like it.

Neo

In this section, we return from the wilderness of solving differential equations to more formal mathematics. In particular, we want to study the notion of measurement, and simultaneous measurements of observables that we alluded to way back in the introduction. There we have been careful to say that we cannot measure with arbitrary accuracy the position and momentum of a particle at the same time. Then when we study the Gaussian Wavepacket in section 5.4, we saw that the product of the dispersions of its position and momentum has the minimum value

\[(\Delta x)^2 (\Delta p)^2 \geq \frac{\hbar^2}{4},\]  \hspace{1cm} (298)

which we proceed to argue should be interpreted as our inability to measure \( p \) and \( x \) to arbitrary accuracy at the same time. In this section, we will show that non-commuting observables will lead to the Heisenberg Uncertainty Principle – one of the pillars of Quantum Mechanics.

7.1 Pure Ensembles and Expectation Values

Postulate 3 tells us that the measurement of an observable \( \hat{O} \) in some state \( \psi = \sum_{n=1}^{\infty} a_n u_n \) yields the eigenvalue \( \lambda_n \) with some probability \( |a_n|^2 \). The state then collapses into \( u_n \). This is all fine and good in theory, the question is: how do we test for this fact?

The way to do this, is to make many repeated measurements of identically prepared states, and plot out a histogram of the results, e.g. we measure \( \lambda_1 \) 6 times, \( \lambda_2 \) 32 times, \( \lambda_3 \) 8 times etc. And then compare this to our theoretical prediction. Of course the more identically prepared states there are, the better our experiment will test the theoretical prediction. Such a set of identically prepared states is called a Pure Ensemble.

Given a pure ensemble, and a set of measurements, we can also ask what is the average value of all the measured eigenvalues. In the limit of a very large number of measurements, this is called the Expectation Value, which is defined to be

\[\langle \hat{O} \rangle_\psi = \psi(x) \cdot (\hat{O}\psi(x)) = \int_{\mathbb{R}^3} \psi^\dagger(x)\hat{O}\psi(x) \, dV.\]  \hspace{1cm} (299)

We can show that this exactly is the average value of the measured eigenvalues

\[\int_{\mathbb{R}^3} \psi^\dagger(x)\hat{O}\psi(x) \, dV = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_{\mathbb{R}^3} a_n^* a_m u_n^*(x)\hat{O}u_m(x) \, dV \]
\[= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_n a_m^* a_n u_n^*(x)u_m(x) \, dV \]
\[= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_n a_m^* a_n \delta_{mn} \, , \text{ and hence} \]
\[\langle \hat{O} \rangle_\psi = \sum_{n=1}^{\infty} \lambda_n |a_n|^2.\]  \hspace{1cm} (300)

It is trivial to show that the expectation value of a Hermitian operator is purely real. Some examples:
• Expectation of $\hat{x}$,
\[
\langle \hat{x} \rangle_\psi = \int dx \, \psi^\dagger(x) \hat{x} \psi(x)
\]
\[
= \int dx \, x |\psi(x)|^2
\]
\[
= \int dx \, x \rho(x), \tag{301}
\]
which is the same as the classical notion of finding the expectation value of $x$ given probability distribution of $\rho(x)$.

• Expectation of $\hat{p}$,
\[
\langle \hat{p} \rangle_\psi = -i\hbar \int dx \, \psi^\dagger(x) \frac{d}{dx} \psi(x)
\]
\[
= -i\hbar \int dx \int dk \int dk' f^\dagger(k') e^{-i k x} \frac{d}{dx} f(k) e^{i k x}
\]
\[
= -i\hbar \int dk \int dk' f^\dagger(k') (2\pi \delta(k - k')) f(k)
\]
\[
= \frac{1}{2\pi} \int dk \, \hbar k |f(k)|^2, \tag{302}
\]
which is the same as the classical notion of finding the expectation value of $x$ given probability distribution of $|f(k)|^2$, in agreement with Eq. (122).

7.2 Commutators and Simultaneous Measurement

What do we mean by “measuring both things at the same time”?

In Classical Mechanics, this simply means that we can set up two different detectors, say $X$ (for $x$ measurement) and $P$ (for $p$ measurement). To make simultaneous measurements, we press the buttons both at the same time or even with some slight difference in time (to account for experimental error). It doesn’t matter which detectors “goes first”, we will get the more or less the same answer.

In Quantum Mechanics, Postulate 3 tells us that the very act of measurement collapses the wavefunction, so now it matters which detector goes first! Given a wavefunction $\psi(x)$, if $X$ goes first then the following sequence of events occurs
\[
\psi(x) \xrightarrow{X} \phi_{x_0}(x) \xrightarrow{P} u_{p_0}(x) \tag{303}
\]
where $\phi_{x_0}(x)$ is a highly localized function around the measured value $x_0$ as discussed previously, and $u_{p_0}(x)$ is some highly localized function around the measured value $p_0$. On the other hand, if $P$ goes first then
\[
\psi(x) \xrightarrow{P} u_{p_0}(x) \xrightarrow{X} \phi_{x_0}(x). \tag{304}
\]
Since $\phi_{x_0} \neq \psi(x)$ and $u_{p_0} \neq \psi(x)$ in general, the measured pair of values will be different – the first measurement has destroyed some information regarding the second observable! This is the root reason of why there exist an uncertainty relation in Quantum Mechanics.

We can now ask: under what conditions will the order of the measurements not matter? Say if we have two observables, $\hat{O}_A$ and $\hat{O}_B$, then we want
\[
\psi(x) \xrightarrow{O_A} \phi(x) \xrightarrow{O_B} \chi(x), \tag{305}
\]
and
\[
\psi(x) \xrightarrow{O_B} \phi(x) \xrightarrow{O_A} \chi(x), \tag{306}
\]
to give the same observed eigenvalues of $\hat{O}_A$ and $\hat{O}_B$. By inspection, it is clear that this will occur if $\phi$ are both eigenfunctions of $\hat{O}_A$ and $\hat{O}_B$, and hence so is $\chi$.

To formalize all these words, we will introduce some new mathematics.

**(Definition) Commutator:** The Commutator of two operators $\hat{O}_A$ and $\hat{O}_B$ is defined by

$$[\hat{O}_A, \hat{O}_B] = \hat{O}_A\hat{O}_B - \hat{O}_B\hat{O}_A.$$  

(307)

This definition means that

$$[\hat{O}_A, \hat{O}_B] = -[\hat{O}_B, \hat{O}_A].$$  

(308)

We now have two possibilities that describe the situation on measurements above:

- **Commuting Observables and Simultaneous Eigenfunctions:** Suppose now $\hat{O}_A$ and $\hat{O}_B$ are two observables. Suppose, further that the wavefunction $\psi(x)$ is a *simultaneous eigenfunction* $\hat{O}_A$ and $\hat{O}_B$ with eigenvalues $a$ and $b$

$$\hat{O}_A\psi(x) = a\psi(x), \quad \hat{O}_B\psi(x) = b\psi(x)$$  

(309)

then

$$[\hat{O}_A, \hat{O}_B]\psi(x) = (\hat{O}_A\hat{O}_B - \hat{O}_B\hat{O}_A)\psi(x) = ab - ba = 0,$$  

(310)

which is to say, “$\hat{O}_A$ and $\hat{O}_B$ commute”. We can write this relation in operator form by dropping $\psi$

**Commuting Observables** : $[\hat{O}_A, \hat{O}_B] = 0$.  

(311)

As have seen in the above example, commuting observables can be measured simultaneously. We call such observables **Compatible Observables** or **Commuting Observables**. Physically, this means that $\hat{O}_A$ and $\hat{O}_B$ has definite eigenvalues in $\psi$.

Now, let’s state an extremely important theorem.

**Theorem (Simultaneous Basis of Eigenfunctions):** Suppose $\hat{O}_A$ and $\hat{O}_A$ commute, then they share (at least) a basis of simultaneous eigenfunctions.

**Proof:** We will prove this Theorem for the special case where at least one of the operator is non-degenerate. Assuming $\hat{O}_A$ is no-degenerate, so it possess a set of eigenfunctions $\{\psi_{i} \}$ with distinct eigenvalues $\{a_{i} \}$. By the eigenvalue equation

$$\hat{O}_A\psi_{i} = a_{i}\psi_{i},$$  

(312)

and operating from the left with $\hat{O}_B$,

$$\hat{O}_B\hat{O}_A\psi_{i} = a_{i}\hat{O}_B\psi_{i},$$  

(313)

and using commutativity $[\hat{O}_A, \hat{O}_B] = 0$,

$$\hat{O}_A(\hat{O}_B\psi_{i}) = a_{i}(\hat{O}_B\psi_{i}),$$  

(314)

which is to say that $\hat{O}_B\psi_{i}$ is also an eigenfunction of $\hat{O}_A$ with eigenvalue $a_i$. But since $\hat{O}_A$ is degenerate, $\hat{O}_B\psi_{i}$ must be the same eigenfunction as $\psi_{i}$ up to a (for the moment possibly complex) number $\lambda$ (recall that there exist an equivalence class of wavefunctions see Eq. (71)) as $\psi_{i}$, i.e.

$$\hat{O}_B\psi_{i} = \lambda\psi_{i}.$$  

(315)

But this is nothing but an eigenvalue equation for $\hat{O}_B$ and we identify $\lambda$ as its eigenvalue, which by Hermiticity is real. Since every eigenfunction of $\hat{O}_A$ is also an eigenfunction of $\hat{O}_B$, it is clear that
\{\psi_a\} forms a complete basis for both operators. In this special case where \(\hat{O}_A\) is non-degenerate, there is only one such basis. \(\square\)
The proof for the case where both operators are degenerate is much more involved. Those interested can see Shankar (pg 45).

Since \(\psi_a\) is also an eigenfunction of both \(\hat{O}_A\) and \(\hat{O}_B\), and we can also give it a \(b\) label \(\psi_{a,b}\), and we say that \(\psi_{a,b}\) are \textbf{Simultaneous Eigenfunctions} of \(\hat{O}_A\) and \(\hat{O}_B\).

\textbf{Example:} Harking back to section 4.1, recall that \(\hat{p}\) and \(\hat{H}_{\text{free}}\) share the same Eigenfunctions \(u_{p,E}(x)\) where now we have democratically label the eigenfunction without prejudice to any operator:

\[\hat{H}_{\text{free}} u_{p,E}(x) = E u_{p,E}(x), \quad \frac{d u_{p,E}(x)}{dx} = p u_{p,E}(x). \quad (316)\]

We will see another case of degeneracy and simultaneous eigenfunctions when we discuss angular momentum in section 8.4.

\textbf{Example:} \(\psi(x)\) is a reflection symmetric potential \(U(x)\), if \(\chi_E(x)\) is an eigenfunction of \(\hat{H}\) with energy \(E\), then so is \(\chi_E(-x)\), then

\[\hat{P}\chi_E(x) = \hat{P}\chi_E(-x) \quad \text{and} \quad \hat{H}\chi_E(x) = \hat{H}\chi_E(-x) \quad \text{and} \quad E\chi_E(-x) = E\chi_E(x)\]

and by Completeness of the eigenfunctions of \(\hat{H}\), the proof is complete. \(\square\)

\textbf{Conservation Laws:} In Classical Mechanics, some observables are \textit{conserved under time evolution} if the potential \(U(x)\) has some symmetry. For example, if \(U(x) = f(r)\) is spherically symmetric, then we know that the total angular momentum \(\mathbf{L}\) is conserved. In Quantum Mechanics, conservation laws are expressed as the \textit{vanishing of the observable with the Hamiltonian}, i.e. if \(\hat{O}\) commutes with \(\hat{H}\)

\[\{\hat{O}, \hat{H}\} = 0 \quad (319)\]

then the observable is conserved under time evolution. In the above example with Parity, you can see from the many examples in section 5 that if a state has a definite parity, then this parity is conserved under time evolution as long as the Potential is symmetric under reflection.

\textbf{Non-commuting Observables:} The definition for non-commuting observables \(\hat{O}_A\) an \(\hat{O}_B\) is simply

\[\text{non – Commuting Observables} : [\hat{O}_A, \hat{O}_B] \neq 0 \quad (320)\]

In words, we say that “\(\hat{O}_A\) and \(\hat{O}_B\) do not commute”.

As you can easily prove to yourself, non-commuting observables do not share eigenfunctions, hence from the example at the start of this section this means that observations of one will now affect the observations of the other.

An example of this is our favorite pair of observables \(\hat{p}\) and \(\hat{x}\). Acting on some generic state \(\psi(x)\) we find

\[\{\hat{x}_i, \hat{x}_j\} \psi(x) = (x_i x_j - x_j x_i) \psi(x) = 0 \quad (321)\]
while

\[ [\hat{p}_i, \hat{p}_j] \psi(x) = (-\hbar)^2 \left[ \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \right] \psi(x) = 0 \] (322)

using the symmetry of mixed partial derivatives. Finally,

\[ [\hat{x}_i, \hat{p}_j] \psi(x) = \left[ x_i \left( -i\hbar \frac{\partial}{\partial x_j} \right) - \left( -i\hbar \frac{\partial}{\partial x_j} \right) x_i \right] \psi(x) \]

\[ = -i\hbar \left( x_i \frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_j} x_i \right) \psi(x) \]

\[ = -i\hbar \left( \frac{\partial f}{\partial x_j} x_i - \frac{\partial f}{\partial x_j} x_i \right) \psi(x) \]

\[ = i\hbar \delta_{ij} \psi(x). \] (323)

We obtain the **Canonical Commutator Relationships** for \( \hat{x}_i \) and \( \hat{p}_i \)

\[ [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0 \] (324)

As we will see in the next section, non-commuting observables lead to the Uncertainty Principle.

*Canonical Quantization*: In the lectures, we have derived Eq. (324) from our definitions of \( \hat{x} \) and \( \hat{p} \), working in the position basis. However, if we take away the basis, we can impose the canonical commutator relations, i.e. specifying Eq. (324) as the starting point for Quantum Mechanics and then deriving the position (or any other) basis operators from there. This is the more usual “modern” view, although our approach of deriving the momentum operator from the properties of translation is, in the view of some, more general. *

### 7.3 Non-commuting Observables and The Uncertainty Principle

As we told the story at the start of this section, if two observables \( \hat{O}_1 \) and \( \hat{O}_2 \) do not commute, then the order of the measurements matter. Indeed, since say when the measurement associated with observable \( \hat{O}_1 \) is made, the wavefunction collapses into one of its eigenstate, some of the information associated with \( \hat{O}_2 \) is “lost” so to speak. In this section, we will quantify this.

**Definition** Uncertainty Operator: The uncertainty of a state \( \psi \) with respect to an observable \( \hat{O} \) is defined as

\[ \Delta \hat{O} = \hat{O} - \langle \hat{O} \rangle_{\psi}. \] (325)

This operator has the following properties

- **\( \Delta \hat{O} \) is Hermitian. Proof**: As \( \hat{O} \) is an observable, it must be Hermitian, and since \( \langle \hat{O} \rangle_{\psi} \) is just a number, \( \Delta \hat{O} \) must also be Hermitian. □

- **Dispersion**: The expectation value of \( (\Delta \hat{O})^2 \) of a state \( \psi \) is known as the dispersion, and has the following form

\[ \langle (\Delta \hat{O})^2 \rangle_{\psi} = \langle \hat{O}^2 \rangle_{\psi} - 2\hat{O}\langle \hat{O} \rangle_{\psi} \] (326)

or

\[ \langle (\Delta \hat{O})^2 \rangle_{\psi} = \langle \hat{O}^2 \rangle_{\psi} - \langle \hat{O} \rangle_{\psi}^2 \] (327)

\[ \langle (\Delta \hat{O})^2 \rangle_{\psi} = \langle \hat{O}^2 \rangle_{\psi} - \langle \hat{O} \rangle_{\psi} \] (328)

i.e. the dispersion of \( \hat{O} \) is the “expectation of the square minus the square of the expectation”, which is consistent with the classical notion of a dispersion of an ensemble.
Furthermore, if χ is a normalized eigenfunction of \( \hat{O} \) then \( \langle \Delta \hat{O}^2 \rangle_\chi = 0 \). Proof is by direct application of Eq. (328):

\[
\langle \hat{O}^2 \rangle_\chi = \chi \cdot (\hat{O}^2 \chi) = \lambda^2 \quad \text{and} \quad \langle \hat{O} \rangle_\chi^2 = (\chi \cdot (\hat{O} \chi)) = \lambda^2
\]

so

\[
\langle \hat{O}^2 \rangle_\chi - \langle \hat{O} \rangle_\chi^2 = 0 \quad \Box.
\]

In other words, if the state \( \chi_A \) is an eigenstate of \( \hat{O}_A \) then the uncertainty is zero and we measure it with probability 1, which is a trivial statement. What is non-trivial is that if \( \hat{O}_B \) is another observable which does not commute with \( \hat{O}_A \), then its uncertainty in any simultaneous measurement on \( \chi_A \) will be infinite! We now state the general form of the **Uncertainty Principle**:

**Uncertainty Principle**: For any given two observables \( \hat{O}_A \) and \( \hat{O}_B \), then the following uncertainty relation holds for any state \( \psi \)

\[
\langle \Delta \hat{O}_A^2 \rangle_\psi \langle \Delta \hat{O}_B^2 \rangle_\psi \geq \frac{1}{4} \langle [\hat{O}_A, \hat{O}_B]^2 \rangle_\psi.
\]

In the case when \( \hat{O}_A = \hat{x} \) and \( \hat{O}_B = \hat{p} \), then using the canonical commutator relation \([\hat{x}, \hat{p}] = i\hbar\), we get the original famous **Heisenberg Uncertainty Principle**

\[
\langle \Delta \hat{x}^2 \rangle \langle \Delta \hat{p}^2 \rangle \geq \frac{\hbar^2}{4}
\]

which you have already seen derived using the Gaussian wavepacket in section 5.4.

We want to prove the Uncertainty Principle Eq. (331) in a snazzy operator way\(^{16}\). To do this, we require 2 useful lemmas.

**Lemma 1 (The Schwarz inequality)**: For any two normalized states \( \psi \) and \( \phi \), then\(^{17}\)

\[
(\psi \cdot \psi)(\phi \cdot \phi) \geq |\psi \cdot \phi|^2.
\]

**Proof**: For any complex number \( \lambda \) and any two normalized states \( \psi \) and \( \phi \), we can construct a state

\[
\Phi = \psi + \lambda \phi
\]

and then

\[
\Phi \cdot \Phi \geq 0 \forall \lambda
\]

since \( \Phi \) is just another state and its norm must be \( \geq 0 \) but \( \leq \infty \) if both \( \psi \) and \( \phi \) are normalizable. If we now set

\[
\lambda = -(\phi \cdot \psi)(\phi \cdot \phi)
\]

and plug it into Eq. (335), we get Eq. (333). \( \Box \)

**Lemma 2**: An **anti-Hermitian operator** is defined to be a **linear** operator which obey the relationship

\[
\int_{\mathbb{R}^3} f^\dagger(x) \hat{C} g(x) \, dV = - \int_{\mathbb{R}^3} \left( \hat{C} f(x) \right)^\dagger g(x) \, dV.
\]

or more compactly

\[
\hat{C} \equiv -\hat{C}^\dagger.
\]

\(^{16}\)See Prof. Nick Dorey’s notes for a perhaps more direct way.

\(^{17}\)This inequality is analogous to the vector space inequality \( |a|^2 |b|^2 \geq |a \cdot b|^2 \) which you might have seen before.
The expectation values of anti-Hermitian operator is purely imaginary. Proof: Suppose χ(x) is a
normalized eigenfunction of Ĉ with eigenvalue λ, then taking expectation values of both Ĉ and Ĉ†

\[ \langle Ĉ \rangle_χ = \int_{\mathbb{R}^3} \chi(x) Ĉ \chi(x) \, dV = λ, \quad \text{and} \]

\[ \langle Ĉ \rangle_χ = \int_{\mathbb{R}^3} \chi(x) Ĉ \chi(x) \, dV \]

\[ = \int_{\mathbb{R}^3} (Ĉ \chi(x))^\dagger \chi(x) \, dV = \lambda^*, \quad \text{(339)} \]

and using Eq. (337) we see that $\lambda + \lambda^* = 0$ so $\lambda \in \mathbb{C}$, i.e. all its eigenvalues are purely imaginary. Using the Completeness property of linear operators, we can expand any state $ψ$ in this basis so it follows that the expectation value $\langle Ĉ \rangle_ψ \in \mathbb{C}$. □

We are now ready to prove Eq. (331).

**Proof (Uncertainty Principle):** Given a state $Ψ$, then operating on this state with the uncertainty operators $Δ O_A$ and $Δ O_B$ yield

\[ ψ = Δ O_A ψ \quad , \quad φ = Δ O_B ψ \quad \text{(341)} \]

where $ψ$ and $φ$ are some other states. Using Hermiticity of $Δ O_A$, we see that

\[ ψ \cdot ψ = \int_{\mathbb{R}^3} (Δ O_A ψ(x))^\dagger Δ O_A ψ(x) \, dV \]

\[ = \int_{\mathbb{R}^3} ψ(x)^\dagger Δ O_A^2 ψ(x) \, dV = \langle Δ O_A^2 \rangle_ψ \]

where we have used the Hermiticity of $Δ O_A$ in the 2nd line. Similarly we can calculate $φ \cdot φ = \langle Δ O_B^2 \rangle_φ$ and $ψ \cdot φ = \langle Δ O_A Δ O_B \rangle_ψ$.

Using Lemma 1, we then take the expectation value around the state $Ψ$ to get

\[ (ψ \cdot ψ)(φ \cdot φ) \geq |ψ \cdot φ|^2 \]

\[ \implies \langle Δ O_A^2 \rangle_ψ \langle Δ O_B^2 \rangle_φ \geq \langle Δ O_A Δ O_B \rangle_ψ^2. \quad \text{(343)} \]

We are halfway through the proof – our next task is to evaluate the RHS of Eq. (343). First we note that the following identity holds

\[ Δ O_A Δ O_B = \frac{1}{2} \left[ Δ O_A Δ O_B + Δ O_B Δ O_A \right] \]

\[ = \frac{1}{2} \left( Δ O_A Δ O_B - Δ O_B Δ O_A + Δ O_B Δ O_A + Δ O_A Δ O_B \right) \]

\[ = \frac{1}{2} [Δ O_A, Δ O_B] + \frac{1}{2} (Δ O_A Δ O_B + Δ O_B Δ O_A). \quad \text{(344)} \]

But the commutator $[Δ O_A, Δ O_B] = [O_A, O_B]$ is anti-Hermitian

\[ ([O_A, O_B]) = (O_A O_B - O_B O_A) = O_B O_A - O_A O_B = -[O_A, O_B] \quad \text{(345)} \]

while the last term on Eq. (344) is Hermitian

\[ \langle Δ O_A Δ O_B + Δ O_B Δ O_A \rangle^\dagger = Δ O_B Δ O_A + Δ O_A Δ O_B. \]

Hence the RHS of Eq. (343) becomes, using Lemma 2 for the expectation value of $[O_A, O_B]$,

\[ \langle Δ O_A Δ O_B \rangle_ψ^2 \geq \frac{1}{2} \left( \langle [O_A, O_B] \rangle \right) + \frac{1}{2} \left( \langle Δ O_A Δ O_B + Δ O_B Δ O_A \rangle \right)^2 \]

\[ \geq \frac{1}{4} \left( \langle [O_A, O_B] \rangle \right)^2 + \frac{1}{4} \left( \langle Δ O_A Δ O_B + Δ O_B Δ O_A \rangle \right)^2 \quad \text{(347)} \]

and since the last term can only make the inequality stronger, the proof is complete. □
7.4 Summary

In this section, we study the notion of simultaneous observations and elaborated on how some observables are inherently incompatible with each other and the measurement of one will destroy information of the other(s). Such incompatibility is encoded in mathematical language as non-commutativity of the operators associated with the observables.

We then show that the Postulates of Quantum Mechanics lead us to the Uncertainty Principle – which is a powerful consequence of Postulate 3 (collapse of a wavefunction after a measurement), restricting our ability to extract information out of a wavefunction. How much information is “destroyed” by the collapse is given by the amount of non-commutativity of the observables as indicated by Eq. (331). Returning to \( \hat{x} \) and \( \hat{p} \), their commutator is \( [\hat{x}, \hat{p}] = i\hbar \delta_{ij} \), i.e. the amount of “lost information” is proportional to the Planck’s Constant \( \hbar \), which sets the scale of Quantum Mechanics. Since Classically, no information is “lost” in any measurement, the “Classical Limit” of a quantum theory can be recovered by taking the limit \( \hbar \to 0 \).

This section marks the end of our formal development of Quantum Mechanics.