Exercise 1 (Error Estimates)

For a proper, lower semi-continuous and convex functional \( J : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\} \) with non-empty subdifferential \( \partial J \), the generalised Bregman distance is defined as

\[
D^p_J(u, v) := J(u) - J(v) - \langle p, u - v \rangle, \quad p \in \partial J(v).
\]

Furthermore we assume that for the \( J \)-minimising solution \( u^\dagger \) (as defined in the lecture) the source condition

\[
\exists w \in V : \quad K^* w \in \partial J(u^\dagger)
\]

is satisfied.

a) Show that \( D^p_J(u, v) \geq 0 \) for all \( u, v \in \mathcal{X} \) and \( p \in \partial J(v) \).

b) Show that for \( f^\delta \in V \) and \( f = Ku^\dagger \), satisfying \( \|f - f^\delta\|_V \leq \delta \), there exists a \( p \) such that the estimate

\[
D^p_J(u_\alpha, u^\dagger) \leq \frac{\delta^2}{2\alpha} + \frac{\alpha\|w\|^2_V}{2}
\]

holds true. Here \( u_\alpha \) is a minimiser of \( \tilde{T}_\alpha \) as introduced in the lecture.

Exercise 2 (Exact recovery)

Let \( u_\lambda \neq 0 \) be a generalised Eigenfunction satisfying \( \lambda K^* Ku_\lambda \in \partial J(u_\lambda) \) and \( \|Ku_\lambda\|_V = 1 \). Further let \( J \) be absolutely one-homogeneous, i.e. \( J(cu) = |c|J(u) \) for all constants \( c \in \mathbb{R} \).

a) Compute \( \lambda \).

b) Show that \( u_\alpha = \max\{\gamma - \lambda \alpha, 0\}u_\lambda \) is a global minimiser of \( \tilde{T}_\alpha \) as introduced in the lecture, for \( f = \gamma Ku_\lambda \).

c) Let \( \lambda_0 := \min_{\|Ku\|_V = 1} J(u) \). Prove that minimisers \( u_\alpha \) of \( \tilde{T}_\alpha \) have a systematic bias of the form

\[
\|Ku_\alpha - f\|_V \geq \alpha \lambda_0
\]

if \( \|f\|_V \geq \alpha \lambda_0 \).
Exercise 3 (Deconvolution)

The discrete two-dimensional convolution of a signal $u \in \mathbb{R}^{n_1 \times n_2}$ with a kernel $g \in \mathbb{R}^{(2r_1+1) \times (2r_2+1)}$ can be described via

$$f_{i_1,i_2} = (Cu)_{i_1,i_2} := \sum_{k_1=-r_1}^{r_1} \sum_{k_2=-r_2}^{r_2} u_{k_1,k_2} g_{i_1-k_1,i_2-k_2},$$

for $i_1, r_1 \in \{1, \ldots, n_1\}$, $i_2, r_2 \in \{1, \ldots, n_2\}$.

a) **Forward problem**: Implement the discrete two-dimensional convolution operator (with periodic boundary conditions) as a MATLAB©-class named convolution. An object $C$ of this class should be initialised via the command $C = \text{convolution}(\text{imsizes}, \text{kernel});$. Here imsize denotes the image dimensions of the two-dimensional image to be convolved ($n_1$ and $n_2$ in (3)), and kernel is the convolution kernel. Moreover, the object should be able to compute the convolution via the call $f = C*u$; where $f$ and $u$ are column vector representations of the convolved image $f$ and the original image $u$. Further should it be possible to compute the adjoint (transpose) convolution via $u = C'f$.

**Hint**: Make use of the discrete convolution theorem that allows to diagonalise the convolution operator in the discrete Fourier transform domain.

b) **Regularised inverse problem**: Implement the following discrete deconvolution model

$$u_\alpha \in \arg \min_{u \in \mathbb{R}^n} \left\{ \frac{1}{2} \|Cu - f\|^2_2 + \alpha \max_{p \in \mathbb{R}^{n_1 \times n_2}} \langle \nabla u, p \rangle \right\}$$

via the primal-dual hybrid gradient method introduced in the lecture. Here $C$ denotes the discrete convolution (3), $n = n_1 n_2$, and $\nabla$ is a forward finite-difference approximation of the gradient.

**Hint**: The class $C$ of Exercise 3 a) is not necessarily required for this task.

c) Take your favourite picture, load it into MATLAB©via imread and convert it into a gray-value image with double-precision intensity values. Create an object of the class you have implemented in Exercise 3 a) to convolve this image with a convolution kernel kernel. Use the MATLAB©-command fspecial to create a convolution kernel of your choice. Subsequently, create different noisy versions of your blurry image for different noise levels.

d) Use the column-vector representations of your noisy images of Exercise 3 c) as the input for your deconvolution algorithm of Exercise 3 b). Compute reconstructions of (4) for different choices of $\alpha$ and visualise your results.
Exercise 4 (Bregman iteration)

With the help of the generalised Bregman distance (1) we can define the following iterative regularisation scheme (also known as Bregman iteration)

\[
\begin{align*}
\alpha u_{\alpha}^{k+1} & \in \arg \min_{u \in \mathcal{U}} \left\{ \frac{1}{2} \| Ku - f \|_{\mathcal{V}}^2 + \alpha J(u) - \langle p_{\alpha}^k, u \rangle \right\}, \\
\alpha p_{\alpha}^{k+1} & = p_{\alpha}^k + \frac{1}{2} K^* (f - Ku_{\alpha}^{k+1})
\end{align*}
\]

for \( \alpha > 0 \) and with \( u_{\alpha}^0 = p_{\alpha}^0 = 0 \) and \( p_{\alpha}^k \in \partial J(u_{\alpha}^k) \) for all \( k \in \mathbb{N} \).

a) Show that the iterates of (5) satisfy \( \| Ku_{\alpha}^{k+1} - f \|_{\mathcal{V}} \leq \| K u_{\alpha}^k - f \|_{\mathcal{V}} \).

b) Argue why the Bregman iteration together with Morozov’s discrepancy principle as a stopping criterion is a useful strategy to find \( J \)-minimising solutions. Give also an estimate for the parameter \( \eta \) in the discrepancy principle.

c) Verify the estimate

\[
D_{J^*}(u^\dagger, u_{\alpha}^k) \leq \frac{\alpha \| w \|_{\mathcal{V}}^2}{2k}
\]

for \( k \in \mathbb{N} \setminus \{1\} \), exact data \( (\delta = 0) \) and the source condition (2).

d) Show that iteration (5) can also be written as

\[
\begin{align*}
\alpha u_{\alpha}^{k+1} & \in \arg \min_{u \in \mathcal{U}} \left\{ \frac{1}{2} \| Ku - f_{\alpha}^k \|_{\mathcal{V}}^2 + \alpha J(u) \right\} ; \\
\alpha f_{\alpha}^{k+1} & = f_{\alpha}^k + f - Ku_{\alpha}^{k+1},
\end{align*}
\]

for \( f_{\alpha}^0 = f \).

e) Implement the Bregman iteration (5) for the deconvolution problem (4) of Exercise 3. Use the discrepancy principle as a stopping criterion, and compute reconstructions for different choices of \( \alpha \).