Branes, Bundles and Attractors

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Eurostrings / MBG-60, Cambridge, April 2006

Abstract

We use the attractor mechanism of IIA string theory to predict the Chern classes of stable bundles on Calabi-Yau threefolds.

Based on work to appear with R. Reinbacher and S.-T. Yau.

1. Introduction

At JHS60 in 2001, I asked whether the number of quasi-realistic string theory vacua is finite or infinite, and raised as an example the question of whether, among the set of CHSW heterotic string compactifications on a given Calabi-Yau M, there is a maximum number of (net) generations (the absolute value of the number of 27's of E_6 minus the number of $\bar{2}7$'s; resp. $SO(10)$, $SU(5)$).

In the large volume limit, this is equivalent to the following mathematical question: among the μ-stable holomorphic bundles V with rank $r = 3, 4, 5$, $c_1(V) = 0$, and $c_2(V) = c_2(M)$, is there a bound on $|c_3(V)|$. This is because these are the conditions for finding a solution of the $d = 10$ Yang-Mills equations with unbroken $N = 1$ supersymmetry (i.e. hermitian Yang-Mills), and with unbroken GUT gauge group, such that the equation

$$dH = \mathrm{tr}\ F^2 - trR^2$$

following from the Green-Schwarz anomaly cancellation mechanism can be solved.

While it is easy just to say "we observe 3 generations" and restrict attention to $c_3(V) = 6$, this question is a simpler analog of interesting questions like "are there a finite number of Calabi-Yau manifolds?" and "are there a finite number of possible hidden sector gauge groups?" If not, then in the absence of other (unknown) selection mechanisms, string theory might make an infinite number of distinct predictions for beyond the standard model physics, and we run the risk that it is not falsifiable. More generally, estimates for numbers of vacua determine the predictivity of string theory.

It is not very obvious whether such a bound on $|c_3|$ exists. We can compare with $c_2(V) = \mathrm{tr}\ F^2$, which for a supersymmetric solution must be positive in some sense (e.g. for a self-dual solution in $d = 4$, $F^2 = F \wedge *F > 0$). Since $c_3 = \mathrm{tr}\ F^3$ there is no positivity argument (e.g. $F \to -F$ produces another valid bundle).

Even if there is no mathematical bound of this type, it is possible that there is a bound after putting in more physics. For example, one can argue that the number of physically relevant Calabi-Yau manifolds must be finite, by appealing to Cheeger's theorem: all but finitely many run off to infinite volume (talk at Strings 2005; to appear with B. Acharya).

As it happens, we do not need this out. M. Maruyama and A. Langer have argued that, given r, $c_1(V)$ and $c_2(V)$, all the higher Chern classes can take at most finitely many distinct values.

These arguments are very mathematical (translation: I don't understand them) and it would be nice to have a more physical argument for this point.

2. Bundles and Attractors

Recently with R. Reinbacher and S.-T. Yau, we studied this problem by relating it to the "attractor conjectures" formulated by Greg Moore and Frederik Denef (based on earlier work of Ferrara Kallosh Strominger).

The idea is to consider a different physical application of holomorphic bundles and solutions of the hermitian Yang-Mills equations. Let us consider the IIa superstring compactified on M.

This theory has $b^{1,1}+1$ vector potentials, arising from the RR vector potential $C^{(1)}$, and the RR three-form integrated over a basis of $H_2(M,\mathbb{Z})$,

$$A_i = \int_{\Sigma_i} C^{(3)}.$$

It also has a spectrum of electrically and magnetically charged BPS states, obtained by wrapping Dirichlet branes on even-dimensional cycles.

Consider a configuration of r $D6$-branes wrapped on M. This will correspond to a BPS bound state if there exists a (strictly) $U(r)$ connection A on M which solves the hermitian Yang-Mills equations. The Chern character of this connection determines the lower D-brane charges, using the world-volume couplings

$$S = \int C^{(2k)} \wedge \mathrm{Tr}\ e^F \wedge \sqrt{\hat{A}(M)}.$$

Thus, the question "what are the allowed Chern characters of solutions of hermitian Yang-Mills on M," is equivalent to the question, "what is the spectrum of charges of BPS states of IIa string on M (in the large volume limit),

$$Q = \mathrm{ch}(V) = \mathrm{Tr}\ e^F$$

(we drop the curvature terms for the time being, and restore them later).

In itself, this is just rephrasing the question, but now we can appeal to a different physical regime of the theory. For a given charge Q, the validity of the D-brane description requires the string coupling g_s to satisfy $g_s|Q| << 1$.

If we continue the string coupling to the opposite regime $g_s|Q| >> 1$, we find a BPS particle with the same charge Q, but with space-time curvature $R \sim 1/l_s^2 g_s|Q| << M_{Planck}^2$.

Thus, this particle can be described as a solution of the $N = 2$, $d = 4$ supergravity obtained from IIa compactification on M. Assuming spherical symmetry, we find a black hole solution, with non-zero horizon area and entropy.

$$ds^2 = -\left(1 - \frac{2f(r)}{r}\right) dt^2 + \left(1 - \frac{2f(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Besides the metric and electromagnetic field strengths, the solution is characterized by the values of the scalar fields $t^i = B^i + iJ^i$ (complexified Kähler moduli). Thanks to supersymmetry, these are governed by a first order equation,

$$\frac{\partial}{\partial u} t^i = -g^{i\bar{j}} \frac{\partial}{\partial \bar{t}^{\bar{j}}} S[Q; t(u)]$$

where $u = 1/r$ and

$$S[Q;t] = e^K |Z[Q;t]|^2 = \frac{\left|\int_M e^{B+iJ} \wedge \mathrm{Tr}\ e^F\right|^2}{\int J^3}$$

would be the mass squared of a BPS brane of charge $Q = \mathrm{Tr}\ e^F$.

Thus, the moduli evolve by gradient flow to a local minimum of $S(Q;t)$. Its value at the minimum determines the area of the event horizon and thus the entropy of the black hole.

Thus, the question of whether a BPS particle of charge Q exists, is in part answered by the question: does the function $S[Q;t]$ have a non-zero local minimum? If it does, then a black hole of this charge exists. Since g_s is in a hypermultiplet, we can vary it to weak coupling, and conclude that a BPS particle of charge Q exists.

The other possibility is that $S[Q;t]$ is zero at the minimum. This can be subdivided into two cases: the minimum is achieved at a singularity of the moduli space, or at a regular point. Either way, no such simple conclusion follows. We discuss this case later, and for now state the "attractor conjecture" (G. Moore):

> If $S[Q;t]$ has a non-zero local minimum for some t, then a BPS particle exists with charge Q.

$$\frac{\partial S}{\partial t} = 0; \qquad S \neq 0.$$

In the case at hand, one can actually find the general solution of the attractor condition. One first observes that it can be rewritten as

$$0 = D_i Z = \left(\frac{\partial}{\partial t^i} + \frac{\partial K}{\partial t^i} \right) Z$$

with

$$Z = \int_M e^t \wedge \mathrm{Tr}\ e^F$$

and

$$K = -\log \int_M e^t \wedge \mathrm{Tr}\ e^{\bar{t}},$$

where $t = t^i \omega_i = B + iJ$.

One then checks that $D_i e^t$ is orthogonal under the integral to e^t (in the mirror IIb language, it is a $(2,1)$ form, while e^t is a $(3,0)$ form). Thus, choosing $e^F = e^t$ would produce a local minimum at t. This is not real, but one can take a sum with $e^{\bar{t}}$ as well to get a real solution.

Thus the general solution of the attractor condition is

$$\mathrm{Tr}\ e^F \cdot \mathrm{Td}\ M = \mathrm{Re}\ \bar{C} e^t$$

where C is a complex number and t is an "allowed" value of the two-form $B + iJ$. Allowed means that J is in the Kähler cone, satisfying $\int_\Sigma J^k > 0$ for any holomorphic submanifold Σ. We also restored the curvature terms in the world-volume action

$$\mathrm{Td}\ M = 1 + \frac{1}{24} R^2 + \ldots$$

Expanding out this solution, we get a fairly simple conjecture for at least a subset of the possible Chern characters of stable holomorphic bundles. Let us see what it says, and compare it to known mathematical results.

3. Comparison

$$Q = \mathrm{Tr}\ e^{F}\ \ \cdot \mathrm{Td}\ M = \mathrm{Re}\ \bar{C} e^{B+iJ}$$

Call the set of charges Q which can be obtained this way ATT, while the set of Chern characters of stable bundles is CH. The conjecture is that $ATT \subset CH$. Is it true?

Although there is no general mathematical classification of stable bundles on Calabi-Yau threefolds, some facts are known.

The first comment to make is that, given a stable bundle V, one can tensor it with a line bundle L to get a stable bundle $V \otimes L$. Physically, this is simply adding a $U(1)$ field strength F solving Maxwell's equations, which is possible for any F satisfying the Dirac quantization condition.

This property is also true of ATT, as we can shift B by an arbitrary amount. While we do not see the quantization condition in this semiclassical expression, this is not essential.

Since we can shift $c_1(V)$ by an arbitrary amount, we can now restrict attention to $c_1(V) = 0$. One then finds that $c_2(V)$ can be the square of any class J in the Kähler cone. In particular, it will satisfy the inequality

$$0 \leq \int_M c_2(V) \wedge J \qquad (\text{for } c_1 = 0).$$

This is the Bogomolov inequality, which is true for all stable bundles. On surfaces this inequality more or less characterizes the set of BPS charges, up to some fine structure (shifts by small finite amounts).

Actually, taking the $\mathrm{Td}\, M$ correction into account, we get a slightly stronger inequality,

$$0 \leq \int_M \left(c_2(V) - \frac{1}{24} c_2(TM) \right) \wedge J,$$

which is true in all the cases we considered.

Given r, $c_1 = 0$ and prescribed $c_2(V)$, this data fixes the choice of $B + iJ$ and the real part of the complex number C, leaving a single remaining parameter, the imaginary part of C, write this as $C = r(1 + i\xi)$.

A series expansion of the attractor formula leads to a formula for c_3 in terms of this data,

$$\begin{aligned} c_3 &= \frac{2r\xi(1+\xi^2)}{3}\lambda^3 \tilde{H}^3 \\ &= \frac{2^{5/2} r}{3} \cdot \tilde{H}^3 \cdot \frac{\xi}{(1+\xi^2)^{1/2}} \end{aligned}$$

where $J = \lambda \tilde{H}$.

Since this expression has a finite $\xi \to \infty$ limit, this leads to a bound,

$$\frac{|c_3|}{r} \leq \frac{2^{5/2}}{3} \left(\frac{c_2}{r}\right)^{3/2}$$

where the square root is defined by finding a two-form $\tilde{H}$ as above. The precise coefficient appearing here is valid if we allow $J \to 0$, which is deep in the stringy regime. Since we dropped world-sheet instanton corrections, we might not believe this. If we put a cutoff at some $J \sim 1$, we get the same bound with a smaller but still order one coefficient.

Thus, we have a physical argument that the set of possible c_3 values for stable bundles could be finite. However it is not a conclusive argument, as we found a sufficient condition for bundles to exist, not a necessary condition.

The condition on c_2 turned out to be the same as the known necessary condition. Could this be a necessary condition as well?

As it turns out, it is easy to construct bundles with c_3 larger than this condition predicts, by the monad (linear sigma model) construction. This can produce

$$c_3 \sim c_2^2$$

as opposed to $(c_2)^{3/2}$. So our condition is not necessary.

We can learn more by comparing with the mathematical bounds of Maruyama and Langer.

Langer's bound has the general form

$$|c_3| \leq P(c_2, c_1, r)$$

where P is a polynomial which (we believe) is quadratic in c_2. So there is a better bound – can we get this from the attractor mechanism?

We need to deal with the case of zero S at the minimum, which as we discussed was not covered by the original attractor conjecture. S can be zero either at a singularity of moduli space, or a regular point. The first case covers BPS particles with zero entropy, but there is no clear criterion for existence. In the second case, a spherically symmetric black hole does not exist.

But – one cannot conclude that no BPS state exists. The loophole is that a non-spherically symmetric, or multi-center solution, can exist. These are described by the "split attractor flows" found by Denef.

4. Split attractors

A multi-center solution is like a bound state of several spherically symmetric solutions. The basic criteria for a bound state of two BPS particles to exist is that

- They must be "mutually nonlocal" – the electric-magnetic DSZ intersection form is nonzero. Order the two objects so this be positive,

$$e_1 m_2 - e_2 m_1 > 0$$

- Then, the ratio of the central charges must have a positive imaginary part,

$$\mathrm{Im}\,\frac{Z_1}{Z_2} > 0.$$

  This follows from supergravity and is also related to the mass squared of the lightest excitation of the particle pair (for D-branes, it leads to a tachyonic open string).

These conditions are a simplification of the conditions for two holomorphic bundles to admit a stable extension (by the index theorem, they imply the existence of the maps needed for an exact sequence, but not necessarily that the maps are injective and surjective).

Similarly, they are a simplification of the split attractor conditions. Checking the full conditions appears to require constructing explicit flows in moduli space.

Nevertheless, we might conjecture that if these conditions are satisfied, a BPS state of charge $Q_1 + Q_2$ will exist, and that adding such charge vectors will give a complete description of the set of all Chern characters of stable bundles.

Unfortunately, if we add in all bound states of pairs of attractors, using the two conditions we just gave, we find that all values of c_3 are possible. So we do not reproduce the (presumed) bound $|c_3| < c_2^2$. Why?

- Any statement we can make from the attractor mechanism (at tree level) is homogeneous, in the sense that the set of allowed charges is invariant under overall rescaling $Q \to NQ$. However, the mathematical arguments for existence of stable bundles do not have this property.

  Perhaps the real bound is not homogeneous under scaling up the charges. When we make it so, it becomes trivial.

- If the bound is homogeneous, then the only way out would be that we oversimplified the split attractor conditions; a better treatment should reproduce the correct bound.

Otherwise, there is a contradiction between the physics predictions and the mathematical theorem.