Hidden Symmetries and Fermions in M-Theory

Axel Kleinschmidt (Albert Einstein Institute, Potsdam)

EuroStrings 2006, Cambridge

April 6, 2006

Based on:

- T. Damour, M. Henneaux and H. Nicolai, E_{10} *and a 'small tension extension' of M-theory*, Phys. Rev. Lett. **89** (2002) 221601 [hep-th/0207267]
- AK and H. Nicolai, E_{10} *and* $SO(9,9)$ *invariant supergravity*, JHEP **0407** (2004) 041 [hep-th/0407101]
- AK and H. Nicolai, *IIB supergravity and* E_{10}, Phys. Lett. B **606** (2005) 391–402 [hep-th/0411225]
- T. Damour, AK and H. Nicolai, *Hidden symmetries and the fermionic sector of eleven-dimensional supergravity*, Phys. Lett. B **634** (2006) 319–324 [hep-th/0512163]
- AK and H. Nicolai, *IIA and IIB spinors from* $K(E_{10})$, [hep-th/0603205]

Plan

- Motivation
- Bosonic $E_{10}/K(E_{10})$ coset model in $D = 1$
- Spectral analysis
- Bosonic correspondence and versatility
- Fermionic correspondence and spinors of $K(E_{10})$
- Fermionic versatility
- Summary and discussion

Introduction: Hidden symmetries

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$D = 11$, $N = 1$ supergravity on T^n [Cremmer, Julia 1979]

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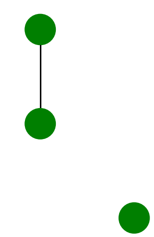

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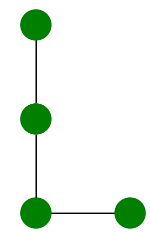

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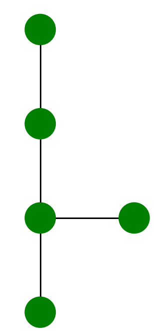

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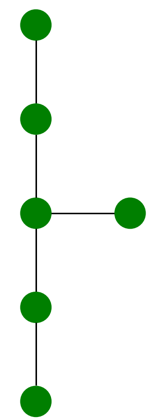

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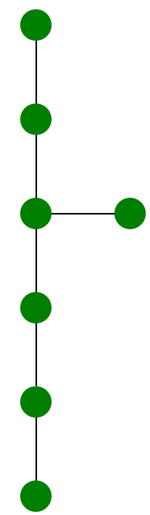

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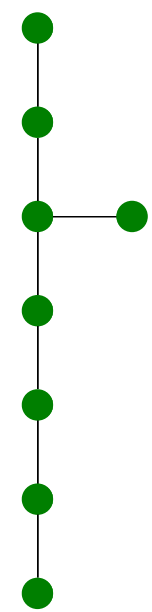

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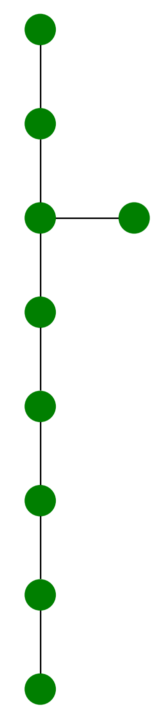

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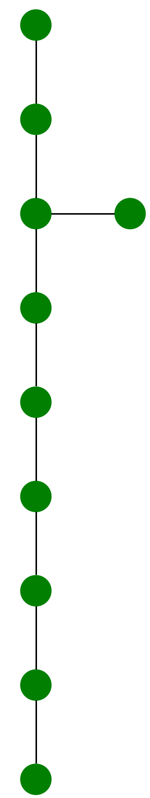

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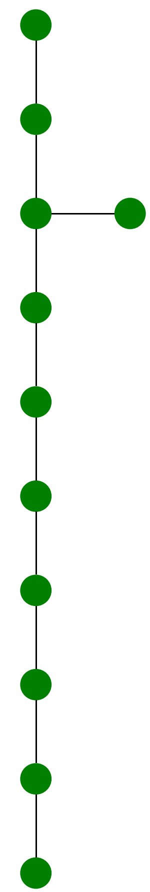

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Null geodesic motion of massless particle on $E_{10}/K(E_{10})$ coset $\mathcal{V}(t)$	Dynamical $\Longleftrightarrow$ equivalence	$D = 11$ SUGRA (or M-theory??) $G^{(11)}_{MN}(t, \mathbf{x})$ $A^{(11)}_{MNP}(t, \mathbf{x})$

[Damour, Henneaux, Nicolai 2002]

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Null geodesic motion of spinning massless particle on $E_{10}/K(E_{10})$ coset $\mathcal{V}(t)$ $\Psi(t)$	Dynamical $\Longleftrightarrow$ equivalence THIS TALK	$D=11$ SUGRA (or M-theory??) $G_{MN}^{(11)}(t,\mathbf{x})$ $A_{MNP}^{(11)}(t,\mathbf{x})$ $\psi_M^{(11)}(t,\mathbf{x})$

[DHN][Damour, AK, Nicolai 2006][de Buyl, Henneaux, Paulot 2006]

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Other approaches and related work

- E_{11} as symmetry before reduction

  [West 2001] [Schnakenburg, West 2001]

  $\Rightarrow$ see also talk by Peter West
- Geodesic E_{11} coset model

  [Englert, Houart 2003]
- Borcherds symmetries and mysterious dualities

  [Julia, Henry-Labordère, Paulot 2002]
- Automorphic forms E_{10} and Borcherds

  [Harvey, Moore 1995]

  [Dijkgraaf, Verlinde, Verlinde 1995]

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- [...]

$E_{10}/K(E_{10})$ coset (I)

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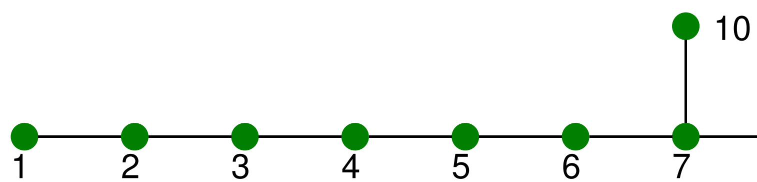
E_{10} is the Kac–Moody group with hyperbolic Kac–Moody Lie algebra $\mathfrak{e}_{10}$ of infinite dimension. Dynkin diagram



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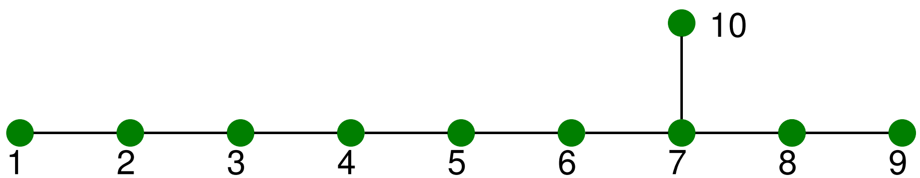

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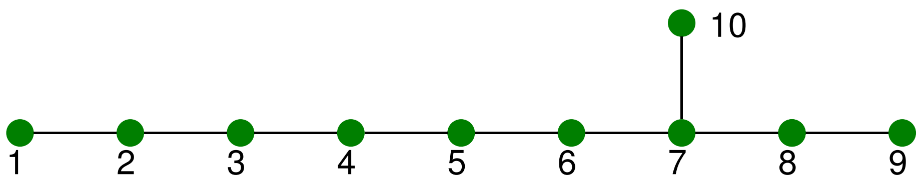

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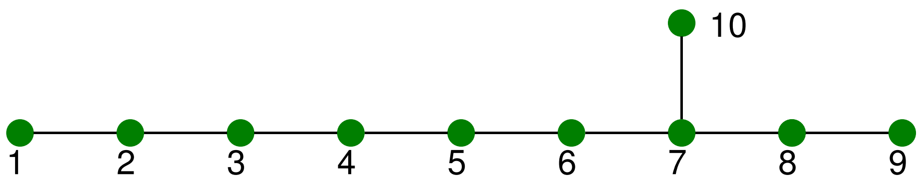

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- (Chevalley) transposition: $(\mathfrak{n}_\pm)^T = \mathfrak{n}_\mp, \quad (\mathfrak{h})^T = \mathfrak{h}$
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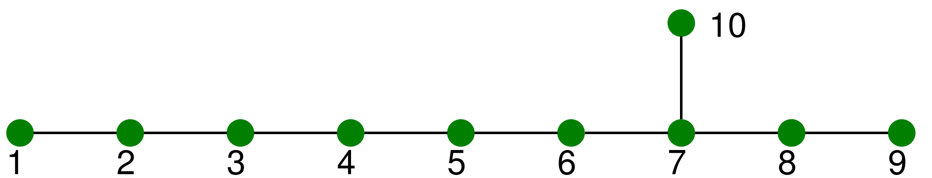

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Define the compact subalgebra $K(\mathfrak{e}_{10}) \subset \mathfrak{e}_{10}$ by

$$K(\mathfrak{e}_{10}) = \{ x \in \mathfrak{e}_{10} : x^T = -x \}$$

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Generalized Iwasawa parametrization (KAN) (Borel gauge)

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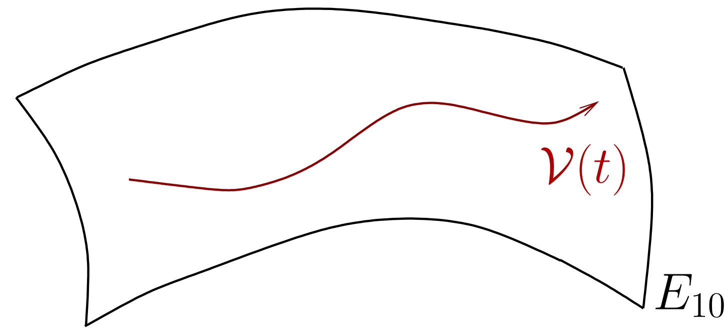

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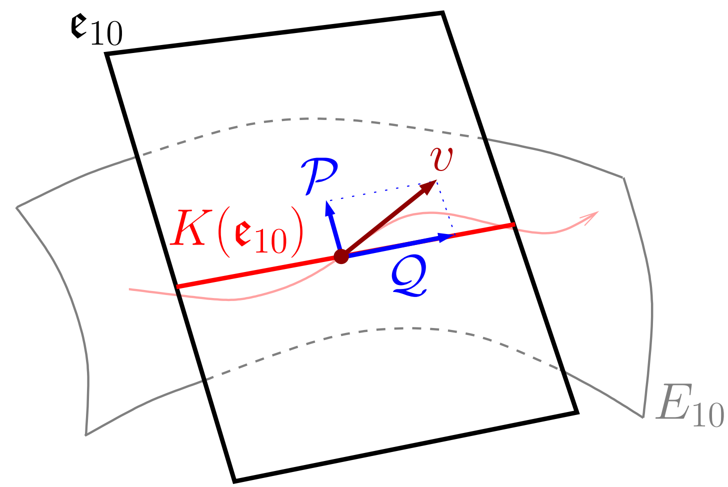

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Velocity $\quad v = \partial_t \mathcal{V}\mathcal{V}^{-1} \in \mathfrak{e}_{10}$

$$\mathcal{P}(t) = \frac{1}{2}\left(\partial_t \mathcal{V}\mathcal{V}^{-1} + (\partial_t \mathcal{V}\mathcal{V}^{-1})^T\right) \in \mathfrak{e}_{10} \ominus K(\mathfrak{e}_{10})$$

$$\mathcal{Q}(t) = \frac{1}{2}\left(\partial_t \mathcal{V}\mathcal{V}^{-1} - (\partial_t \mathcal{V}\mathcal{V}^{-1})^T\right) \in K(\mathfrak{e}_{10})$$

Transformation with $g \in E_{10}$ and $k(t) \in K(E_{10})$

$$\mathcal{V}(t) \to k(t)\mathcal{V}(t)g^{-1} \quad \Rightarrow \quad \mathcal{P} \to k\mathcal{P}k^{-1}, \quad \mathcal{Q} \to k\mathcal{Q}k^{-1} + \partial_t k k^{-1}$$

Bosonic dynamics: $D = 1$ σ-model

Lagrange function on coset $E_{10}/K(E_{10})$ [DHN]

$$\mathcal{L} = \mathcal{L}(t) = \frac{1}{2n(t)} \langle \mathcal{P}(t) | \mathcal{P}(t) \rangle$$

$\mathcal{L}$ is invariant under time reparametrizations, local $K(E_{10})$ and global E_{10} transformations.

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$\mathcal{D} = \partial_t - \mathcal{Q}$ is the $K(E_{10})$ covariant derivative:

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The system is classically (formally) integrable. [Lax pair]

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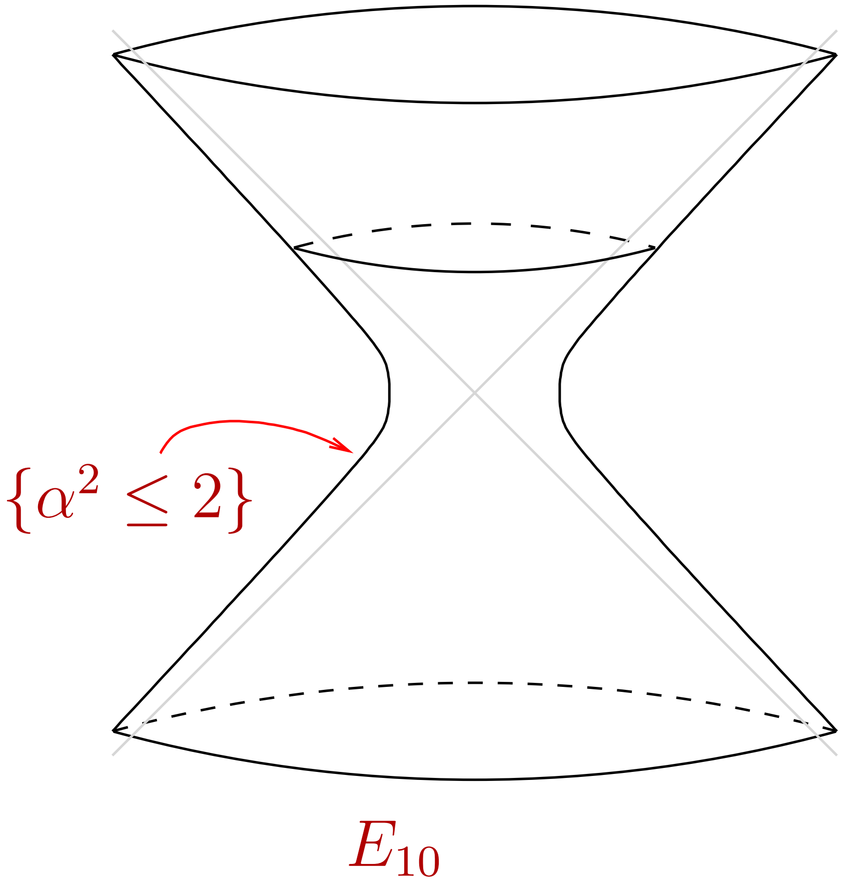

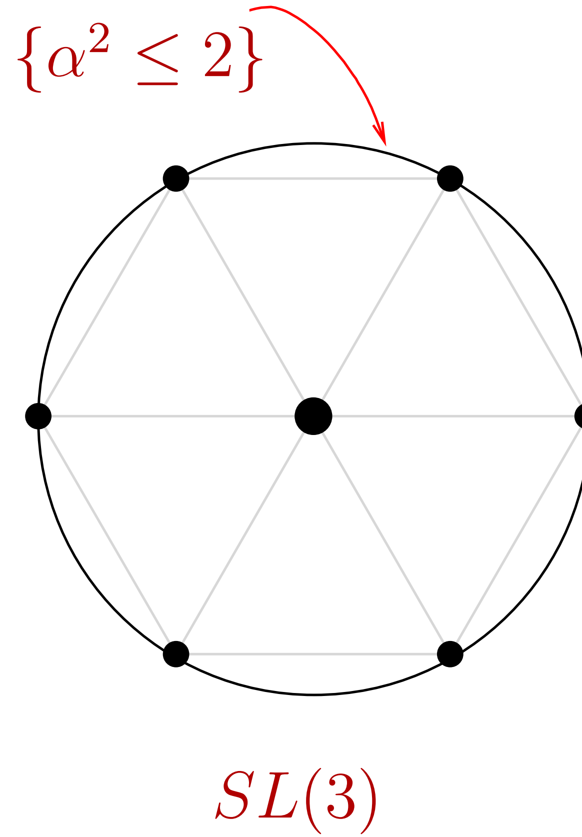

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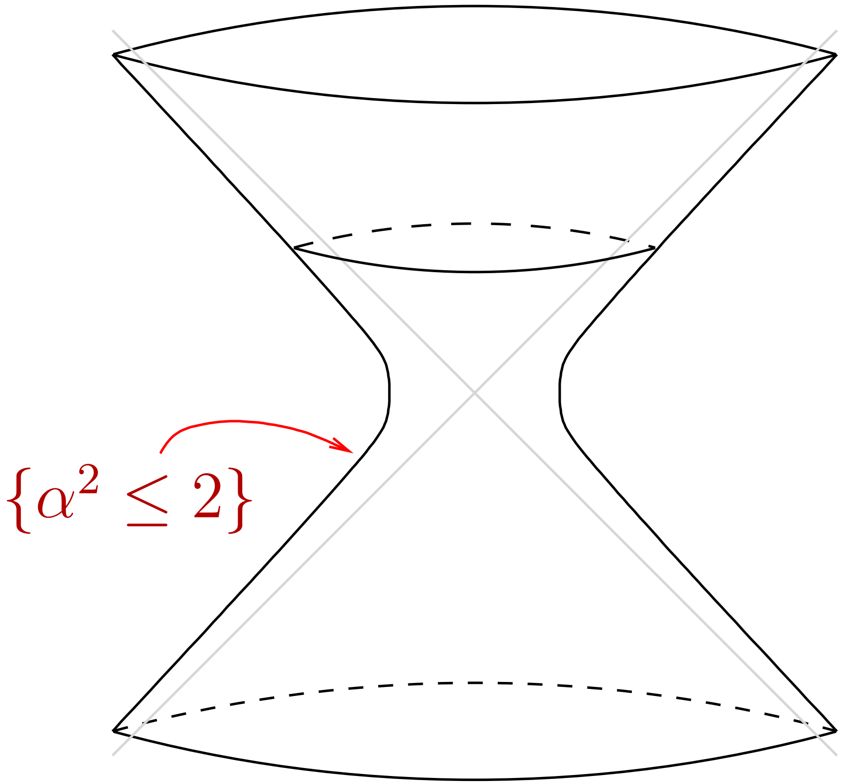
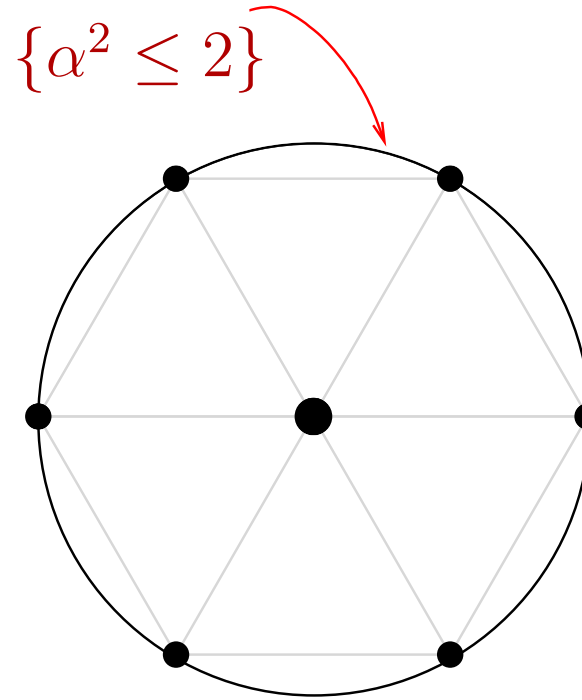


E_{10}



$SL(3)$

FACT: No closed formula known describing the root multiplicities and structure constants of $\mathfrak{e}_{10}$.

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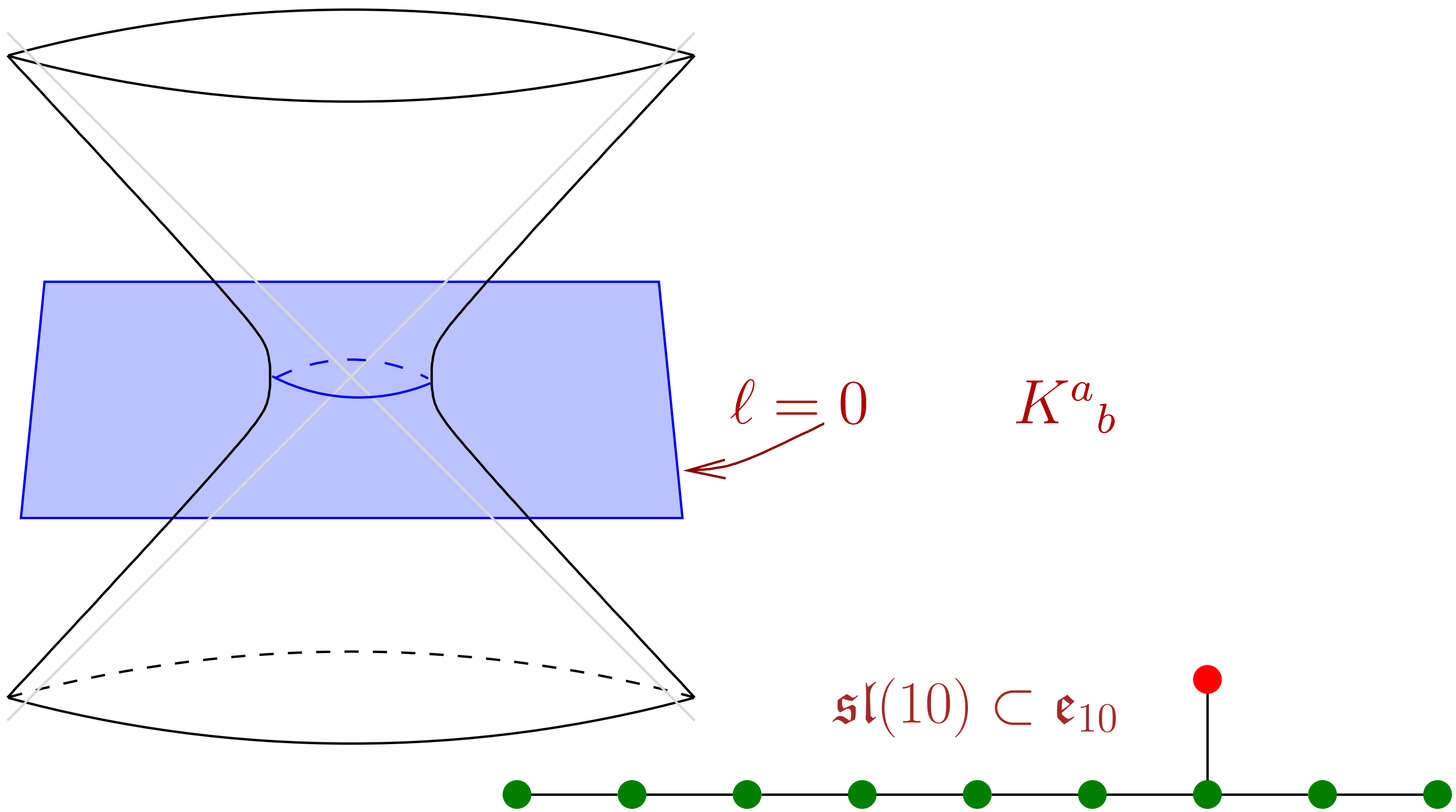

Follow recursive approach: [DHN]

Level Decomposition under finite subalgebra!

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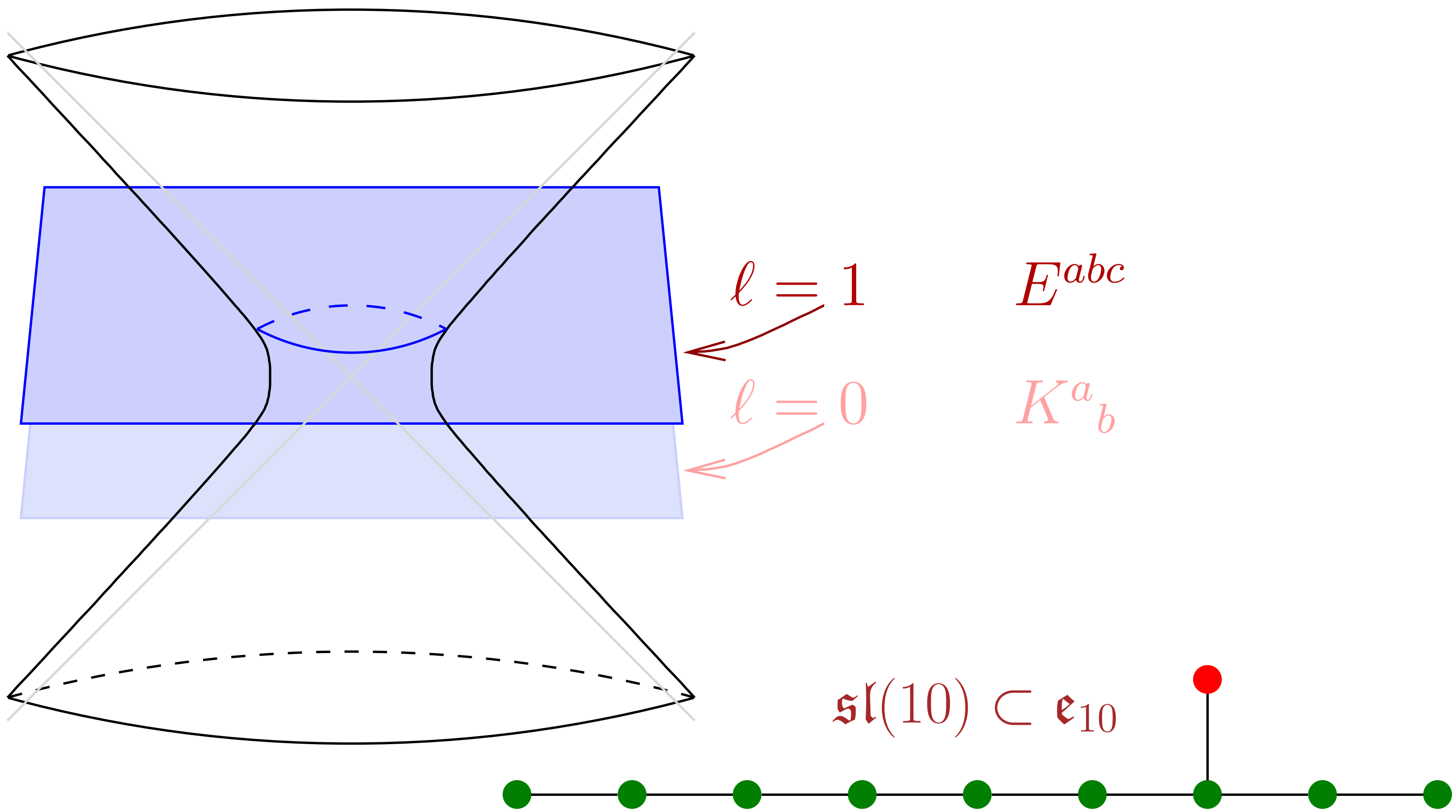

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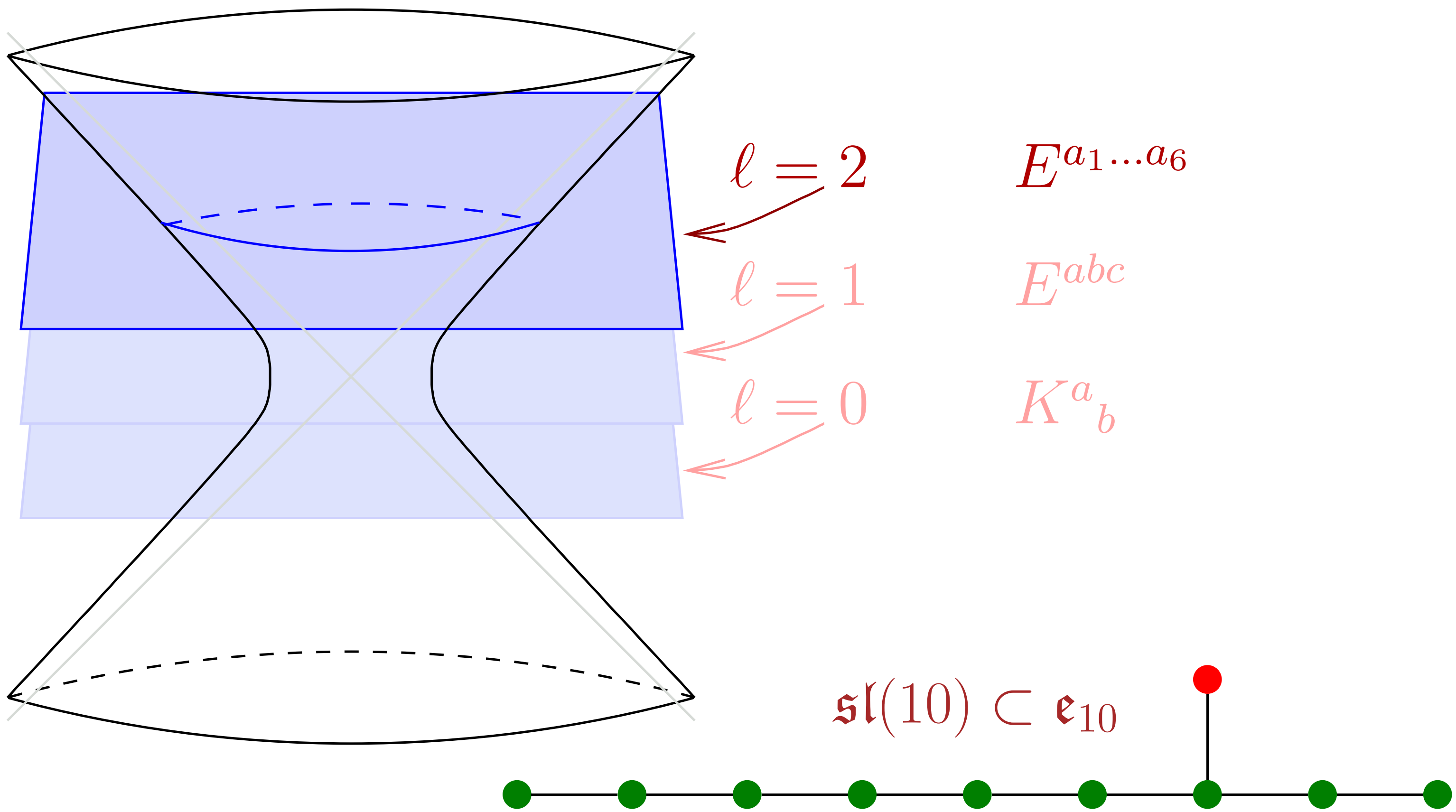

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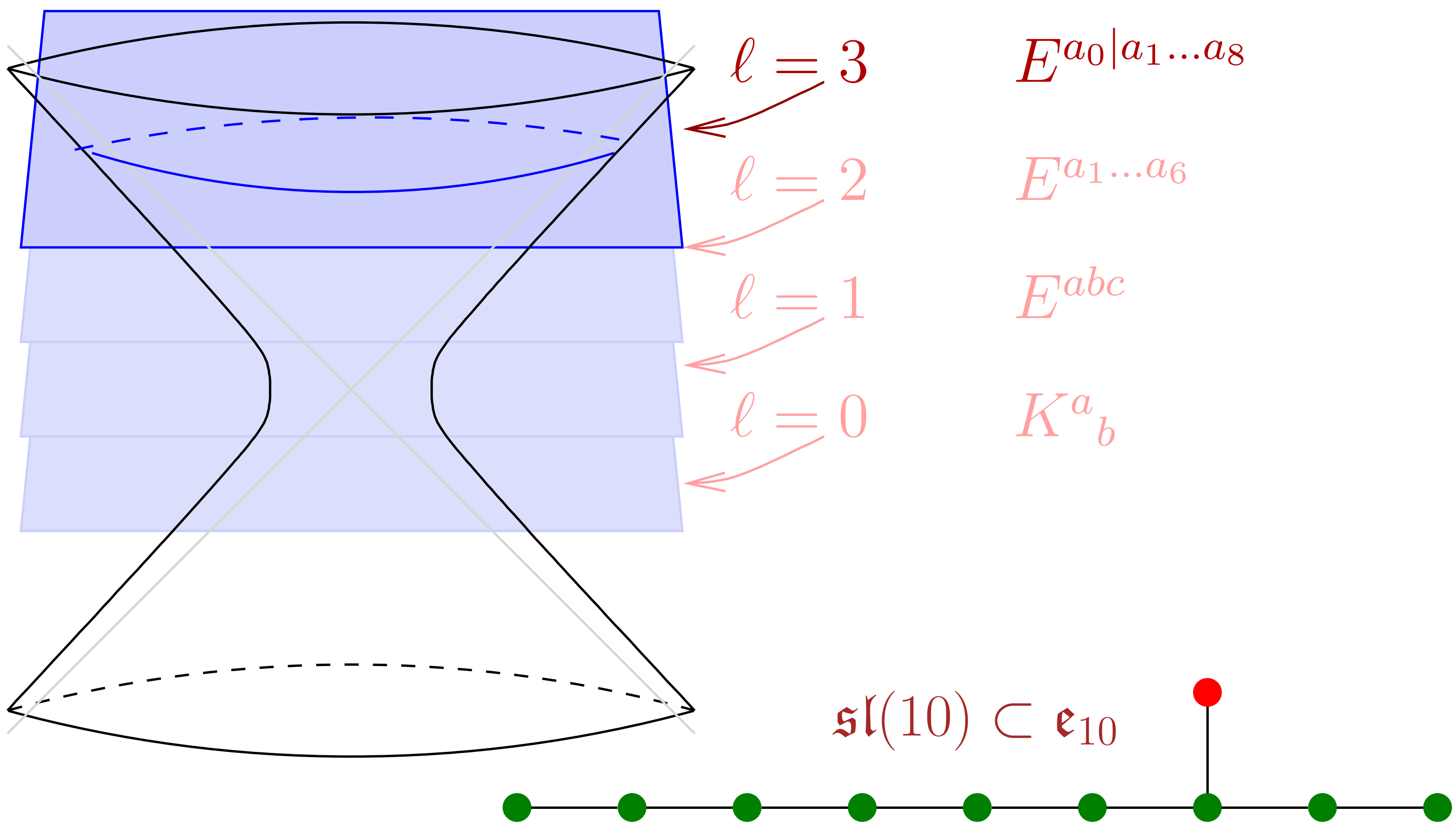

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Bosonic dynamical versatility

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- DHN correspondence for $\mathfrak{sl}(10)$ [DHN;Damour, Nicolai 2004]

$\mathcal{D}(n^{-1}\mathcal{P}) = 0$	$\Longleftrightarrow$	$D = 11$ (SU)GRA
consistent truncation	dictionary	EOM truncation

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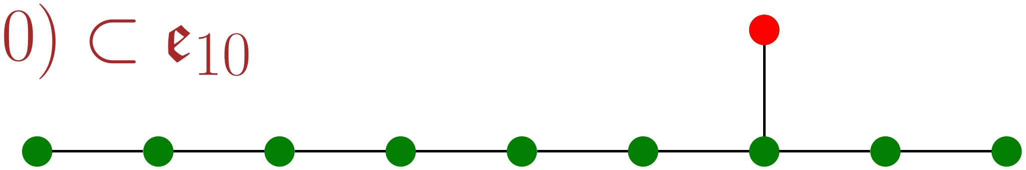
$\mathfrak{sl}(10) \subset \mathfrak{e}_{10}$



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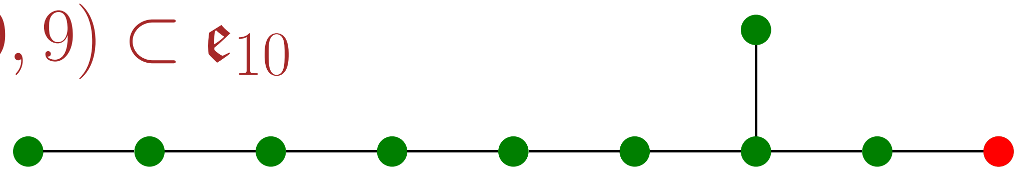
$\mathfrak{so}(9,9) \subset \mathfrak{e}_{10}$



$D = 10$, massive $N = 2$ type IIA supergravity

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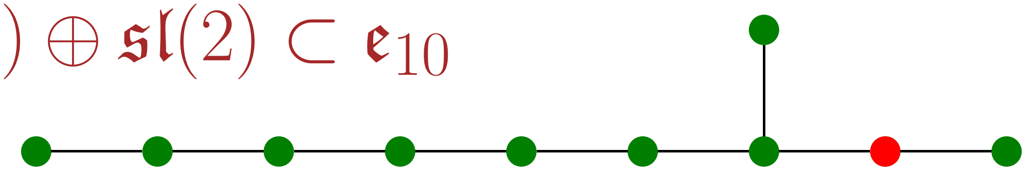
$\mathfrak{sl}(9) \oplus \mathfrak{sl}(2) \subset \mathfrak{e}_{10}$



$D = 10$, $N = 2$ type IIB supergravity

Bosonic dynamical versatility

- DHN correspondence for $\mathfrak{sl}(10)$ [DHN;Damour, Nicolai 2004]

$\mathcal{D}(n^{-1}\mathcal{P}) = 0$	$\Longleftrightarrow$	$D = 11$ (SU)GRA
consistent truncation	dictionary	EOM truncation

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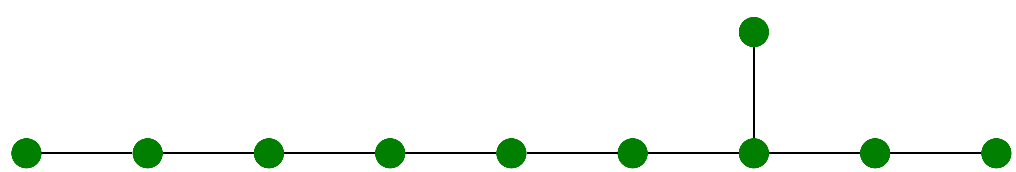

- Similar results have been known for E_{11}.
  [West 2001][Schnakenburg, West 2001]

Incorporating fermions

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- For hidden symmetries, fermions transform under the maximal compact subgroup $K(E_n)$ in $E_n/K(E_n)$.

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$$K(\mathfrak{e}_{10}) := \left\{ x \in \mathfrak{e}_{10} : x^T = -x \right\}$$

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  A useful analogy is $\mathfrak{so}(n) \subset \mathfrak{gl}(n)$.

- However: $K(\mathfrak{e}_{10})$ is not a (generalized) Kac–Moody algebra. No readily available representation theory.

Low $\mathfrak{sl}(10)$ level structure of $K(\mathfrak{e}_{10})$

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- Commutation relations [Damour, AK, Nicolai 2006] [West 2003]

$$\begin{aligned}
\left[J^{ab}, J^{cd}\right] &= \delta^{bc} J^{ad} + \delta^{ad} J^{bc} - \delta^{ac} J^{bd} - \delta^{bd} J^{ac} \equiv 4\delta^{bc} J^{ad} \\
\left[J^{a_1a_2a_3}, J^{b_1b_2b_3}\right] &= J^{a_1a_2a_3b_1b_2b_3} - 18\delta^{a_1b_1}\delta^{a_2b_2} J^{a_3b_3} \\
\left[J^{a_1a_2a_3}, J^{b_1\dots b_6}\right] &= J^{[a_1|a_2a_3]b_1\dots b_6} - 5!\, \delta^{a_1b_1}\delta^{a_2b_2}\delta^{a_3b_3} J^{b_4b_5b_6} \\
\left[J^{a_1\dots a_6}, J^{b_1\dots b_6}\right] &= -6\cdot 6!\, \delta^{a_1b_1}\cdots\delta^{a_5b_5} J^{a_6b_6} + \dots \\
\left[J^{a_1a_2a_3}, J^{b_0|b_1\dots b_8}\right] &= -336\left(\delta^{b_0b_1b_2}_{a_1a_2a_3} J^{b_3\dots b_8} - \delta^{b_1b_2b_3}_{a_1a_2a_3} J^{b_4\dots b_8b_0}\right) + \dots \\
\left[J^{a_1\dots a_6}, J^{b_0|b_1\dots b_8}\right] &= -8!\left(\delta^{b_0b_1\dots b_5}_{a_1\dots a_6} J^{b_6b_7b_8} - \delta^{b_1\dots b_6}_{a_1\dots a_6} J^{b_7b_8b_0}\right) + \dots \\
\left[J^{a_0|a_1\dots a_8}, J^{b_0|b_1\dots b_8}\right] &= -8\cdot 8!\left(\delta^{a_1\dots a_8}_{b_1\dots b_8} J^{a_0b_0} - \delta^{a_1\dots a_8}_{b_0b_1\dots b_7} J^{a_0b_8} - \delta^{a_0a_1\dots a_7}_{b_1\dots b_8} J^{a_8b_0}\right. \\
&\quad \left. +8\,\delta^{a_0}_{b_0}\delta^{a_1\dots a_7}_{b_1\dots b_7} J^{a_8b_8} + 7\delta^{a_1}_{b_0}\delta^{a_0a_2\dots a_7}_{b_1\dots b_7} J^{a_8b_8}\right) + \dots
\end{aligned}$$

with

$$J^{ab} = K^a{}_b - K^b{}_a \qquad J^{a_1a_2a_3} = E^{a_1a_2a_3} - (E^{a_1a_2a_3})^T$$

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- Representations?? $\Rightarrow$ Use input from supergravity!

Fermionic correspondence

Fermionic correspondence

- Know: Bosonic correspondence for $SL(10) \subset E_{10}$

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- Postulate: Fermionic correspondence for $SL(10) \subset E_{10}$

$\mathcal{D}(\Psi(t)) = 0$	$\Longleftrightarrow$	$D = 11$ SUGRA
KM truncation		EOM truncation

$\Psi(t)$ is a $K(\mathfrak{e}_{10})$ spinor and $\mathcal{D}$ is $K(\mathfrak{e}_{10})$ covariant

Fermions: Supergravity equation

- Equation of motion for $D = 11$ gravitino (flat indices) [CJS] $(A, B = 0, \ldots, 10, \quad a, b = 1, \ldots, 10)$

$$\Gamma^B \left[\left(D_A(\omega) + \mathcal{F}_A\right) \psi_B^{(11)} - \left(D_B(\omega) + \mathcal{F}_B\right) \psi_A^{(11)} \right] = 0$$

with $\mathcal{F}_A := +\frac{1}{144}\left(\Gamma_A{}^{BCDE} - 8\delta_A^B \Gamma^{CDE}\right) F_{BCDE}^{(11)}$.

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with $\mathcal{F}_A := +\frac{1}{144}(\Gamma_A{}^{BCDE} - 8\delta_A^B \Gamma^{CDE}) F^{(11)}_{BCDE}$.

- Use $D = 11$ supersymmetry to fix $\psi_0^{(11)} = \Gamma_0 \Gamma^a \psi_a^{(11)}$ and rescale $\psi_a^{(10)} := e^{1/2} \psi_a^{(11)}$. $\psi_a^{(10)}$ has 320 components.
- Fix pseudo-Gaussian gauge for the vielbein

$$E_M{}^A = \begin{pmatrix} N & 0 \\ 0 & e_m{}^a \end{pmatrix}$$

- Evaluate spatial a component of supergravity equation...

Making the fermionic correspondence

SUGRA

$$
\begin{aligned}
0 = \mathcal{E}_a := \;& \partial_t \psi_a^{(10)} + \omega_{t\,ab}^{(11)} \psi^{(10)b} + \frac{1}{4}\omega_{t\,cd}^{(11)} \Gamma^{cd} \psi_a^{(10)} \\
& -\frac{1}{12} F_{tbcd}^{(11)} \Gamma^{bcd} \psi_a^{(10)} - \frac{2}{3} F_{tabc}^{(11)} \Gamma^{b} \psi^{(10)c} + \frac{1}{6} F_{tbcd}^{(11)} \Gamma_a{}^{bc} \psi^{(10)d} \\
& +\frac{N}{144} F_{bcde}^{(11)} \Gamma^0 \Gamma^{bcde} \psi_a^{(10)} + \frac{N}{9} F_{abcd}^{(11)} \Gamma^0 \Gamma^{bcde} \psi_e^{(10)} - \frac{N}{72} F_{bcde}^{(11)} \Gamma^0 \Gamma_{abcdef} \psi^{(10)f} \\
& +N \left(\omega_{a\,bc}^{(11)} - \omega_{b\,ac}^{(11)} \right) \Gamma^0 \Gamma^b \psi^{(10)c} + \frac{N}{2} \omega_{a\,bc}^{(11)} \Gamma^0 \Gamma^{bcd} \psi_d^{(10)} - \frac{N}{4} \omega_{b\,cd}^{(11)} \Gamma^0 \Gamma^{bcd} \psi_a^{(10)} \\
& +N e^{1/2} \Gamma^0 \Gamma^b \Big(2\partial_a \psi_b^{(11)} - \partial_b \psi_a^{(11)} - \frac{1}{2} \omega_{c\,cb}^{(11)} \psi_a^{(11)} - \omega_{0\,0a}^{(11)} \psi_b^{(11)} + \frac{1}{2} \omega_{0\,0b}^{(11)} \psi_a^{(11)} \Big)
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\end{aligned}
$$

Postulate

$\mathcal{D}(\Psi(t)) = 0$	$\Longleftrightarrow$	$\mathcal{E}_a = 0$
Kac–Moody		SUGRA

Spinor representation $\Psi = (\psi_a, \ldots)$, $\mathcal{D} = \partial_t - \mathcal{Q}$

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$$

Kac–Moody

$$
\begin{aligned}
0 \quad = \quad & \Big(\partial_t - \frac{1}{2} Q_{ab} \overset{\text{vs}}{J}{}^{ab}_{(0)} - \frac{1}{2} \cdot \frac{1}{3!} Q_{a_1a_2a_3} \overset{\text{vs}}{J}{}^{a_1a_2a_3}_{(1)} - \frac{1}{2} \cdot \frac{1}{6!} Q_{a_1\ldots a_6} \overset{\text{vs}}{J}{}^{a_1\ldots a_6}_{(2)} \\
& - \frac{1}{2} \cdot \frac{1}{9!} Q_{a_0|a_1\ldots a_9} \overset{\text{vs}}{J}{}^{a_0|a_1\ldots a_8}_{(3)} + \ldots \Big) (\psi_a, \ldots)
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Making the fermionic correspondence

SUGRA

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$$0 = \Big(\partial_t - \frac{1}{2} Q_{ab} \overset{\mathrm{vs}}{J}{}^{ab}_{(0)} - \frac{1}{2} \cdot \frac{1}{3!} Q_{a_1 a_2 a_3} \overset{\mathrm{vs}}{J}{}^{a_1 a_2 a_3}_{(1)} - \frac{1}{2} \cdot \frac{1}{6!} Q_{a_1 \ldots a_6} \overset{\mathrm{vs}}{J}{}^{a_1 \ldots a_6}_{(2)}$$
$$- \frac{1}{2} \cdot \frac{1}{9!} Q_{a_0 | a_1 \ldots a_9} \overset{\mathrm{vs}}{J}{}^{a_0 | a_1 \ldots a_8}_{(3)} + \ldots \Big) (\psi_a, \ldots)$$

Bosonic dictionary [DHN]

$$Q_{ab} \leftrightarrow -\omega_{t\,ab}^{(11)} \qquad Q_{a_1 a_2 a_3} \leftrightarrow 2F_{t a_1 a_2 a_3}^{(11)}$$
$$Q_{a_1 \ldots a_6} \leftrightarrow -\frac{2}{4!} N \epsilon_{a_1 \ldots a_6 b_1 \ldots b_4} F_{b_1 \ldots b_4}^{(11)} \qquad Q_{a_0 | a_1 \ldots a_8} \leftrightarrow \frac{3}{2} N \epsilon_{a_1 \ldots a_8 b_1 b_2} \tilde{\Omega}_{b_1 b_2\, a_0}^{(11)}$$

$K(\mathfrak{e}_{10})$ spinor representations: $SO(10)$

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- From $\mathcal{D}(\Psi(t)) = 0 \iff \mathcal{E}_a = 0$ for $\psi_a(t) \leftrightarrow \psi_a^{(10)}(t, \mathbf{x}_0)$ deduce $K(\mathfrak{e}_{10})$ action [DKN][de Buyl, Henneaux, Paulot 2006]

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\begin{aligned}
J_{(0)}^{a_1 a_2} \cdot \psi^b &= \frac{1}{2}\Gamma^{a_1 a_2}\psi^b + 2\delta^{b[a_1}\psi^{a_2]} \\
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- Similarly, can prove existence of an unfaithful Dirac representation of $K(\mathfrak{e}_{10})$ of dimension 32. [dBHP,DKN]

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- Related to discussion of ‘generalized holonomies’.

Extended $D = 1$ σ-model

- Lagrange function: Bosons

$$\mathcal{L} = \mathcal{L}(t) = \frac{1}{2n} \langle \mathcal{P} | \mathcal{P} \rangle$$

Extended $D = 1$ σ-model

- Lagrange function: Bosons and fermions

$$\mathcal{L} = \mathcal{L}(t) = \frac{1}{2n}\langle \mathcal{P} | \mathcal{P} \rangle - i(\Psi | \mathcal{D}\Psi)_{vs}$$

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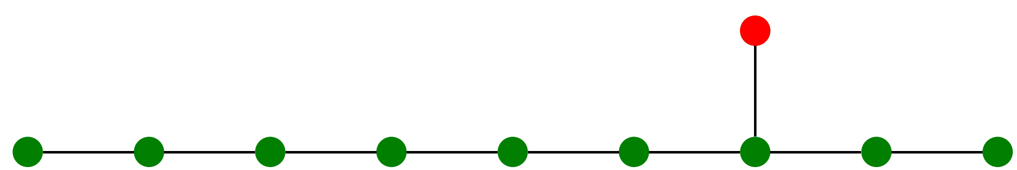
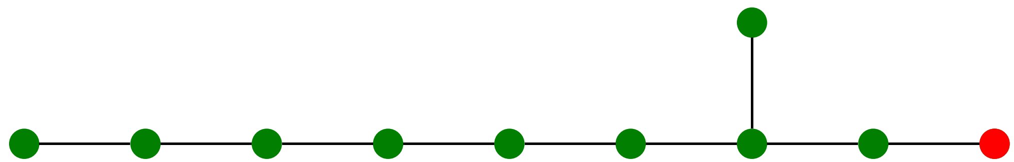
- Invariant symmetric form $(\cdot|\cdot)_{vs}$ has been constructed for **320** representation of $K(\mathfrak{e}_{10})$. [DKN]
- This was checked for $D = 11$ supergravity



Fermionic versatility: $SO(9) \times SO(9)$



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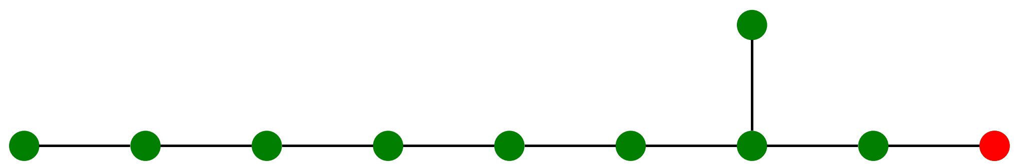

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$$X^{ij} := \frac{1}{2}\left(J^{ij}_{(0)} + J^{ij10}_{(1)}\right), \quad X^{\bar{i}\bar{j}} := \frac{1}{2}\left(J^{ij}_{(0)} - J^{ij10}_{(1)}\right)$$

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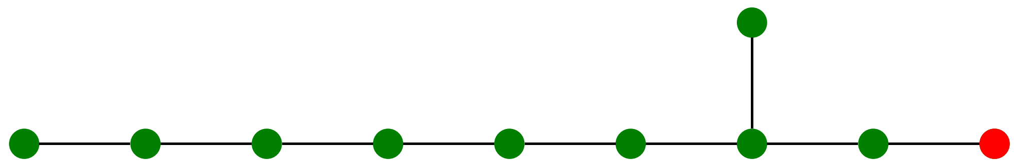

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- Use basis for Γ-matrices with: $\Gamma^{10} = \begin{pmatrix} \mathbf{1}_{16} & 0 \\ 0 & -\mathbf{1}_{16} \end{pmatrix}$

Projectors: $P_{\pm} := \frac{1}{2}\left(\mathbf{1}_{16} \pm \Gamma^{10}\right)$

Fermionic versatility: $SO(9) \times SO(9)$

- Define [AK, Nicolai 2004] $(k = 1, \ldots, 9)$

$$\begin{aligned} \tilde{\psi}_k &= \psi_k + \frac{1}{2}\Gamma_k \Gamma^{10} \psi_{10} \\ \tilde{\psi}_{10} &= -\frac{3}{2}\psi_{10} - \Gamma_{10}\Gamma^k \psi_k \end{aligned}$$

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- From $K(\mathfrak{e}_{10})$ action and $SO(9) \times SO(9)$ basis e.g.

$$X^{ij} \cdot P_- \tilde{\psi}_{10} = \frac{1}{2}\Gamma^{ij} P_- \tilde{\psi}_{10}$$
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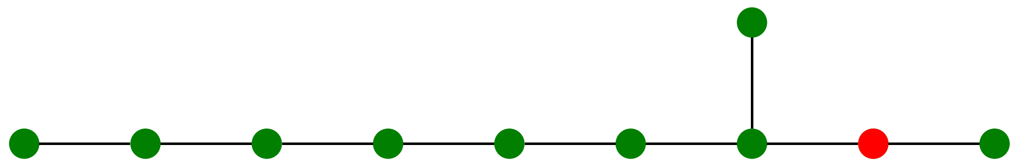
- Implies decomposition to IIA fermions!

$$\mathbf{320} \quad \longrightarrow \quad (\mathbf{9}, \mathbf{16}) \oplus (\mathbf{1}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{9}) \oplus (\mathbf{16}, \mathbf{1})$$

Fermionic versatility: $SO(9) \times SO(2)$



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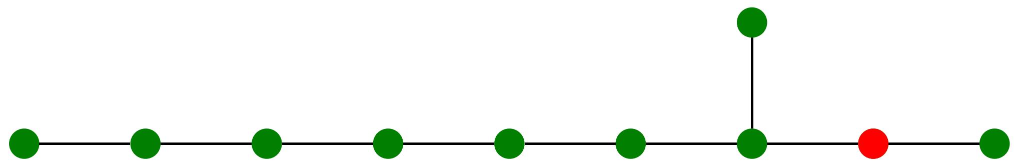

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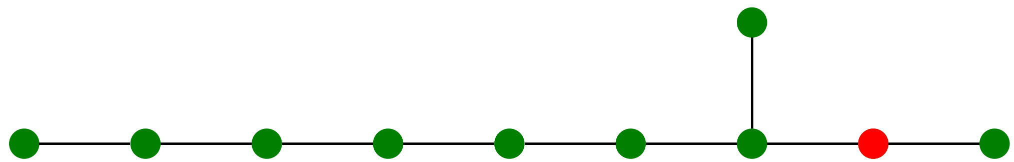

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- Imaginary unit: $\Gamma^* := \Gamma^9\Gamma^{10}$ satisfies $(\Gamma^*)^2 = -\mathbf{1}$

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- Define [AK, Nicolai 2006] $(r = 1, \ldots, 8)$

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\lambda_1 &:= P_- \left(\psi_9 - \Gamma^* \psi_{10}\right) \\
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\chi_2^r &:= P_- \Gamma^* \left(-\frac{4}{3}\psi^r - \frac{1}{3}\Gamma^r \Gamma^9 \psi_9 - \frac{1}{3}\Gamma^r \Gamma^{10} \psi_{10}\right)
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$\longrightarrow$ Conclusions

Spinors for $K(E_{11})$

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- At the dynamical level:
  What is the $K(E_{11})$ covariant spinor equation ??

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Thank you for your attention

Gradient conjecture

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- A_9 spectrum of $\mathfrak{e}_{10}$ contains infinite series of gradient representations:

ℓ	A_9 module	Generator
$3k+1$	$[k00000100]$	$E_{a_1\dots a_k}{}^{b_1b_2b_3}$
$3k+2$	$[k00100000]$	$E_{a_1\dots a_k}{}^{b_1\dots b_6}$
$3k+3$	$[k10000001]$	$E_{a_1\dots a_k}{}^{b_0\vert b_1\dots b_8}$

- Have the right structure to correspond to spatial gradients, $\partial_{a_1}\cdots\partial_{a_k}F_{tb_1b_2b_3}(t,\mathbf{x}_0)$ (possibly non-local).

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- Have the right structure to correspond to spatial gradients, $\partial_{a_1}\cdots\partial_{a_k}F_{tb_1b_2b_3}(t,\mathbf{x}_0)$ (possibly non-local).
- Lie-algebraic mechanism for emergence of space-time?
- Spatial gradients are only a *tiny* subset of all representations: For each ℓ there is one gradient representation — total number of representations grows exponentially with ℓ.

$\longrightarrow$ Conclusions

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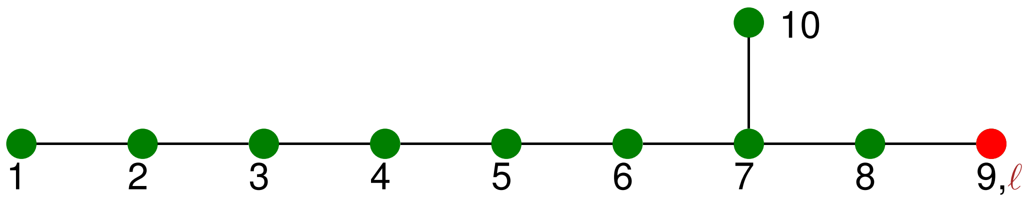

$$\Longleftrightarrow \mathfrak{so}(9,9) \equiv D_9 \subset \mathfrak{e}_{10}$$

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ℓ	D_9 module	Tensor
0	$[010000000] \oplus [000000000]$	M^{IJ}, T ($I = 1, \ldots, 18$)
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Lorentz group is hidden in

$$SO(9)_{\rm diag} \subset SO(9) \times SO(9) \subset SO(9,9)$$

Under this group $\ell = 1$ decomposes as

$$E_A \to \mathbf{9} + \mathbf{84} + \mathbf{126} + \mathbf{36} + \mathbf{1},$$

i.e. all anti-symmetric tensors of odd degree: $C^{(p)}$ with $p = 1, 3, 5, 7, 9$. These are just the *massive* IIA supergravity RR potentials. NSNS fields on $\ell = 0, 2$.

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Using the $SO(9)$ ϵ-symbol these can also be viewed as even degree forms! $\Longrightarrow$ T-duality.

Bosonic versatility: $A_8 \bigoplus A_1$

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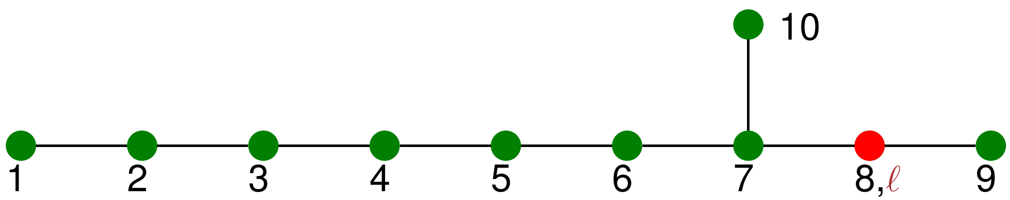

$\Longleftrightarrow \mathfrak{gl}(9) \oplus \mathfrak{sl}(2) \subset \mathfrak{e}_{10}$

The diagram shows an explicit $SL(2,\mathbb{R})$ already hinting at a IIB interpretation.

Bosonic versatility: $A_8 \oplus A_1$

ℓ	$A_8 \oplus A_1$ module	Tensor
0	$([10000001], \mathbf{1}) \oplus ([00000000], \mathbf{3})$	$K^a{}_b,\ J_i$ ($a = 1, \ldots, 9,\ i = 1, 2, 3$)
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2	$([00001000], \mathbf{1})$	$E^{a_1 \ldots a_4}$
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- Compute the σ-model equations of motion. Compare with IIB equations
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  The matching implies/requires self-duality! [No supersymmetry has been used!]

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Similar spectrum calculation in E_{11} revealed existence of a nine-brane $SL(2,\mathbb{R})$ *quadruplet* and *doublet* [AK, Schnakenburg, West 2003][AK, Nicolai 2004]. Consistency with the IIB supergravity algebra was shown by [Bergshoeff et al. 2005].

$\longrightarrow$ Conclusions

Higher Levels Example: IIA

(ℓ_1, ℓ_2)	A_8 module	E_{10} element	α^2	mult(α)	μ	Object
(1,0)	[0,0,0,0,0,0,1,0]	(0,0,0,0,0,0,0,0,0,1)	2	1	1	NSNS2
(0,1)	[0,0,0,0,0,0,0,1]	(0,0,0,0,0,0,0,0,1,0)	2	1	1	RR1
(1,1)	[0,0,0,0,0,1,0,0]	(0,0,0,0,0,0,1,1,1,1)	2	1	1	RR3
(2,1)	[0,0,0,1,0,0,0,0]	(0,0,0,0,1,2,3,2,1,2)	2	1	1	RR5
(3,1)	[0,1,0,0,0,0,0,0]	(0,0,1,2,3,4,5,3,1,3)	2	1	1	RR7
(4,1)	[0,0,0,0,0,0,0,0]	(1,2,3,4,5,6,7,4,1,4)	2	1	1	RR9=MASS
(2,2)	[0,0,1,0,0,0,0,0]	(0,0,0,1,2,3,4,3,2,2)	2	1	1	NSNS6
(3,2)	[0,1,0,0,0,0,0,1]	(0,0,1,2,3,4,5,3,2,3)	2	1	1	DUAL GRAV
(3,2)	[1,0,0,0,0,0,0,0]	(0,1,2,3,4,5,6,4,2,3)	0	8	1	DUAL DIL
(4,2)	[1,0,0,0,0,0,1,0]	(0,1,2,3,4,5,6,4,2,4)	2	1	1	GRAD NSNS2
(4,2)	[0,0,0,0,0,0,0,1]	(1,2,3,4,5,6,7,4,2,4)	0	8	1	?
(5,2)	[0,0,0,0,0,1,0,0]	(1,2,3,4,5,6,8,5,2,5)	2	1	1	?
(3,3)	[1,0,0,0,0,0,0,1]	(0,1,2,3,4,5,6,4,3,3)	2	1	1	GRAD RR1
(3,3)	[0,0,0,0,0,0,0,0]	(1,2,3,4,5,6,7,5,3,3)	0	8	0	?

$\longrightarrow$ Conclusions