Factorization

and

String Field Theory

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Background Independent Open String Field Theory

- Critical CFT on the 2d surface with boundary (disk) coupled to b, c ghosts

- Space of local operators in the interior: $\mathcal{O}$

- Space of 2d boundary actions:

$$I_b = I_{CFT} + I_{ghosts} + b_{-1} \int_{C \to boundary} \mathcal{O}$$

$$b_{-1} = \int_{C \to boundary} b(v); \qquad b(v) = v^i b_{ij} \epsilon^j_k d\sigma^k$$

v-Killing vector field generating the rotation of the disk.

- BRST operator Q: $\qquad Q = \int_{C \to boundary} J_{BRST}$

- String Field Theory action:

$$dS = < \int d\sigma d\mathcal{O}(\sigma) \{Q, \int d\sigma' \mathcal{O}(\sigma')\} >$$

- If ghosts and matter decouple → $\mathcal{O} = cV(matter)$:

$$S = -\beta^i(t) \frac{\partial}{\partial t^i} Z(t) + Z(t)$$

t^i - parameterize boundary interactions $\partial_i V(t)_{|t=0} = V_i$
V_i - an operator from CFT inserted on the boundary
β^i - beta function for coupling t^i.

There are two major difficulties with this description:

1. If ghosts and matter do not decouple the space of 2d boundary theories above is smaller than the space of operators $\mathcal{O}$.

2. Locality - even if restricted to the case when ghosts and matter decouple it is not clear what is the meaning of locality.

In this talk we address second question.

“Local” deformations

Suppose target is $R^{(1,25)}$ and consider all possible boundary interactions given by expansion in derivatives of X^μ:

$$\int_{boundary} V(X) = \int [T(X) + A_\mu(X)\partial X^\mu + ...]$$

- 1st term, local in $X(\sigma)$ - open string tachyon

- 2nd, with one derivative $\partial_\sigma X(\sigma)$ - open string gauge field

- Others, with higher derivatives - open string massive modes.

S is a functional of open string modes $T, A_\mu, ...$: $S(T, A_\mu, ...)$.

But $V(X)$ is an infinite expansion; in what sense it is local?

Example - general quadratic interaction on boundary:

$$\int V_2(X) = \int d\sigma d\sigma' X^\mu(\sigma) u_{\mu\nu}(\sigma - \sigma') X^\nu(\sigma')$$

which is clearly non-local is of the same type of infinite series:

$$\int V_2(X) = \sum_n \int t^n_{\mu\nu} X^\mu(\sigma) \partial^n X^\nu(\sigma)$$

with

$$u_{\mu\nu}(\sigma - \sigma') = \sum u^n_{\mu\nu} e^{in(\sigma-\sigma')}; \qquad u^n_{\mu\nu} = \sum_m (in)^m t^m_{\mu\nu}$$

Moreover - boundary part of matter world-sheet action is non-local itself in $X^\mu(\sigma)$!

If we substitute

$$X(z,\bar{z}) = X_0(z,\bar{z}) + X_h(f)$$

where $X_0 = 0$ on boundary and $X_h(f)$ is a harmonic function with $f(\sigma)$ on boundary - solution of $\Delta X_{cl} = 0$:

$$\int dz d\bar{z} \partial X^\mu \bar{\partial} X^\mu = \int dz d\bar{z} \partial X_0^\mu \bar{\partial} X_0^\mu + 2 \int dz d\bar{z} \partial X_h^\mu(f) \bar{\partial} X_0^\mu$$

$$+ \int dz d\bar{z} \partial X_h^\mu(f) \bar{\partial} X_h^\mu(f) =$$

$$\int dz d\bar{z} \partial X_0^\mu \bar{\partial} X_0^\mu + \int d\sigma d\sigma' f^\mu(\sigma) H(\sigma - \sigma') f^\mu(\sigma')$$

Here H is a Hilbert transform - non-local operator (sometimes also called Newmann operator):

$$H(\sigma - \sigma') = \int d\sigma' |n| e^{in(\sigma - \sigma')}$$

$$I_{boundary} = I(X_{cl}(f)) = \int f H(f)$$

Operator H: $C^{\infty}(S^1, R) \rightarrow C^{\infty}(S^1, R)$ can be understood as follows.

Suppose f from $C^{\infty}(S^1, R)$ can be decomposed uniquely as $f = f_+ + f_-$ such that f_+ admits holomorphic extension inside the disk and f_- - outside the disk with $f_-(\infty) = 0$. Than:

$$H(f) = f'_+ - f'_-$$

This boundary term can be rewritten in more pleasant form using the holomorphic decomposition:

$$I_{boundary} = \int_{boundary} f_+^{\mu} df_-^{\mu}$$

This boundary action is 2-cocycle on $U(1)^{26}$. 2-cocycle on $U(1)$ is:

$$\alpha_2(X, Y) = \int X dY$$

and here we have it for $X = f_+$ and $Y = f_-$.

What we called before the boundary deformation for open string modes is in this language the boundary deformation of this boundary action with "local" functionals of $f(\sigma)$.

More generally we see that it is natural to deform with functionals of f_+ and f_-, but these will be "non-local".

What about ghost sector?

It can be shown in a very similar manner that in case of b, c system (as well as any β, γ system of arbitrary spin $j, 1-j$) there is a splitting between bulk and boundary modes

(**Hong Liu, Leon Takhtajan, S. Sh., 2001, unpublished**):

Let H^b, H^c be spaces of smooth sections of b, c restricted to boundary of the disk and $H_-^{b,c}$ - subspaces consisting of boundary values of zero modes of $\bar{\partial}$ - solutions of classical equations of motion:

$$H_-^b = \{z^2, z^3, ...\}$$

$$H_-^c = \{z^{-1}, 1, z, ...\}$$

We parametrize ghosts as:

$$b = B + \beta; \qquad c = C + \gamma$$

where bulk fields B, C lie in $H_+^{b,c} = H^{\beta,c} - H_-^{b,c}$ and β, γ are holomorphic with boundary values from $H_-^{b,c}$. Similarly for $\bar{b}, \bar{c}$. One needs to impose further boundary conditions on bulk fields, but independently on these world-sheet action splits into:

$$I(b, c) = I_{bulk}(B, C) + I_{boundary}(\beta, \gamma)$$

$$I_{boundary} = \int_{S^1} \beta H(\gamma)$$

and

$$\beta(\sigma) = \sum_{n=2}^{\infty} (b_n \pm \bar{b}_{-n}) e^{in\sigma}$$

$$\gamma(\sigma) = \sum_{n=-1}^{\infty} \gamma_n e^{in\sigma}$$

$$\gamma_n = c_n \mp \bar{c}_{-n}; \qquad n \geq 2; \qquad \gamma_0 = \mp\bar{\gamma}_0; \qquad \gamma_1 = \mp\bar{\gamma}_{-1}$$

where in polar coordinates β means $\beta_{\theta,\theta}$ and γ is γ^θ (or $\beta_{r\theta}, \gamma^r$ - there is certain freedom which depends on choice made on bulk fields). H acts on arbitrary function on S^1, $\Gamma(\sigma)$:

$$\Gamma(\sigma) = \sum_{n=2}^{\infty} (\Gamma_n e^{in\sigma} + \Gamma_{-n} e^{-in\sigma}) + \Gamma_1 e^{i\sigma} + \Gamma_0 + \Gamma_{-1} e^{-i\sigma}$$

as

$$H(\gamma) = \Gamma_+ - \Gamma_- = i \sum_{n=2}^{\infty} (\Gamma_n e^{im\sigma} - \Gamma_{-n} e^{-in\sigma})$$

1. Fields on world-sheet are naturally decomposed into bulk and boundary fields

$$\begin{array}{llll} bulk - & X_0(z,\bar{z}), & B(z,\bar{z}), & C(z,\bar{z}) \\ boundary - & f(\sigma), & \beta(\sigma), & \gamma(\sigma) \end{array}$$

Boundary fields are arbitrary functions on S^1 with the values in target space. Arbitrary map from world-sheet (disk) to target is described by this set of fields.

2. World-sheet path integral, disk partition function, is a product of D-instanton partition function Z_D (integral over bulk fields) and boundary partition function (integral over boundary fields) Z_{bndry}.

$$Z_{disk} = Z_D(x_0) \times Z_{bdry}$$

where x_0 denotes closed string background, flat in this case.

3. Boundary perturbations, which define string field theory action S, enter only in the second factor since operators are inserted on the boundary and are functionals of boundary fields only:

$$Z_{disk}^{deformed} = Z_D(x_0) \times Z_{bndry}^{deformed}(x_0, t)$$

Boundary partition function - different for different closed string backgrounds ⇔ dependence on x_0.

4. Boundary action is non-local in boundary fields.

Question 1: *is such factorization a generic property of CFT or it only holds for free theory?*

Meaningful only if the boundary theory, though given by non-local action on S^1, still has some universal properties, e.g. - boundary action has no α' corrections and is given by "classical formula" s.t. boundary field defines the generalized Dirichlet boundary condition for equations of motion:

$$I_{bndry}(f, \beta, \gamma) = I_{cl}(X_{cl}(f), b_{cl}(\beta), c_{cl}(\gamma))$$

One can reformulate this statement for disk partition function for CFT:

$$\int_{\Phi|_{\partial\Sigma}=\Phi_b(\theta)} D[\Phi]\, e^{\frac{i}{\alpha'} I_\Sigma(\Phi)} = \mathbf{Z}_D(x_0) e^{\frac{i}{\alpha'} I_{bdry}(\Phi_b)}$$

$I_\Sigma(\Phi)$ - Lagrangian for the world-sheet conformal field theory on a 2d surface with boundary

$Z_D(x_0)$ - the D-instanton partition function which is given by same formula for $\Phi_b = 0$

$$I_{bndry} = I_\Sigma(\Phi_{cl}(\Phi_b))$$

This should be true only for restricted class of 2d QFT - mainly, for CFT.

Note - the latter form of conjecture (on-shell version) is holographic.

One can modify the string field theory action S by dividing on Z_D - independent of deformation parameters t:

$$\tilde{S}(x_0, t) = -\beta^i(t) \frac{\partial}{\partial t^i} Z_{bndry}(x_0, t) + Z_{bndry}(x_0, t)$$

Question 2: *is it possible that action* $\tilde{S}(x_0, t)$ *has two different fixed points,* t_1^* *&* t_2^**, such that they correspond to different closed string background* x*?.*

$$\tilde{S}(x_0, t_1^* + \delta t) = \tilde{S}(x, t_2^* + \delta t)$$

LHS - n-point open string corr. functions in string theory with background (x_0, t_1^*) as coefficients of expansion in δt;

RHS - same for background (x, t_2^*).

Simplest known example - constant B-field (closed string background) ⇔ constant **EM** field (open string background):

$$\tilde{S}((G, B); \delta t) = \tilde{S}((G, 0); t^*(= B) + \delta t)$$

Since this action at fixed points is boundary partition function this would mean that:

$$Z_{bndry}(x_0, t_1) = Z_{bndry}(x, t_2)$$

In other words - can one get the boundary partition function of second closed string background as boundary deformation of the first one, even if “non-local”?

Question 3. Can one reconstruct bulk background x from boundary theory? With some, controllable, ambiguity?

Curved backgrounds - WZW model

Non-trivial case with curved closed string background - **WZW**.

$$I_{WZW}(g) = -\frac{\kappa}{4\pi}\int_{\Sigma} tr(\partial_\mu g g^{-1})^2 - \frac{i\kappa}{12\pi}\Gamma(g)$$

This is a sigma model with target being a group manifold G. $\Gamma(g) = \int_{\partial X = \Sigma} tr(dgg^{-1})^3$

For simplicity consider $G = SL(2,R)$, the case of AdS_3 target. In this case $H^3(G) = 0$ and there is no problem defining the theory on the manifold with boundary - three form $w_3 = tr(dgg^{-1})^3$ is exact and can be written as:

$$w_3 = dw_2(g)$$

for some $w_2(g)$ which is defined up to an exact from:

$$w_2(g) = w_2^*(g) + d\beta(g)$$

where w_2^* is a favorite 2-form, and $\beta(g)$ is arbitrary 1-form. So, the WZW theory on the disk is defined via action (section - family, parameterized by 1-form β):

$$I^{\beta}_{WZW}(g) = -\frac{\kappa}{4\pi}\int_{\Sigma} tr(\partial_\mu g g^{-1})^2 - \frac{i\kappa}{12\pi}\int_{\Sigma} w_2^*(g) + \int_{S^1}\beta(g)$$

If the group has non-trivial $H^3(G)$ situation is more subtle but final answer still is well-defined.

Factorization for this CFT turns out to take the form (**M. Baumgartl, I. Sachs, S. Sh., 2005**):

$$\mathbf{Z}(g|_{\partial\Sigma} = f(\sigma)) = \int_{g|_{\partial\Sigma}=f(\theta)} D[g]\, e^{iI^{\beta}_{WZW}(g)} =$$

$$= \mathbf{Z}(g|_{\partial\Sigma} = 1) e^{i\alpha_2(f_+,f_-)}$$

$\alpha_2(g_1, g_2)$ - is a group 2-cocyle for current algebra;

$f_+(\sigma), f_-(\sigma)$ - complexified group elements

f_+ - holomorphic in the interior; f_- - in the exterior

f_+, f_- - solutions of Rieman-Hilbert problem for $f(\sigma)$: $S^1 \to G$ (generalized Dirichlet for classical equations of motion):

$$f(\sigma) = f_+(\sigma)f_-(\sigma)$$

α_2 - mathematically well-defined object for current algebra.

It is completely determined by two group elements from $S^1 \to G$ for any group G, but up to trivial cocycle.

When we evaluate 2-cocycle on holomorphic and anti-holomorphic elements f_+, f_- ambiguity due to trivial cocycle $\Leftrightarrow$ exactly the ambiguity reflected by arbitrary ("local") 1-form β.

$$\alpha_2(f_+, f_-) = \alpha_2^0(f_+, f_-) + \int_{S^1} \beta(f) = I^{\beta}_{WZW}(g_{cl}(f(\sigma))$$

Here α_2^0 is our favorite representative of 2-cocycle and β - as before. In some sense above formula can be considered as a definition of classical action.

If we represent the group elements f_+, f_- :

$$f_+ = e^{iX_+^a \lambda^a}; \qquad f_- = e^{iX_-^a \lambda^a}$$

with λ - Lie algebra generators.

2-cocycle up to commutator terms (non-abelian corrections) has the form:

$$\alpha_2(f_+, f_-) = \int X_+^a dX_-^a + ...(commutators)$$

Thus, this again is "(non-Abelian) Hilbert transform"!

Conclusion: factorization holds for **WZW** model:

$$\int [dg(z,\bar{z})] e^{iI^{\beta}_{WZW}(g)} = \int_{g_{|\partial\Sigma}=1} [dg(z,\bar{z})] e^{iI_{WZW}(g)} \times$$
$$\int [df(\sigma)] e^{i\alpha_2(f_+,f_-)} = Z_D \times Z_{bndry}$$

LHS and RHS depend on 1-form β, so above formula is defined up to the deformation on the boundary with β - family of theories. Choice of β corresponds to choice of boundary condition, so for some β above formula defines conformal invariant boundary conditions.

For any β the resulting boundary partition function is in a space of ("non-local") boundary deformations of free theory of X^a's - thus these non-local boundary deformation change the close string background.

Factorization for WZW model

Following identity on the level of diff. forms holds for $w(g) = tr(\partial g g^{-1} \bar{\partial} g g^{-1}) + w_2^*(g)$ for any choice of $w_2^*(g)$:

$$d\gamma(g_1, g_2) = d[w(g_1 g_2) - w(g_1) - w(g_2) + tr g_1^{-1} \partial g_1 \bar{\partial} g_2 g_2^{-1}] = 0$$

This is a cocycle property (PW-identity) and we conclude that $\gamma(g_1, g_2)$ is a closed 2-form.

Thus if we integrate $\gamma(g_1, g_2)$ over disk we get the functional which depends only on the maps from boundary of the disk to the group - this maps we denote by $f_1(\sigma), f_2(\sigma)$:

$$\int_\Sigma \gamma(g_1, g_2) = \alpha_2(f_1, f_2)$$

Evaluate **WZW** action on the product:

$$g(z, \bar{z}) = g^0(z, \bar{z}) g_{cl}(z, \bar{z})$$
$$g^0_{\partial\Sigma} = 1, \quad g_{\partial\Sigma} = f(\sigma)$$

and $g_{cl}(f)$ is a solution of **WZW** equations. This is arbitrary map from disk to group. Since $\alpha_2(1, f) = 0$:

$$I^\beta_{WZW}(g) = I_{WZW}(g_0) + I^\beta_{WZW}(g_{cl}(f)) - \frac{\kappa}{2\pi i} \int tr g_0^{-1} \bar{\partial} g_0 \partial g_{cl} g_{cl}^{-1}$$

Note that β enters only in classical action since it doesn't depend on extension of $f(\sigma)$ into interior, and the action for g_0 gives **WZW** theory with boundary condition in conjugacy class of identity.

Main problem in defining **WZW** theory on disk thus is hidden in the classical action.

Classical solutions are:

$$g_{cl}(f) = g_+(z)g_-(\bar{z})$$

Such that holomorphic part $g_+(z)$ is $f_+(\sigma)$ on the boundary and anti-holomorphic part $g_-(\bar{z})$ is $f_-(\sigma)$ on boundary with:

$$f_+(\sigma)f_-(\sigma) = f(\sigma)$$

Integral over g_0 gives dependence on boundary field $f(\sigma)$ only through cross term - one needs to compute the following correlation in the **WZW** theory of g_0:

$$< e^{-\int dz d\bar{z} \quad \bar{J} \times \partial g_+(z) g_+^{-1}(z)} >$$

This cross term is anti-holomorphic current $\bar{J}$ in **WZW** theory with boundary condition in identity conjugacy class coupled to entirely holomorphic object $\partial g_+(z)g_+^{-1}(z)$.

Explicit computation (or very general argument) shows:

$$< e^{-\int dz d\bar{z} \quad \bar{J} \times \partial g_+(z) g_+^{-1}(z)} > = < 1 >$$

Thus cross term doesn't contribute and entire path integral factorizes into the integral over $g_0(z, \bar{z})$ and $f(\sigma)$ with former being standard **WZW** theory with identity conjugacy class and latter - given by classical action!

Classical - is given by 2-cocycle as a result of direct computation and definition of α_2.

τ-function

It is interesting to think about WZW model as induced from fermions coupled to gauge field in 2d (**Polyakov-Wiegman, 1984**):

$$\mathcal{L} = i\bar{\psi}_-(\partial + A)\psi_- + i\bar{\psi}_+(\bar{\partial} + \bar{A})\psi_+ - A\bar{A}$$

If we put this theory on the disk and consider it from **WZW** point of view on classical solutions $A = g_+^{-1}(z)\partial g_+(z); \bar{A} = \bar{\partial} g_-(\bar{z}) g_-^{-1}(\bar{z})$ than previois considerations correspond to computing following matrix element in fermionic Fock space:

$$\tau(g) =< 0|g >$$

where $g(\sigma) = g_+(\sigma)g_-(\sigma) = f_+ f_-$.

For chiral fermions ψ_+, ψ_- vacuum $|0 >$ is defined in standard way, and the state $|g >$ is a vacuum state for fermions in external field A_μ. In first quantized language this corresponds to rotation $\psi_D \to g\psi_D$ since we consider the on-shell case.

According the methods of second quantization there is an unitary operator which acts in second quantized Hilbert space and represents this rotation - linear canonical transformation, Bogolubov transformation: $g(\sigma) \to V(g)$.

$$|g >= V(g)|0 >$$

Now, if we solve Rieman-Hilbert problem for $g(\sigma) = f_+ f_-$ since for each group element there should be a linear canonical transformation:

$$f_+(\sigma) \rightarrow V(f_+)$$
$$f_+(\sigma) \rightarrow V(f_-)$$
$$f_+ f_- = g \rightarrow V(f_+ f_-) = V(g)$$

$$\tau(g) =< 0|g >=< 0|V(g)|0 >=$$
$$=< 0|V(f_+)V(f_-)e^{i\alpha_2(f_+,f_-)}|0 >=$$
$$=< f_+|f_- > e^{i\alpha_2(f_+,f_-)}$$

We used the fact that the group acts projectively - it is in fact not a loop group but its central extension.

Because of holomorphicity of f's the states $|f_+ >, |f_- >$ are very special - corresponding matrix of canonical transformation (**Berezin**) contains the overlaps $\int_{S^1} \bar{u}_n f_\pm u_m$ and many of the matrix elements vanish. As a result

$$< f_+|f_- >\sim 1$$

is some constant independent of f, and

$$\tau(g) \sim e^{i\alpha_2(f_+,f_-)}$$

which gives the dependence of **WZW** path integral on boundary values of group element - classical action.