

# Adventures in $G_2$ topological theories

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based on

\* Topological Membranes

hep-th/0507089

with Lilia Anguelova and Paul de Medeiros

\*  $G_2$  Foam

with Paul de Medeiros

\* Open  $G_2$  string and topological membranes

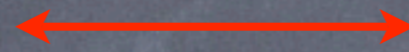
with Jan de Boer, Paul de Medeiros and  
Sheer El-Showk

# Topological Strings on CY

## A-model

holomorphic maps  
Kähler structure  $H^{1,1}$

mirror



## B-model

constant maps  
complex structure  $H^{2,1}$

Gromov-Witten invariants



## Crystal quantum foam

6D  $U(1)$  gauge theory  
Donaldson-Thomas invariants

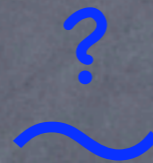
Open-Closed Duality  
Chern-Simons theory

# $G_2$ Topological Theories

Topological membrane

on associative 3-cycles

$G_2$  de Rham cohomology



Topological  $G_2$ -string

constant maps

$G_2$  de Rham cohomology

$G_2$  invariants?

$G_2$  quantum foam

Abelian 2-form gauge theory

under construction...

Open-Closed Duality?  
Chern-Simons theory?

# Topological M-theory:

- topological theory on 7-manifolds with  $G_2$ -structure
- suggested as unification on A/B model topological strings
- unification on lower dim form theories of gravity

# Hitchin's theory

Manifold  $M$  with  $G_2$  holonomy  $\leftrightarrow$  associative 3-form  $\Phi$

$$d\Phi = 0 \quad d*\Phi = 0$$

Hitchin's theory: build a classical action which extremizes on these conditions:

Metric is constructed as

$$\sqrt{g}g_{jk} = \frac{1}{144} \Phi_{ji_1i_2} \Phi_{ki_3i_4} \Phi_{i_5i_6i_7} \epsilon^{i_1\dots i_7}$$

Is 3-form suitable to construct an action in 7 dimensions?

Yes, because of stability.

## Hitchin's action

$$V(\Phi) = \int_{M_7} \Phi \wedge * \Phi$$

$\Phi$ : stable 3-form,  $*$  is in terms of  $\Phi$ .

Variation:

$$d\Phi = 0$$

$$\Phi = \Phi_0 + dB, \quad d\Phi_0 = 0$$

$\Phi$ : field strength on 2-form gauge potential. Vary only exact piece. Equation of motion:  $d * \Phi = 0$

Choosing 4-form  $G = * \Phi$

$$dG = 0, \quad G = G_0 + dC, \quad \text{vary } C$$

$$d * G = 0$$

# Reduction to six dimensions

$$M_7 = M_6 \times S^1$$

Hitchin's ansatz

$$\Phi = \rho + k \wedge dt$$

$$*\Phi = \hat{\rho} \wedge dt + \frac{1}{2} k \wedge k$$

with  $\Omega = \rho + i\hat{\rho}$

$$\begin{aligned} V(\Phi) &= \int \Phi \wedge *\Phi = \int (\rho + k \wedge dt) \wedge (\hat{\rho} \wedge dt + \frac{1}{2} k \wedge k) = \\ &= \frac{1}{2} \int k \wedge k \wedge k \wedge dt - \frac{i}{4} \int \Omega \wedge \bar{\Omega} \wedge dt \end{aligned}$$

Reduction gives A and B+ $\bar{B}$  model target space actions.

# Topological theory of membranes?

## Strategy:

Similar to topological A-model.

Consider space of maps from 3-manifold to  $G_2$ .

Construct topologically invariant path integral that localizes on associative 3-cycles.

Mathai-Quillen formalism

# Mathai-Quillen formalism

- integral representation of Euler class of vector bundles
- constructs of topological invariants of infinite dimensional vector bundles

Manifold  $M$ , vector bundle  $E \rightarrow M$

$$e_{\nabla, s}(E) = \frac{1}{(2\pi)^n} \int d\chi e^{-\frac{1}{2}s \cdot s + i ds \cdot \chi + \frac{1}{2} \Omega_{\nabla} \chi \cdot \chi}$$

regularized Euler number for infinite dimensional vector bundles

# Topological membranes

- consider maps from membrane world volume to  $G_2$  manifold
- localize path integral on maps of associative 3-cycles

associative 3-cycles calibrated

$$\Phi|_{X_3} = \text{vol } X_3 \quad \Leftrightarrow \quad *\Phi_{IJKL} dx^J dx^K dx^L|_{X_3} = 0$$

defines section  $\Theta_I$

# Mathai-Quillen action

$$I_M = \int_{\Sigma_3} d^3\sigma \left( \frac{1}{2} \Theta \cdot \Theta + i\chi \cdot d\Theta - \frac{1}{4} R_{IJKL} \Psi^I \Psi^J \chi^K \chi^L \right)$$

invariant under BRST

$$\begin{aligned} \delta x^I &= \Psi^I & \delta \Psi^I &= 0 \\ \delta \chi^I &= i\Theta^I - \Gamma_{JK}^I \Psi^J \chi^K \end{aligned}$$

$$I_M = \delta \left( -\frac{i}{2} \int_{\Sigma_3} d^3\sigma \chi^I \Theta_I \right)$$

Full action  $S_M = I_M + \int_{\Sigma_3} x^*(\Phi)$

global U(1) ghost number symmetry

$$(x, \Psi, \chi) \sim (0, 1, -1)$$

Observables  $\mathcal{O}_\omega = \frac{1}{p!} \omega_{I_1 \dots I_p}(x) \Psi^{I_1} \dots \Psi^{I_p}$

$$\delta \mathcal{O}_\omega = \mathcal{O}_{d\omega}$$

BRST cohomology  $\cong$  de Rham cohomology

# Connection to $G_2$ string?

- open string boundary conditions:  
open strings can end on 3-cycles or 4-cycles
- gauge theory on 3-branes: Chern-Simons theory (Jan de Boer, Sheer El-Showk)
- normal modes: calibration preserving deformation of 3-cycles  
membrane action contains normal modes for open  $G_2$  string?

work in progress

# A-model quantum foam

- reformulation as 6D U(1) gauge theory

$$S = \frac{1}{g_s^2} \int k \wedge k \wedge k$$

fix background  $k = k_0 + g_s F$

$$dF = 0 \quad F^{(2,0)} = 0 \quad F^{(1,1)} \wedge k_0 \wedge k_0 = 0$$

embed into U(1) gauge theory

supersymmetrize (BRST)

$$S_{\text{bosonic}} = \int_{\mathcal{X}} \frac{1}{2} \text{tr} \left( d\Phi \wedge *d\Phi + [\Phi, \bar{\Phi}]^2 + |F^{(2,0)}|^2 + |F^{(1,1)}|^2 \right) + \frac{1}{2} \text{tr} \left( F \wedge F \wedge k_0 + \frac{ig_s}{3} F \wedge F \wedge F \right)$$

# $G_2$ quantum foam

Hitchin action  $S = \int \Phi \wedge * \Phi$

Fixed background  $\Phi = \Phi_0 + H$

$d\Phi = 0$      $dH = 0$      $H = dB$     2-form gauge theory

$$\begin{aligned} \Phi \wedge * \Phi = & \quad \Phi_0 \wedge \Phi_0 + \frac{1}{2} H_{abc} \Phi_0^{abc} + \alpha (H_{abc})^2 \\ & + \beta \underbrace{(*\Phi_{abcd} H^{bcd})^2}_{\underline{7}} + \gamma \underbrace{(\Phi_{abc} H^{abc})^2}_{\underline{1}} + \dots \end{aligned}$$

complete BRST structure

$$\underline{35} = \underbrace{\underline{27}}_{\text{metric deformation}} + \underline{7} + \underline{1}$$

# Localization

$$1) \quad \Phi_{abc} H^{abc} = 0$$

$$2) \quad * \Phi_{abcd} H^{abcd} = 0$$

reduction  $\Phi = k \wedge dt + \rho$

$$\rho = \text{Re } \Omega$$

$$* \Phi = (1/2) k \wedge k + \hat{\rho}$$

$$\hat{\rho} = \text{Im } \Omega$$

$$H = F \wedge dt + \hat{H}$$

$$1) \quad \underline{k} F + \rho \hat{H} = 0$$

$$F^{(1,1)} \wedge k_0 \wedge k_0 = 0$$

$$2) \quad \underline{\hat{\rho}} F + (1/2) k \wedge k \hat{H} = 0$$

$$F^{(2,0)} = F^{(0,2)} = 0$$

couple Kähler and complex structure in an interesting way

# Instanton moduli space

$$1) \quad \Phi_{abc} H^{abc} = 0$$

$$2) \quad *\Phi_{abcd} H^{abcd} = 0$$

$$H = d\delta\mathcal{B} \quad \xi_a = \Phi_{abc}\delta\mathcal{B}^{bc}$$

$$d\xi = 0$$

$$d^+\xi = 0$$

$$TM_{G_2} \simeq H^1(M_0, \mathbb{R})$$

work in progress



Happy Birthday Michael!