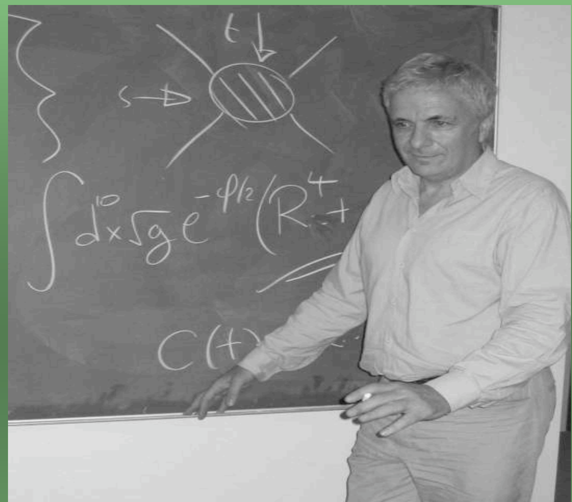
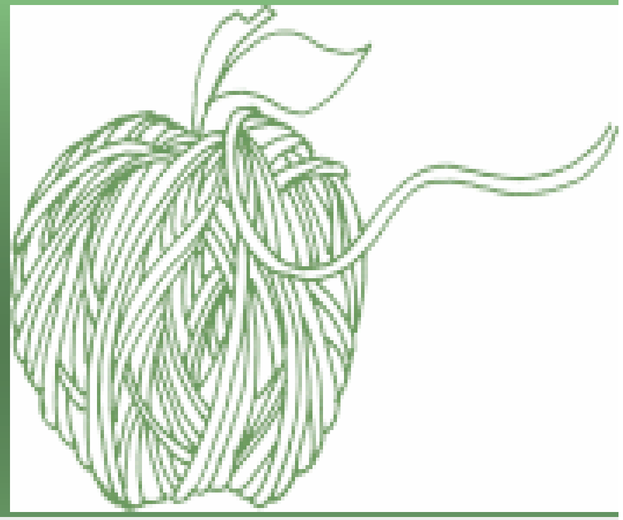
MBG-60: Cambridge, 7- 8 April, 2006

Supersymmetry & Combinatorics: an intriguing connection





based on

Jacek Wosiek & GV hep-th/0512301, hep-th/0603045, Enrico Onofri, J.Wosiek & GV math-ph/0603082,

work in progress, and instructive discussions with

Don Zagier (CdF)

Original motivations

Claimed planar equivalence between QCD_{OR} and SYM has interesting physical consequences, e.g.

$< \bar{\square}\square >$ in $N_f=1$ & $N_f=3$ QCD

(A. Armoni, M. Shifman, G. Shore, GV)

- One of the original motivations of VW was to check planar equivalence, and compute its accuracy at finite N, in a simple QM case
- This has not been done yet (may soon follow)
- However, we stumbled on an amusing model with unexpected physical and mathematical properties
- Hope that Mike will be amused too..

Outline

1. Planar equivalence and QCD quark condensate
2. Planar QM: a Hamiltonian approach
3. An intriguing SUSY matrix model
4. Mathematical implications of the model
5. Physical implications of the maths.

Planar QM: a Hamiltonian approach

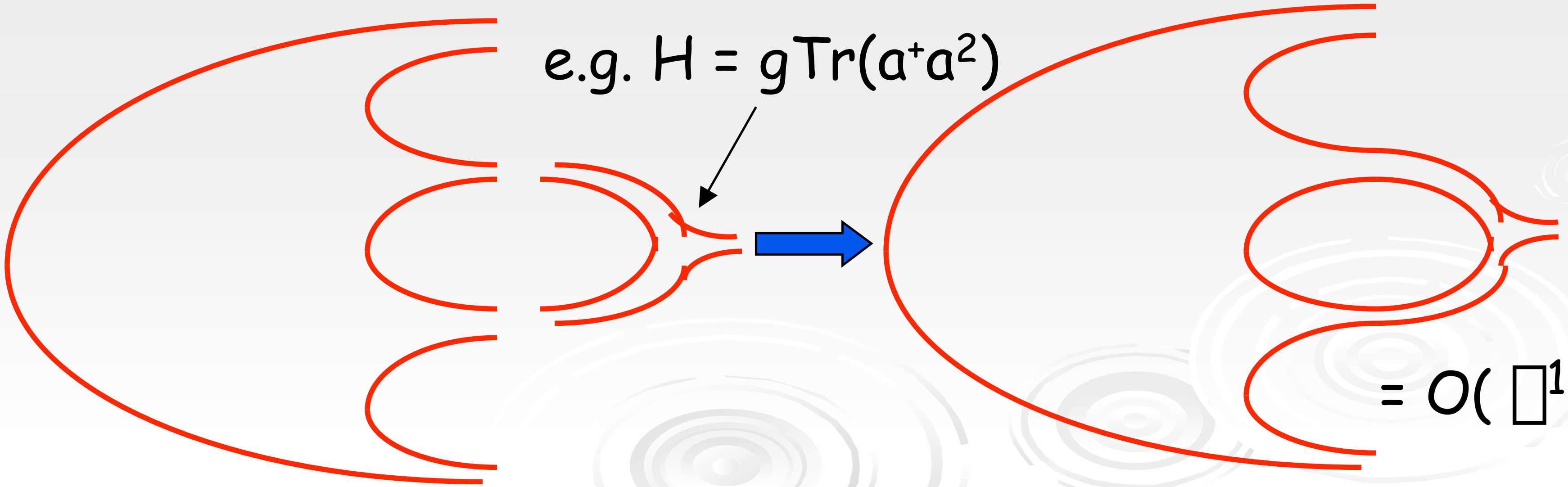
- In the large-N limit of a U(N) matrix theory singlet states are related to single-trace operators
- In SUSY-QM with a single bosonic matrix *a* and a single fermionic matrix *f* these operators look like binary necklaces (necklaces with two kinds of beads, a boson and a fermion)
- In the specific model of VW leading-order states are of the type

$$|\{n_i, m_j\}\rangle \sim Tr[a^{n_1} f^{m_1} \dots a^{n_k} f^{m_k}]^\dagger |0\rangle$$

where |0> is the usual empty Fock vacuum

Planar Hamiltonians are also taken to be single-traces. A trace with n factors will be multiplied by g^{n-2}. The action of the Hamiltonian on each state gives, to leading order, a linear combination of single trace states with coefficients that depend only on $\lambda = g^2N$

$$\left|\{n_i, m_j\}\right\rangle \sim N^{-(\sum n_i + \sum m_j)/2} Tr\left[a^{n_1} f^{m_1} \ldots a^{n_k} f^{m_k}\right]^{\dagger} |0\rangle$$



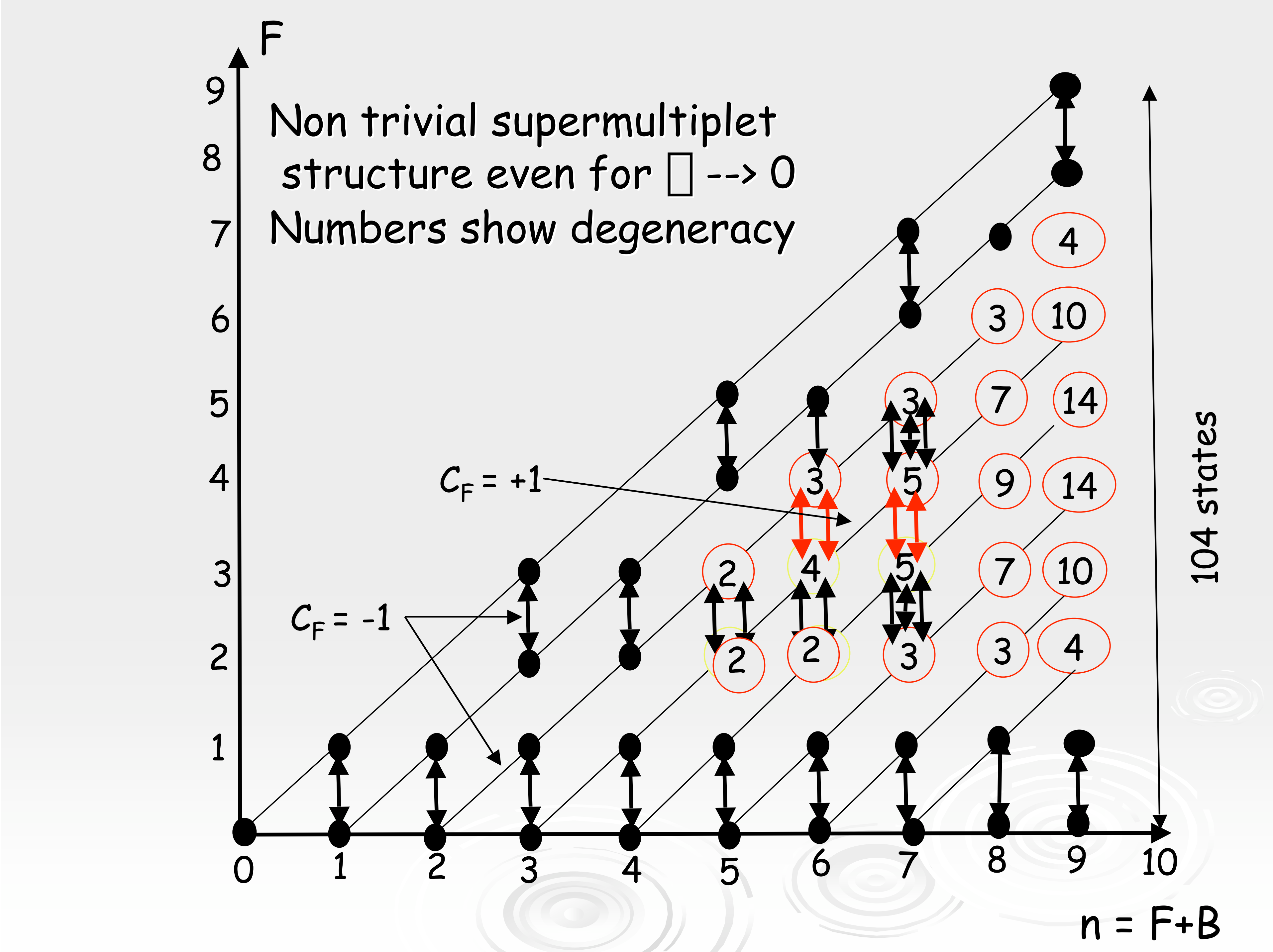
An intriguing SUSY matrix model

- Take the SUSY charges to be, quite simply:

$$Q = Tr\left(fA^{\dagger}(a^{\dagger})\right) = Tr\left(f(a^{\dagger} + ga^{\dagger 2})\right) \;,\; Q^2 = 0$$

$$H = \{Q^{\dagger}, Q\} \;,\; C = [Q^{\dagger}, Q] \;,\; C^2 = H^2 \;,\; F = Tr(f^{\dagger} f)$$

- Diagonalize, H, C, F
- Trivial E=0 vacuum, |0>: SUSY is unbroken
- E > 0 SUSY must be organized in doublets of states with same $C_F = (-1)^F C$
- States w/ C = +(-) E are annihilated by $Q^{+}(Q)$



F
Non trivial supermultiplet
structure even for --> 0
Numbers show degeneracy
C_F = +1
C_F = -1
104 states
n = F+B
0 1 2 3 4 5 6 7 8 9 10
1 2 3 4 5 6 7 8 9

Connection with

Binary Necklaces (BNL)

&

Linear-Feedback Shift Registers (LFSR)

(E.Onofri, J.Wosiek & GV,math-ph/0603082)

- Having constructed, counted, and paired the states in SUSY doublets, we searched for something similar known in math.
- Google hinted at a possible connection with Binary Necklaces (cyclic sequences of zeroes and ones) Their number (on-line enc. of integer sequences):
- A000031(n) = Number of n-bead necklaces with 2 colours when turning over is not allowed

  is given by MacMahon's formula (see below).

  But there was a problem:
- The number of even and odd binary necklaces clearly doesn't match! Example @ n=2:

  (aa), (ff), (af) = (fa) => 2 bosons, 1 fermion, ..

- Q: How can supersymmetry work if $n_B \neq n_F$?
- A: Pauli's exclusion principle kills some BNL giving back the balance between bosons and fermions

  $N(n) = N_{PAN}(n)$ (PAN = Pauli-allowed necklaces)
- Playing still with our sequence of integers we found that $N_{PAN}(n) = 2\text{x}A000016(n)$, where:
- A000016(n+1) = Number of distinct (infinite) output sequences from Binary n-stage Shift Register which feeds back the complement of the sum of its contents
- We were able to prove that A000016(n+1) are in one-to-one correspondence with PAN(n+1) w/ F odd.

Binary Necklaces, Pauli Necklaces, LFSR   n= B+F

B \ F	even	odd
even	PFN	
	PAN $\neq$ BNL	PAN = BNL = LFSR
odd	PAN = BNL	PAN = BNL = LFSR

SUSY

$$N_{BNL}(n) = \frac{1}{n}\sum_{d|n} \varphi(d)\, 2^{n/d}$$

$$N_{PAN}(n) = \frac{1}{n}\sum_{\substack{d|n \\ d\,odd}} \varphi(d)\, 2^{n/d}$$

$$N_{PFN}(n) = \frac{1}{n}\sum_{\substack{d|n \\ d\,even}} \varphi(d)\, 2^{n/d}$$

$$N_{LFSR}(n) = \frac{1}{2} N_{PAN}(n)$$

Math. def. of PFN = BNL w/ Z_k symmetry, k even & F/k odd

A Pauli-forbidden necklace
(courtesy of Google)

Z_{10} symmetry, 10 = even & F/10= 1 =odd

N_{PFN} fluctuates a lot!

$B \downarrow$ \ $F \rightarrow$	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
4	0	1	1	2	0	3	1	4	0	5	1	6	0	7	1	8	0	9	1
6	0	1	0	4	0	7	0	12	0	19	0	26	0	35	0	46	0	57	0
8	0	1	1	5	1	14	2	30	0	55	3	91	1	140	4	204	0	285	5
10	0	1	0	7	0	26	0	66	0	143	0	273	0	476	0	776	0	1197	0
12	0	1	1	10	0	42	4	132	0	335	7	728	0	1428	12	2586	0	4389	19
14	0	1	0	12	0	66	0	246	0	715	0	1768	0	3876	0	7752	0	14421	0
16	0	1	1	15	1	99	5	429	1	1430	14	3978	2	9690	30	21318	0	43263	55
18	0	1	0	19	0	143	0	715	0	2704	0	8398	0	22610	0	54484	0	120175	0
20	0	1	1	22	0	201	7	1144	0	4862	26	16796	0	49742	66	130752	0	312455	143
22	0	1	0	26	0	273	0	1768	0	8398	0	32066	0	104006	0	297160	0	766935	0
24	0	1	1	31	1	364	10	2652	0	14000	42	58786	4	208012	132	643856	0	1789515	335
26	0	1	0	35	0	476	0	3876	0	22610	0	104006	0	400024	0	1337220	0	3991995	0
28	0	1	1	40	0	612	12	5538	0	35530	66	178296	0	742900	246	2674440	0	8554275	715
30	0	1	0	46	0	776	0	7752	0	54484	0	297160	0	1337220	0	5170604	0	17678835	0
32	0	1	1	51	1	969	15	10659	1	81719	99	482885	5	2340135	429	9694845	1	35357670	1430
34	0	1	0	57	0	1197	0	14421	0	120175	0	766935	0	3991995	0	17678835	0	68635478	0
36	0	1	1	64	0	1463	19	19228	0	173593	143	1193010	0	6653325	715	31429068	0	129644790	2704
38	0	1	0	70	0	1771	0	25300	0	246675	0	1820910	0	10855425	0	54587280	0	238819350	0
40	0	1	1	77	1	2126	22	32890	0	345345	201	2731365	7	17368680	1144	92798380	0	429874830	4862

TABLE 1. N_{PFN}, the number of Pauli-forbidden necklaces calculated from eq. (??)(entries with odd F and/or odd B vanish identically).

$$N_{PFN}(B,F) = N_{BNL}(B/2^k, F/2^k) = N_{PAN}(B/2^k, F/2^k)$$

where k is the unique +ve integer (if it exists) for which $F/2^k$ is odd and $B/2^k$ is integer (otherwise n_{PFN} is zero).

Strong coupling surprises in F=0,1 sectors

(JW & GV hep-th/0512301)

1. There is a (1st order?) phase transition at λ =1: the weak-coupling energy gap disappears

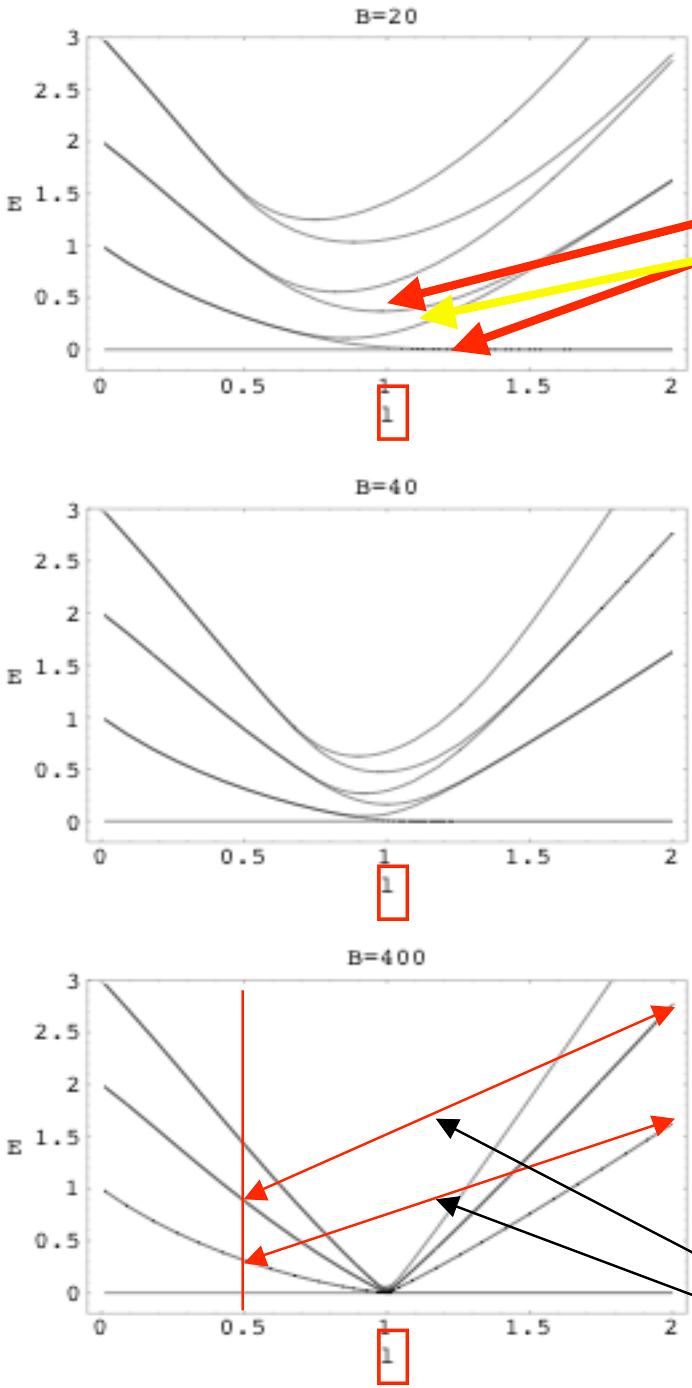
2. The spectrum becomes discrete again for λ >1;

   the eigenvalues at λ are related to those at 1/ λ by a strong-weak duality formula:

$$E(1/\lambda) + 1 = \lambda^{-2}(E(\lambda) + 1)$$

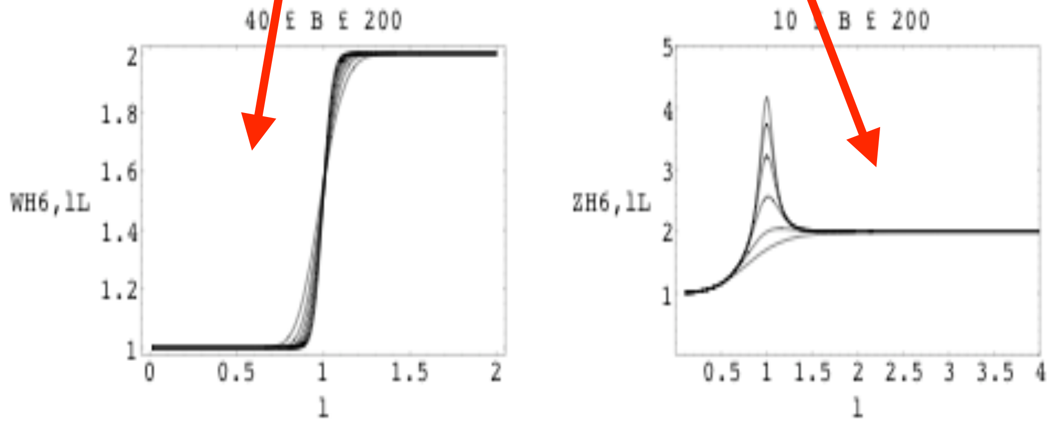
3. The spectrum can be computed analytically in terms of the zeroes of some ${}_1F_2$ function. Duality and phase transition can be studied analytically

4. At □ > 1 a second E=0 bosonic ground state pops up making Witten's index jump by one unit (within F=0,1 sectors).
5. First found numerically. The analytic form of the 2nd ground state can be formally given at all □ but is only normalizable at □ >1



Lowest bosonic and fermionic states as a funtion of □ for different values of the cutoff B (NB swapping of SUSY partners at finite cutoff)

Witten index and free energy as functions of □

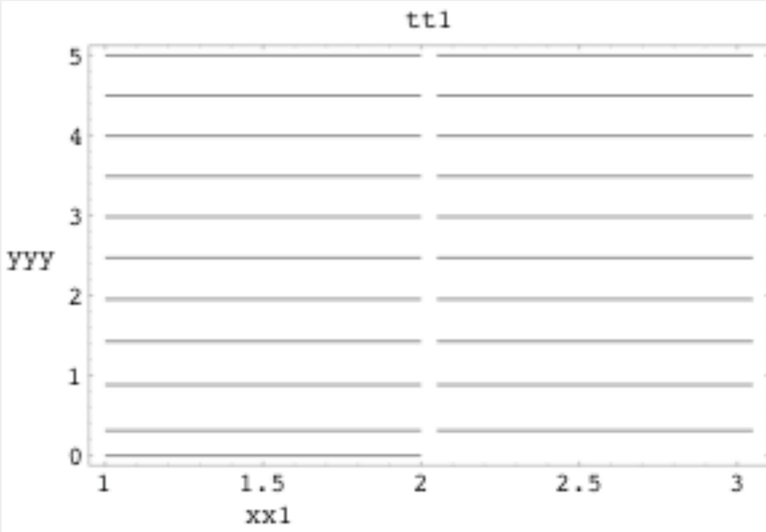


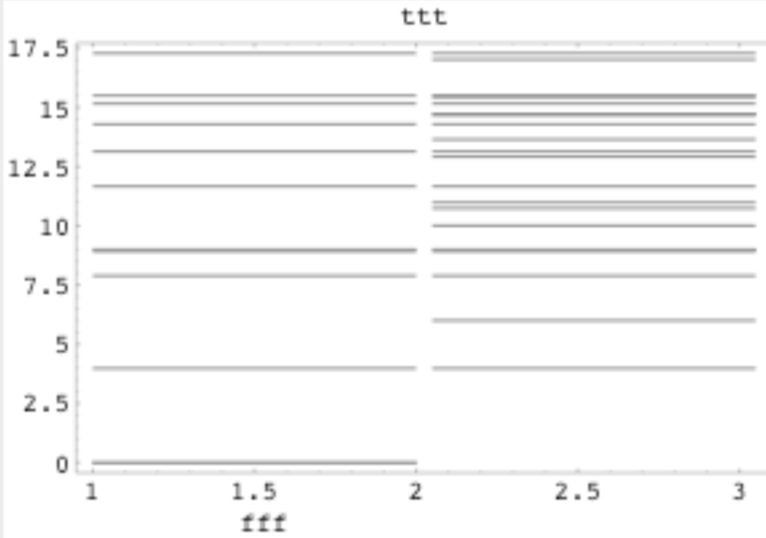
Energies related by □^{2} (E(1/ □)+1) = E(□)+1

More surprises in F=2,3 sectors

(JW & GV hep-th/0603045 & to appear)

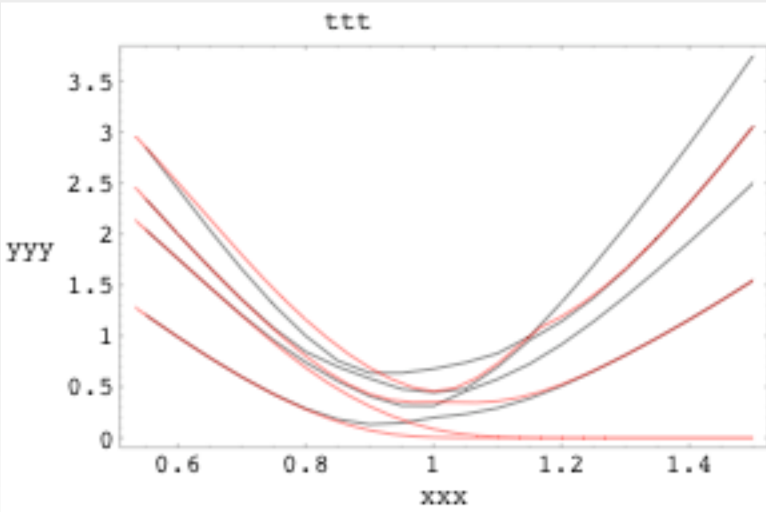
1. We fully confirmed SUSY expect.ns @ small □
2. Rearrangement of doublets more involved than for F=0,1 (F>3 needed to complete SUSY)
3. At large □ : two more E=0 states pop up => Witten index jumps by two units around □ =1



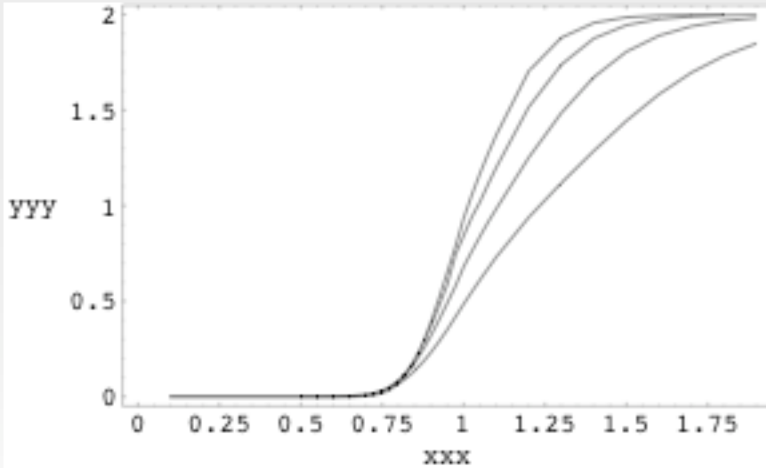


F=0 F=1

F=2 F=3



F=2,3

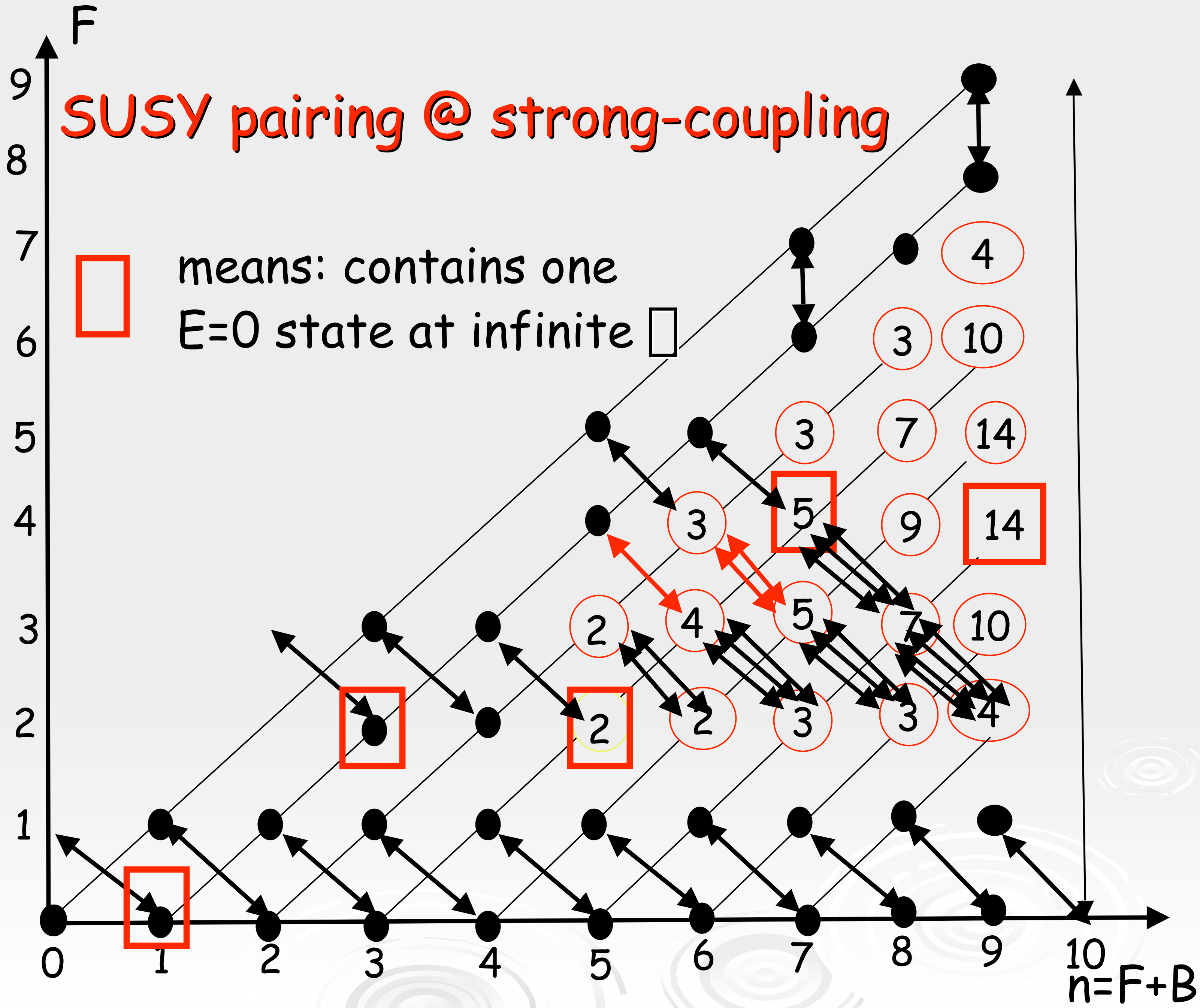


F=2,3

In order to understand what was going on we took -> infinity:

1. Hamitonian becomes block-diagonal in F and B.
2. SUSY doublets contain states with the same B+2F
3. SUSY acts in (B+F,F) plane along diagonals.
4. Some states can't find partners => must have E=0!
5. "Experimentally", lack of SUSY balance only occurred on diagonals with B + 2F = ± 1 (mod 6).
6. This has now been proven by D. Zagier (see note added to OVW paper)

SUSY pairing @ strong-coupling



The null states appear to form an infinite staircase!

Witten-like supertraces

From weak coupling:

$$W(n;m) \equiv \sum_{\substack{B+F=n \\ F\leq m}} (-1)^{F-m} N_{PAN}(B,F) \geq 0 \ , \ W(n;n)=0$$

From strong coupling

$$\tilde{W}(n;m) \equiv \sum_{\substack{B+2F=n \\ F\leq m/2}} (-1)^{F-m/2} \left(N_{PAN}(B,F) - \frac{(\delta_{F,B+1}+\delta_{F,B-1})}{2}(1+(-1)^F) \right) \geq 0$$

In particular:

$$\tilde{W}(n;n) = \sum_F (-1)^F N_{PAN}(n-2F,F) = \delta_{n=1(mod 6)} + \delta_{n=-1(mod 6)}$$

Verified numerically for n up to few thousands and/or proven

Conjecture (proven by now?): as L -> infty there is one and only one E=0 bosonic eigenstate in each (B,F) block with |B-F| = 1

F →, B ↓

B ↓ \ F →	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	2	3	3	3	4	5	5	5	6	7	7	7	8	9	9	9	10	11
3	1	1	2	4	5	7	10	12	15	19	22	26	31	35	40	46	51	57	64	70	77
4	1	1	[illegible]	5	[illegible]	14	20	30	43	55	70	91	115	140	168	204	245	285	330	385	445
5	1	1	3	7	14	26	42	66	99	143	201	273	364	476	612	776	969	1197	1463	1771	2126
6	1	1	3	10	22	42	76	132	217	335	497	728	1038	1428	1932	2586	3399	4389	5601	7084	8866
7	1	1	4	12	30	66	132	246	429	715	1144	1768	2652	3876	5538	7752	10659	14421	19228	25300	
8	1	1	4	15	42	99	[illegible]	429	809	1430	2424	3978	6308	9690	14520	21318	30667	43263	60060		
9	1	1	5	19	55	143	335	715	1430	2704	4862	8398	14000	22610	35530	54484	81719	120175			
10	1	1	5	22	73	201	497	1144	[illegible]	4862	[illegible]	16796	29414	49742	81686	130752	204347				
11	1	1	6	26	91	273	728	1768	3978	8398	16796	32066	58786	104006	178296	297160					
12	1	1	6	31	115	364	1028	2652	6310	14000	[illegible]	58786	[illegible]	208012	371384						
13	1	1	7	35	140	476	1428	3876	9690	22610	49742	104006	20801	400024							
14	1	1	7	40	172	612	1932	5538	14550	35530	81686	178296	[illegible]								
15	1	1	8	46	204	776	2586	7752	21318	54484	130752	297160									
16	1	1	8	51	244	969	3384	10659	30666	81719	204248										
17	1	1	9	57	285	1197	4389	14421	43263	120175											
18	1	1	9	64	335	1463	5601	19228	60115												
19	1	1	10	70	385	1771	7084	25300													
20	1	1	10	77	445	2126	8844														
21	1	1	11	85	506	2530															
22	1	1	11	92	578																
23	1	1	12	100																	
24	1	1	12																		

742900

208012

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786,

$$N_{PAN}(F \pm 1, F) = \frac{2m!}{m!(m+1)!}; m = min(F, F \pm 1)$$

Our s.c. H can be related, in those magic sectors, to that of the XXZ spin chain. Connection with combinatorics of ASMs via RS conjectures (RS = Razumov-Stroganov)

Conclusions

- The simple model I have discussed illustrates how SUSY can teach us something about some non-trivial combinatorial problems
- Viceversa, combinatorial methods have non-trivial implications on the dynamics of SUSY models
- Combination of analytic & numerical work has been crucial for progress
- Many opens Qs: e.g. we still do not quite understand the phase transition at □=1
- Extending the approach to (semi) realistic QFTs w/ or w/out SUSY remains the final physics goal of this (otherwise just amusing) mathematical game

One lesson:

You can find (almost) everything with Google!

But you have to be careful!

See what my daughter recently found:

You still can listen to it at any commercial on "CSI:Miami", it's a sentence saying: "According to the Quantum Theory of Veneziano, everything is connected: you only need to find the connection".

Ok, ok. WTF is the Quantum Theory of Veneziano… I don't know… BUT, and here comes the surprise, Veneziano is not a fake.

Gabriele Veneziano is an italian theoretical physicist leading a group at CERN for the ATLAS experiment

Modern String Theory was started with his paper on dual resonances in Quantum Chromodynamics where he developed some "hacking" called later the Veneziano Model (starting point of Superstrings and Supersymmetry)It's also nice to listen to his conferences…(is it?)