Light-cone quantum string Bethe equations

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based on work with Sergey Frolov and Jan Plefka

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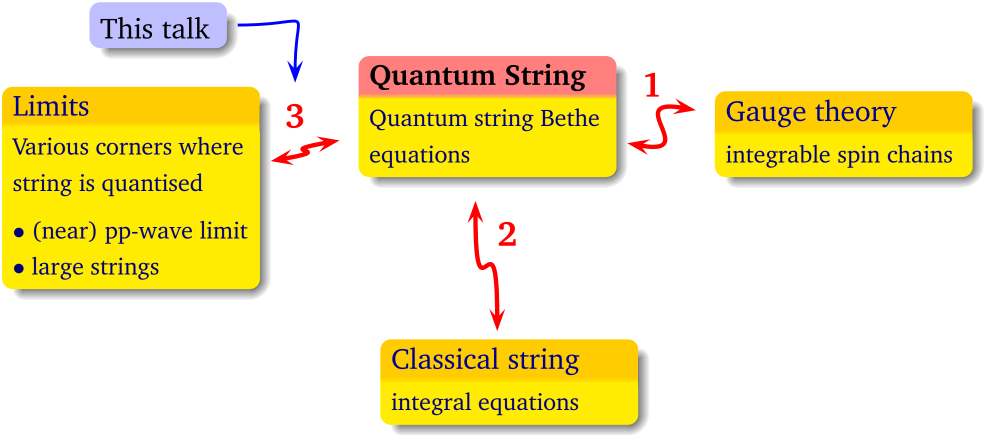
I Introduction

Central problem: How to quantise the string in $AdS_5 \times S^5$ background ?

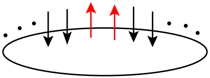
It seems that this can be rephrased as the question of solving a particular set of algebraic equations — quantum string Bethe equations.



1. Gauge theory

- Study two point functions of operators, i.e. anomalous dimension at n-loop
  → problem of diagonalisation of *long range* spin chain hamiltonian H_{sc}

$$\mathrm{tr}\big(\cdots XYYX\cdots X\big) \quad \leftrightarrow$$



[Minahan, Zarembo; Beisert, Kristjansen, Staudacher]

- Get various types of H_{sc} (depending on sector and loop order)

BUT they all appear to be *integrable*: diagonalisation is "simple"

(enough to solve two-magnon problem)

Using:

- explicit, higher loop H_{sc} where known (i.e. Bethe eqs)
- assuming BMN scaling
- assuming integrability

[Beisert, Dippel, Staudacher]

→ (ASYMPTOTIC) ALL LOOP (QUANTUM) BETHE EQUATIONS

2. Classical string theory

Classical σ-model (with fermions) $\longrightarrow$ integrable (infinite set of (local) charges)

Charges read off from analytic properties of **resolvent** $G(x)$,

$$\boxed{G(x+i\epsilon) - G(x-i\epsilon) = 2 \fint_{C_k} d\xi \frac{\rho(\xi)}{x-\xi} = 2\pi n_k - 2\pi \left(\frac{\mathcal{J}+m}{x+1} + \frac{\mathcal{J}-m}{x-1} \right)}$$

Can (in principle) be solved $\longrightarrow$ get (all) classical string solutions

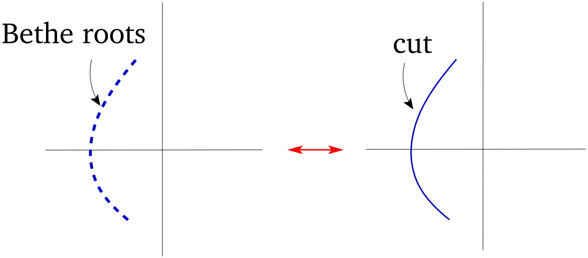
$\longrightarrow$ INTEGRAL EQS FOR THE STRING ON AdS$_5 \times S^5$

[Kazakov, Marshakov, Minahan, Zarembo]

1. $\leftrightarrow$ 2. Gauge theory vs. Classical string theory

Look at the **thermodynamic limit** of all loop gauge theory Bethe eqs

$\longrightarrow$ great similarity with integral eqs. in gauge theory!



gauge:
$$2\,⨍\, dx'\,\frac{\rho(x')}{x-x'} = \frac{1}{x}\,\frac{1}{1-\frac{\lambda'^2}{2x^2}} + \frac{\lambda'^2}{x}\int dx'\,\frac{\rho(x')}{x'^2}\,\frac{1}{1-\frac{\lambda'^2}{xx'}} + 2\pi n_k$$

different

string:
$$2\,⨍\, dx'\,\frac{\rho(x')}{x-x'} = \frac{1}{x}\,\frac{1}{1-\frac{\lambda'^2}{2x^2}} + \frac{\lambda'^2}{x}\int dx'\,\frac{\rho(x')}{x'^2}\,\frac{1}{1-\frac{\lambda'^2}{x^2}} + 2\pi n_k\,.$$

3. Quantum string Bethe equations

Start with string theory integral equations $\longrightarrow$ **d i s c r e t i s e**

$$e^{iLp_k} = \prod_{j \neq k} S_{kj} \times S_{\text{dressing}}(p_k, p_j)$$

$u_k(p_k)$ as for
all loop Bethe eqs

as in all loop
gauge theory

Quantum string Bethe equations

independent of type of excitation

[Arutyunov, Frolov, Staudacher]

[Staudacher; Beisert, Staudacher]

$$\epsilon(p_k) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p_k}{2}\right)} - 1$$

The $(\text{near})^n$ pp-wave limit

- Can we “derive” the unusual dispersion relation (“lattice”)?
- Can we “derive” Bethe equations i.e. the dressing factor?
- What is the nature of magnons (i.e. particles) on the string side?

Try to adress these questions by considering a limit where we know how to quantise strings: $(\text{near})^n$ pp-wave limit.

What is this limit?

- pp-wave: geometry seen by (massless) string moving with $J \to \infty$ s. t. $\lambda' = \lambda/J^2 = \text{const.}$
  → can be obtained by expanding string action in $1/J, \lambda' = const$

[Berenstein, Maldacena, Nastase]

- near pp-wave: as above, but keeping first $1/J$ correction.

[Callan et al.]

Constructing the Hamiltonian

Goal:

Start with GS action on $AdS_5 \times S^5$ $\rightarrow$ find gauge fixed Hamiltonian $\rightarrow$ expand in $1/J$ $\rightarrow$ quantise perturbatively in $1/J$

Up to now: $\longrightarrow$ expand Langrangian, $\longrightarrow$ construct Hamiltonian

not suitable for higher order computations; unusable for other limits.

- Have non-linear σ-model on super coset

  isometry group of (super) $AdS_5 \times S^5$: $$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$$

- Using the super current $g \in \mathrm{PSU}(2,2|4)$

$$A = -g^{-1}\mathrm{d}g = \underbrace{A^{(0)} + A^{(2)}}_{\text{even}} + \underbrace{A^{(1)} + A^{(3)}}_{\text{odd}},$$

write the action [Metsaev, Tseytlin]

$$\mathcal{L} = -\frac{1}{2}\sqrt{\lambda}\,\mathrm{Str}\Big(\gamma^{\alpha\beta}\mathrm{A}^{(2)}_\alpha \mathrm{A}^{(2)}_\beta + \epsilon^{\alpha\beta}A^{(1)}_\alpha A^{(3)}_\beta\Big)\,,$$

WZW term

Constructing the $AdS_5 \times S^5$ Hamiltonian

- Choose particular representation for the coset element

$$g(\chi, x, t, \phi) = \Lambda(t, \phi) g(\chi) g(x) \, .$$

bosonic part gives metric

$$\begin{aligned} ds^2 &= -G_{tt}(z) dt^2 + G_{\phi\phi} d\phi^2 + \frac{1}{(1 - \frac{z^2}{4})^2} dz_i dz_j + \frac{1}{(1 + \frac{y^2}{4})^2} dy_i dy_j \, . \\ G_{tt} &= \left(\frac{1 + \frac{z^2}{4}}{1 - \frac{z^2}{4}} \right)^2 \qquad G_{\phi\phi} = \left(\frac{1 - \frac{y^2}{4}}{1 + \frac{y^2}{4}} \right)^2 \, , \end{aligned}$$

singled out t and ϕ, i.e. manifest $SO(4) \times SO(4)$ symmetry.

Constructing the $AdS_5 \times S^5$ Hamiltonian

- Gauge fixing:

(a) Fix reparametrisation invariance with *uniform light-cone gauge*

[Arutyunov, Frolov]

light-cone:
in AdS two inequivalent null-geodesics: around $S^1 \in S^5$ and in the radial direction of AdS_5 ➔ two inequivalent l.c. gauge choices ➔ take one on the sphere

$$x_+ = \tau\,, \quad x_\pm = \frac{1}{2}(\phi \pm t)$$

uniform:
momentum density p_+ const on the world sheet

$$p_+ = P_+ = E + J = const$$

(b) appropriate fixing of κ-symmetry

Constructing the $AdS_5 \times S^5$ Hamiltonian cont.

- First-order form of the Lagrangian

  (a) bosonic example

$$\mathcal{L} = -\frac{\sqrt{\lambda}}{2}\gamma^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu}(x)$$

is equivalent to

$$\mathcal{L} = p_\mu \dot{x}^\mu + \frac{1}{\sqrt{\lambda}}\gamma^{\tau\tau}\Big(\underbrace{\tfrac{1}{2}\, p_\mu\, p_\nu\, g^{\mu\nu} + \tfrac{\lambda}{2}\, x'^\mu\, x'^\nu\, g_{\mu\nu}}_{C_1}\Big) + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\Big(\underbrace{p_\mu\, x'^\mu}_{C_2}\Big)$$

solving constraints $\qquad x^\mu = (x^+, x^-, x^m)$

$$C_2 = 0 \quad \Rightarrow \quad x'_- = -\frac{1}{P_+}x'^m p_m \qquad \Rightarrow \quad \int_0^{2\pi} d\sigma\, p_m\, x'^m = 0$$

level matching condition

$$C_1 = 0 \quad \Rightarrow \quad P_-(x^m, p^m) = \ldots$$

light-cone Hamiltonian

Constructing the $AdS_5 \times S^5$ Hamiltonian cont.

(b) inclusion of fermions complicates story significantly:

- (a) presence of WZW term
  (b) inversion of $p_\mu(\dot{x}^\nu, \theta)$ hard
  (c) non-trivial symplectic structure

$$\mathcal{L}_{g.f.} = \mathcal{L}_{kin} - \mathcal{H}_{l.c.}(x_m, p_m, \theta, \partial_\sigma \theta)$$

$$\mathcal{L}_{kin} = p_m \dot{x}^m + f(x_m, p_m, \theta) \text{“}\theta \partial_t \theta\text{”} + h(x_m, p_m, \theta) \text{“}\partial_\sigma \theta \partial_t \theta\text{”}$$

plus level matching condition

$\longrightarrow$ $\mathcal{H}_{l.c.}$ *exact, light-cone* Hamiltonian for $AdS_5 \times S^5$

Constructing the $AdS_5 \times S^5$ Hamiltonian cont.

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plus level matching condition

$\longrightarrow$ $\mathcal{H}_{l.c.}$ *exact, light-cone* Hamiltonian for $AdS_5 \times S^5$

- Expand $\mathcal{H}_{l.c.}$ in $1/P_+$ keeping $\tilde{\lambda} = \dfrac{4\lambda}{P_+^2} = \text{const.}$
- Redefine (perturbatively in $1/P_+$) fermions to get canonical Poisson structure.

The near-pp wave Hamiltonian

Quartic Hamiltonian: $\mathcal{H}_4 = \mathcal{H}_{bb} + \mathcal{H}_{bf} + \mathcal{H}_{ff}\,,$

(and introduce complex fields $Z_1 = z_2 + iz_1$ etc.)

Boson-boson interactions:

$$\mathcal{H}_{bb} = \frac{\tilde{\lambda}}{4P_+}\left(Y'_{5-a}Y'_a Z_{5-b}Z_b - Y_{5-a}Y_a Z'_{5-b}Z'_b + Z'_{5-a}Z'_a Z_{5-b}Z_b - Y'_{5-a}Y'_a Y_{5-b}Y_b\right)$$

Fermion-fermion interactions

$$\mathcal{H}_{ff} = -\frac{\tilde{\lambda}}{4P_+}\mathrm{tr}\,\Sigma\left(\eta'^\dagger\eta\eta'^\dagger\eta + \eta^\dagger\eta'\eta^\dagger\eta' + \eta'^\dagger\eta^\dagger\eta'^\dagger\eta^\dagger + \eta'\eta\eta'\eta - \left[\eta\to\theta\right]\right)$$

Boson-fermion interactions

$$\begin{aligned}
\mathcal{H}_{bf} \;=\; & \frac{1}{2P_+}\mathrm{tr}\Big[\frac{\tilde{\lambda}}{2}(Z_{5-a}Z_a - Y_{5-a}Y_a)\left(\eta'^\dagger\eta' + \theta'^\dagger\theta'\right) \\
& - \frac{\tilde{\lambda}}{2}Z'_m Z_n\left[\Gamma_m,\Gamma_n\right]\left(P_+(\eta\eta'^\dagger - \eta'\eta^\dagger) - P_-(\theta^\dagger\theta' - \theta'^\dagger\theta)\right) \\
& + \frac{\tilde{\lambda}}{2}Y'_m Y_n\left[\Gamma_m,\Gamma_n\right]\left(-P_-(\eta^\dagger\eta' - \eta'^\dagger\eta) + P_+(\theta\theta'^\dagger - \theta'\theta^\dagger)\right) \\
& - \frac{i\kappa}{2}\sqrt{\tilde{\lambda}}\,(Z_n P^z_m)'\left[\Gamma_n,\Gamma_m\right]\left(P_+(\eta^\dagger\eta^\dagger + \eta\eta) + P_-(\theta^\dagger\theta^\dagger + \theta\theta)\right) \\
& + \frac{i\kappa}{2}\sqrt{\tilde{\lambda}}\,(Y_n P^y_m)'\left[\Gamma_n,\Gamma_m\right]\left(P_-(\eta^\dagger\eta^\dagger + \eta\eta) + P_+(\theta^\dagger\theta^\dagger + \theta\theta)\right) \\
& + 4i\tilde{\lambda}Z_m Y_n\left(-P_-\Gamma_m\eta'\Gamma_n\theta' + P_+\Gamma_m\theta'^\dagger\Gamma_n\eta'^\dagger\right)\Big]
\end{aligned}$$

The near-pp wave Hamiltonian–spectrum

Looks complicated**!** Still manageable *analytically* due to good gauge choice...

- First, solve the free, i.e. **quadratic** $\mathcal{H}_2$
- Second, compute corrections state $|\Psi\rangle$ due to $\mathcal{H}_4$. Need matrix elements

$$\langle \Psi_A | \mathcal{H}_4 | \Psi_B \rangle$$

Simplifications in subsectors ...

The near-pp wave spectrum – subsectors

What are subsectors?

Space is anisotropic $\rightarrow$ excitations of string in various directions inequivalent.

- $\mathbf{su(1|1)}$ sector: 1 boson and 1 fermion

gauge theory | string theory

$$O_{\mathbf{su(1|1)}} = \mathrm{tr}(Z^{J-M/2}\psi^M) \longleftrightarrow |\Psi_{\mathbf{su(1|1)}}\rangle = \theta^+_{1,n_M}\theta^+_{1,n_{M-1}}\cdots\theta^+_{1,n_1}|0\rangle$$

$\rightarrow$ $\mathcal{H}_4^{\mathbf{su(1|1)}} \equiv 0$ $\rightarrow$ Free theory! Miracle of the gauge choice

BUT we want $\mathbf{E}$, the energy w.r.t to global time

$$H_{l.c.}(P_+) = \mathbf{E} - J \qquad P_+ = \mathbf{E} + J = const.$$

$$\mathbf{E} = J + H_{l.c.}(\mathbf{E} + J)$$

have to solve this for $\mathbf{E}$!

[Arutyunov, Frolov]

The near-pp wave: spectrum of subsectors

Solve perturbatively in $1/J$, keeping $\lambda' = \lambda/J^2 = const.$

$$\mathbf{E}_{\mathbf{su(1|1)}} - J = \sum_{k=1}^{M} \bar{\omega}_k - \frac{\lambda'}{4J} \sum_{k=1}^{M} \sum_{j=1}^{M} \frac{n_k^2 \bar{\omega}_j^2 + n_j^2 \bar{\omega}_k^2}{\bar{\omega}_k \bar{\omega}_j}$$

$$\bar{\omega}_i = \sqrt{1 + \lambda' n_i^2}$$

the same as result of Callan et al. in static gauge

Nontrivial spectrum, for M-impurity (particle) state, came from free theory!

The near-pp wave: spectrum of subsectors

- other rank one sectors, easy for M-impurities
  - $\mathbf{su(2)}$ — fluctuations in $S^3 \in S^5$ two bosons Z, Y:

$$\delta E_{\mathbf{su(2)}} = \langle \Psi_{\mathbf{su(2)}} | \mathcal{H}_4 | \Psi_{\mathbf{su(2)}} \rangle \neq 0$$

  - $\mathbf{sl(2)}$ — fluctations in AdS_3 mirror to $\mathbf{su(2)}$

$$\delta E_{\mathbf{sl(2)}} = -\delta E_{\mathbf{su(2)}}$$

  effect of the opposite curvature

- higher rank sectors — *new* features:

  degeneracy and nontrivial mixing $\rightarrow$ diagonalise matrix $\langle \Psi_A | \mathcal{H}_4 | \Psi_B \rangle$

From pp-wave to light-cone Bethe equations

Q: Why did we bother going through the previous mess?

Interesting physics can be extracted ...

From AdS/CFT correspondence:

$\Delta - J \sim H_{\text{spin chain}}$ acting on lattice of letters (spin)	$\longleftrightarrow$	energy
"particles" combinations of letters (changing letter $\leftrightarrow$ flipping of spin)	$\longleftrightarrow$	excitations created by oscillators $\alpha^\dagger$
spectrum from Bethe eqs	$\longleftrightarrow$	string Bethe eqs **?**

From pp-wave to light-cone Bethe equations

Lightning review of coordinate Bethe ansatz:

$$H_{\text{spin chain}}|\Psi\rangle = E|\Psi\rangle$$

(1) infinite lattice, solve two-body system, ansatz:

$$\left.\begin{aligned}
&|\Psi(p_1,p_2)\rangle = \sum_{x_1,x_2} \Psi(x_1,x_2)\alpha^\dagger_{x_1}\alpha^\dagger_{x_2}|0\rangle \\
&x_1 \ll x_2 \qquad \Psi(x_1,x_2) = e^{i(p_1x_1+p_2x_2)} \\
&x_1 \gg x_2 \qquad \Psi(x_1,x_2) = S(p_1,p_2)e^{i(p_1x_2+p_2x_1)}
\end{aligned}\right\} \quad \text{elastic scattering}$$

→ form of S fixed

→ $E = \sum_i \frac{\lambda}{8\pi^2}\sin^2\left(\frac{p}{2}\right) + \dots$

non-relativistic

on the lattice

From pp-wave to light-cone Bethe equations

Lightning review of coordinate Bethe ansatz:

(2) periodic boundary conditions on Ψ

$$e^{ip_k L} = S(p_1, p_2)\,, \quad k = 1, 2$$

(3) bootstrap to M-body system

$$\boxed{e^{ip_k L} = \prod_{i=1, i\neq k}^{M} S(p_k, p_i)\,, \qquad k = 1, \ldots, M}$$

Bethe equations

Energy additive $E_{tot}(p_1 \ldots p_M) = \sum_{i=1}^{M} E(p_i)$

Cyclicity of trace $\sum_{i=1}^{M} p_i = 0$.

From pp-wave to light-cone Bethe equations

String side:

Assume similar structures exist, for *finite* volume string world sheet,

→ read-off S-matrix

[cf. Staudacher]

- **Assume** light-cone "Bethe" eqs

$$e^{ip_k \frac{P_+}{2}} = \prod_{i=1, i\neq k}^{M} S(p_k, p_i) \qquad L \leftrightarrow P_+/2$$

- **Assume** the dispersion relation

$$E_{l.c.}(p_k) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p_k}{2}\right)}$$

→ compute from Bethe eqs $\delta E_{l.c.}$ for two magnons in the limit

$$P_+ \to \infty \,, \; \lambda \to \infty \quad \text{s.t.} \qquad \tilde{\lambda} = \frac{4\lambda}{P_+^2} = \text{const.}$$

From pp-wave to light-cone Bethe equations

$$\delta E_{l.c.} = \frac{\tilde{\lambda}\, P_+}{2\pi} \sum_{k,j=1,k\neq j}^{M} \frac{-i n_k}{\sqrt{1+\tilde{\lambda} n_k^2}} \log S\left(\frac{4\pi}{P_+} n_k, \frac{4\pi}{P_+} n_j\right) \longleftrightarrow \delta E_{l.c}^{\text{semi-class.}}$$

- solve for S-matrix to accuracy $1/P_+$
- natural extension to all orders in $1/P_+$

$$e^{i p_k \frac{P_+}{2}} = \prod_{j=1, j\neq k}^{M} \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \sqrt{\frac{x_j^+ \, x_k^-}{x_j^- \, x_k^+}} \right)^s , \quad s = \begin{cases} 1 & \text{su}(2) \\ 0 & \text{su}(1|1) \\ -1 & \text{sl}(2) \end{cases}$$

light-cone string Bethe equations

in natural variables

$$x^{\pm}(p) = \frac{e^{\pm i \frac{p}{2}}}{4 \sin^2 \frac{p}{2}} \left(1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \right)$$

Light-Cone Bethe equations–consistency checks

✓ correct classical integral equations in $P_+, M \to \infty, \;\; M/P_+ = \text{const.}$

✓ correct strong coupling limit $\lambda \to \infty, \;\; J = \text{const.}$

? L.C. Bethe equations vs. AFS?

? Finite-size effects? Origin of the “lattice”, can we see $\sin^2(p/2)$?

$$\epsilon(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}$$

? How to get finite number of states in fixed J sectors in string theory?