Nonlinear evolution of unstable fluid interface

S.I. Abarzhi
Department of Applied Mathematics and Statistics
State University of New-York at Stony Brook
LIGHT FLUID ACCELERATES HEAVY FLUID
misalignment PRESSURE and DENSITY gradients
INSTABILITY TURBULENT MIXING

Rayleigh-Taylor instability sustained acceleration (gravity)
Richtmyer-Meshkov instability impulsive acceleration (shock)

- thermonuclear flashes on surface of stars; supernova explosion
- inertial confinement fusion; interaction of laser with matter

**Basic objective:** reliable description of turbulent mixing

**Fundamental issues:**
- the cascades of energy
- the dynamics of small-scale structures
- the dynamics of the large-scale coherent structure

**Coherent structure**

an array of bubble and spikes periodic in the plane normal to the direction of acceleration (shock)

- Dynamics of 3D and 2D nonlinear structures in RMI
- Properties of the 3D-2D dimensional crossover in RMI
INTERFACE  active regions         passive regions
Aref\textsuperscript{1989}  small scales     large scales
        intensive vorticity     simple advection

large-scale coherent motion          scalar fields
spectral approach                    group theory

Abarzhi\textsuperscript{1996} (RTI)

\textbf{Group theory}

coherent structure       periodicity

group of invariance  17 plane crystallographic symmetry groups

\[ G \text{ translations in the plane + rotations + reflections} \]

The COHERENT STRUCTURE is OBSERVABLE

\begin{itemize}
  \item A significant part of the fluid energy is concentrated in the coherent motion
    a DOMINANT mode \( K \) governs macroscopic dynamics
  \item The structure is stable under modulations
    \[ K + \xi: \quad \phi(K + \xi) \approx \phi(K) + F(K)\xi^2 \quad K \leftrightarrow -K \]
    a scalar macroscopic function
\end{itemize}

\( G \) is a \textbf{symmorphic group} with \textbf{inversion} in the plane

3D: p6mm, p4mm, p2mm, cmm, p2  2D: pm\textsubscript{11}
LARGE-SCALE COHERENT MOTION


time \( t \) potential \( \Phi(x, y, z, t) \) free surface \( z^*(x, y, t) \)

\[
\Delta \Phi = 0, \quad \nabla \Phi \bigg|_{z=+\infty} = 0
\]

\[
\frac{\partial z^*}{\partial t} + \nabla z^* \nabla \Phi - \frac{\partial \Phi}{\partial z} \bigg|_{z=z^*} = 0, \quad \frac{\partial \Phi}{\partial t} \bigg|_{z=z^*} = 0, \quad \frac{1}{2} (\nabla \Phi)^2 + g(t)z \bigg|_{z=z^*} = 0
\]

g \geq 0 - instability: RT \quad g > 0 \quad \text{RM} \quad g = 0

Initial conditions: \( z^*(x, y, t_0) \) \( \nu(x, y, t_0) \)

length scale \( \lambda (\sim \lambda_{\text{max}}) \) time scale \( \tau \sim \lambda/|\nu_0| \)

Symmetry: periodic, symmorphic + inversion in the plane \((x, y)\)

difficulty SINGULARITY interplay of harmonics

2D RTI: Taylor\(^{1950}\), Fermi\(^{1952}\), Layzer\(^{1955}\), Garabedian\(^{1957}\),
Birkhoff\(^{1957}\), Zuffiria\(^{1986}\), Inogamov\(^{1990}\), Tanveer\(^{1993}\), Hazak\(^{1997}\)
3D RTI: Abarzhi\(^{1995}\)

LOCAL EXPANSIONS ASYMPTOTIC SOLUTIONS

2D & 3D RMI: Shvarts\(^{1995}\), Inogamov\(^{1995}\), Mikaelian\(^{1998}\),
Zhang\(^{1998}\), Abarzhi\(^{2000}\)

Layzer-type approach single-mode approximation
• Expansion in terms of orthogonal functions:

\[
\Phi = \sum_{n=1}^{\infty} \Phi_n(t) \left( \frac{1}{\gamma_n} \exp \left( -\gamma_n(z - z_0(t)) + i \sum_j k_j r n_j \right) + c.c. \right)
\]

irreducible representations of group \(G\) \(\text{wave-vectors} \ k, G_k\) project operators \(\text{Fourier expansion}\)

• Local expansion at a highly symmetric point of the interface \(x \approx 0, y \approx 0, z \approx z_0(t)\):

\[
z^* (x, y, t) = z_0(t) + \sum_{i+j=1}^{\infty} \zeta_{ij}(t)x^{2i}y^{2j}, \quad N = i + j = 1, 2, \ldots \infty
\]

Dynamical system of ordinary differential equations

\[
\sum_{i+j=1}^{\infty} D_{ij}(\dot{M}, M, \zeta)x^{2i}y^{2j} = 0 \quad \sum_{i+j=1}^{\infty} K_{ij}(\dot{\zeta}, M, \zeta)x^{2i}y^{2j} = 0
\]

\[
\zeta = \{\zeta_{ij}\}; \quad M = \{M_n\} \text{ moments } M_n = \sum_{m=1}^{\infty} \Phi_m (km)^n
\]

• Local dynamics, any time \(t\); the length scale(s) \(\lambda\) is invariable

• Multiple harmonics presentation

• 3D flows with general type of symmetry and 2D flows

• Desired accuracy, \(x \approx 0, y \approx 0, z \approx z_0(t), \ N = i + j = 1, 2, \ldots \infty\)
REGULAR ASYMPTOTIC SOLUTIONS

\[ \sum_{i+j=1}^{\infty} D_{ij}(\dot{M}, M, \zeta)x^{2i}y^{2j} = 0 \]
\[ \sum_{i+j=1}^{\infty} K_{ij}(\dot{\zeta}, M, \zeta)x^{2i}y^{2j} = 0 \]

regular asymptotic solutions \( t/\tau \gg 1 \)

Richtmyer-Meshkov bubble: \( v(t) \sim \lambda/t, \zeta(t) \sim 1/\lambda \)

Layzer-type expansion: regular asymptotic solutions are absent in general case

- non-linearity is non-local
- singularities determine the interplay of harmonics

At a fixed length scale(s) \( \lambda \), shape of the regular bubble is free and is parameterised by the principal curvature(s)

number of the parameters \( N_p \) symmetry of the 3D (2D) flow

\[ N_p \leq 3 \]

2D pm11, 3D p4mm, p6mm \( N_p = 1 \) 3D p2mm \( N_p = 2 \)

✓ to capture the interplay of harmonics
✓ to show existence and convergence for solutions in the family
✓ to involve all bubbles allowed by symmetry of the flow
✓ to choose the physically dominant (i.e. the fastest stable) solution
RICHTMYER-MESHKOV bubbles

Curvature radius $R$ (radii $R_{x,y}$) $kR_{cr} \leq kR \leq \infty$

Velocity $v = L(k, R)/t$ surface variables $\zeta_n = \zeta_n(k, R)$

Fourier amplitudes $\Phi_n = \Phi_n(k, R)/t$

$|\Phi_n/\Phi_1| \sim \exp(-pn)$

Asymptotic stability $v - L(k, R)/t \sim t^{\beta-1}$, $\zeta_n(t) - \zeta_n(k, R) \sim t^\beta$

$\beta = \beta(kR)$ for stable solutions $\text{Re}[\beta] < 0$

Properties

1. The physically dominant solution in the family corresponds to a bubble with a flattened surface, $kR \rightarrow \infty$

2. The bubble flattens in time as $kR \sim (t/\tau)^{\beta_{\infty}}$

3. For highly symmetric 3D flows: $kR_{3D} \sim (t/\tau)^{\beta_{\infty}}$, $v_{3D} \sim 4/kt$

4. The local dynamics of 3D highly symmetric flows is universal; near-circular contour $z^* \sim \zeta_1(x^2 + y^2)$

5. 3D anisotropic bubbles tend to conserve a near-circular contour

6. 3D anisotropic bubbles are unstable

7. 3D Layzer-type “square” solution is the point of bifurcation

8. The dimensional crossover is discontinuous, $\beta_{3D-2D} > 0$

9. NO 2D flows
Family of regular asymptotic solutions in RMI

Velocity $v$ as the function on the radius of curvature $R$

Three-dimensional flows with hexagonal (3D$_h$) and square (3D$_s$) symmetry and two-dimensional flow (2D); $k$ is the wave-vector, $t$ is time, $N$ is order of approximation.

Black circles mark the Layzer-type solutions with

$R_L = 4/k$, $v_L = 1/kt$ in 3D and $R_L = 3/k$, $v_L = 2/3kt$ in 2D.
Family of regular asymptotic solutions in RMI

Exponential decay of the Fourier-amplitudes with an increase in their number.

Three-dimensional flows with hexagonal symmetry p6mm (3D$_h$);
\[ \Phi_{\text{max}} = \Phi_1 (kR \equiv \infty) \]; black circle corresponds to the Layzer-type bubble.
Family of regular asymptotic solutions in RMI

Stability analysis for the family of regular asymptotic solutions
Real parts of exponents $\beta$ as functions on the radius of curvature $R$
Dashed lines correspond to $N=1$, solid lines – to $N=2$, black circle corresponds to the Layzer-type solution.
Evolution of the bubble front in RMI
Highly symmetric 3D and 2D coherent structures

time scale $\tau \sim 1/v_0 k$

$t \ll \tau$: curvature $\zeta_1(t) \sim -k t/\tau$, velocity $v(t) - v_0 \sim v_0 t/\tau$

$t \sim \tau$: curvature $\zeta_1(t) \sim -k$

$t \gg \tau$: curvature $\zeta_1(t) \sim -k (t/\tau)^{-|\beta_\infty|}$, velocity $v(t) \sim C_\infty / k t$

Dynamic trajectories

Solid line corresponds to multiple harmonic solution, and black square - to the flattened bubble. Dashed line corresponds to Layzer-type single-mode solution, and black circle – to the Layzer-type bubble.
Evolution of the bubble front in RMI

RM bubbles flatten  RM bubbles decelerate

• Qualitative agreement with experiments

• Bubble velocity \( v_\infty \sim C_\infty /kt \) \( v_L \sim C_L /kt \)
  \( C_\infty /C_L \sim 3 - 4 \) \( \Delta h \sim C \ln(t/\tau) \)

• Bubble shape \( \zeta_1(t) \sim -k(t/\tau)^{-|\beta_\infty|} \) reliable parameter

• Existence of an exact analytical solution

• a rigid body curvature \( \sim 1/R \) drag force = \( \rho v^2 R^2 \)

For a two-fluid system, Atwood number < 1

• the Layzer-type approach requires MASS FLUX through the interface

• Flattened RM bubble is a multiple-harmonic solution with NO MASS FLUX through the interface
Dependence of velocity $v = L(R_{x,y}, k_{x,y})/t$ on the bubble shape

Low-symmetric bubbles with rectangular symmetry $3D_r$, two-parameter family; various values of the aspect ratio; the highest curve $3D_s$ is the family of solutions for 3D square bubbles with $R_y = R_x$ and $k_x/k_y = 1$; the lowest curve $2D$ is the family of solutions for 2D bubbles flat in the $y$-direction with $R_y \equiv \infty$
Family of regular asymptotic solutions in RMI

Bifurcation of the Layzer-type square solution (black point) for nearly symmetric flows with $k_x \sim k_y$ and $R_x \sim R_y$
Family of regular asymptotic solutions in RMI

Stability analysis for low-symmetric RM bubbles

Dashing lines corresponds to highly symmetric 3D square solutions with \( k_x = k_y \) and \( R_x = R_y \). Solid lines correspond to nearly symmetric solutions with \( k_x \sim k_y \) and \( R_x \sim R_y \). Non-symmetric solutions are unstable.
Evolution of the bubble front in RMI

Low-symmetric 3D coherent structures

The dimensional 3D-2D crossover

2D bubbles under 3D modulations

time scale $\tau \sim 1/|v_0|k_x$

$t << \tau \quad \zeta_{1x}(t) \sim -k_x t/\tau \quad \zeta_{1y} \sim -k_y t/\tau$

$t >> \tau \quad \zeta_{1x}(t) \sim -k_x (t/\tau)^{-|\beta_\infty|} \quad \zeta_{1y} \sim -k_x (t/\tau)^{\beta_{3D-2D}}$

the dimensional crossover is discontinuous, $\beta_{3D-2D} > 0$

Secondary instabilities

• Secondary instabilities in RMI are “slow” in contrast to RTI

SINGULAR ASYMPTOTIC SOLUTIONS

Richtmyer-Meshkov spikes small-scale structure dynamics

Singular asymptotic solutions to dynamical system

Zhang\textsuperscript{1998}, Abarzhi\textsuperscript{2000}

$t >> \tau: \quad \text{shape } z^*(t) \sim \sum_{n=1} C_n \left( \exp(t/\tau) r^2 \right)^n, \quad \text{velocity } v(t) \sim -v_0$

• Tanveer 1993, Baker and Meiron 1989, Pullin\textsuperscript{2001} …

• For a two-fluid system, Atwood number < 1, the singular asymptotic solutions requires mass flux through the interface
Conclusion

✓ Large-scale coherent motion in RMI
✓ Separation of scales active regions passive regions
✓ Group symmetry large-scale coherent motion
✓ Local dynamics of regular bubbles and singular spikes
✓ Consideration of 3D flows with general type of symmetry
✓ Singularity – interplay of harmonics – shape of the bubble
✓ Family of regular asymptotic solutions – symmetry of the flow
✓ The physically dominant solution in the family
✓ Multiple harmonic solution
✓ Universality of local dynamics for 3D highly symmetric flows
✓ Conservation of a near-circular contour of 3D bubbles
✓ Discontinuous 3D-2D dimensional crossover
✓ Singular asymptotes
✓ Comparison between the local dynamics in RTI and RMI
✓ Different types of the bubble front evolution in RTI and RMI
✓ Layzer-type bubbles in RTI and RMI
✓ New type of the evolution of the bubble front in RMI
✓ Integral (velocity) and internal (shape) diagnostic parameters
✓ Theory works effectively for a two-fluid system

Discussion

??? turbulent mixing in RMI and RTI