Toy models for Rayleigh-Taylor instability:

Stuart Dalziel

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

International Workshop on the Physics of Compressible Turbulent Mixing
14 December 2001
(9:10 – 9:30)

with thanks to
Joanne Holford (DAMTP)
David Youngs (AWE)
The growth question:

\[ h = \alpha Agt^2, \text{ where } A = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \]

But what is \( \alpha \)?

\( \diamond \) 0.10, … 0.07, 0.06, … 0.03, 0.02 ?

Timescale:

\[ T = \sqrt{\frac{H}{Ag}} \]

If \( \delta = h/H \), and \( \tau = t/T \),

then \( \delta = \alpha \tau^2 \)
Experiments
Appropriate modelling (?)
Growth

Dimensional analysis/similarity theory

\[ h = \alpha \, Ag \, t^2. \]

**Single mode**

Layzer (1955)

For

\[ \zeta(x, y) = a_0 \cos \frac{2\pi x}{\lambda} \]

if

\[ \frac{dh}{dt} = w, \]

then

\[ (2 + E)\frac{dw}{dt} = Ag(1 - E) - C_D \frac{w^2}{\lambda}, \]

where

\[ E = \exp \left( \frac{-6\pi h}{\lambda} \right). \]

Experimentally \( C_D \sim 10 \)

◊ Does this make sense?

Early time → linear theory

\[ \frac{d^2 h}{dt^2} = \frac{2\pi Ag}{\lambda} h \]

Late time → constant velocity

\[ w_\infty = \sqrt{\frac{Ag \lambda}{C_D}} \]

\[ \Rightarrow h \rightarrow w_\infty(t - t_0) \]
Structure

Often described as ‘bubbles’…
…but more like ‘thermals’ in miscible fluids
Thermals

Self-similar

\[ r = \theta z \]
\[ V = \gamma r^3. \]

Buoyancy conserved

\[ g'V = g'\gamma r^3 = g'\gamma V_0. \]

Constant Froude number

\[ F^2 = \frac{w^2}{g'r} \]

Integrating \( w = \frac{dz}{dt} \)

\[ \frac{\gamma^{1/2}\theta}{2F(g'_0V_0)^{1/2}} z^2 = t \]

Experimental results \( \rightarrow F \approx 1.2. \)
Rayleigh-Taylor as thermals

Froude number $\sim 1.2$ (aspect ratio $0.72$) \( \Rightarrow \) \( C_{\text{Thermal}} \approx 1.3 \).

Rayleigh-Taylor bubbles a little like thermals \( \Rightarrow \) \( C_D \approx 1.3 \)

But in Rayleigh-Taylor environment
- Density field not hydrostatic in ambient
  - Hydrostatic in mean density \( \Rightarrow \) halve buoyancy force \( \Rightarrow C_D \approx 2.6 \)
- Flow around bubble affected by bubble moving in opposite direction
  - Drag due to twice rise speed of bubble \( \Rightarrow C_D \approx 10.4 \)

In agreement with single mode experiments

BUT natural R-T has more than one mode
**Multi-mode**

What happens if $\lambda$ grows with $h$?

Let $\lambda = \psi h$

Late times approximation:

$$\frac{dh}{dt} = \left( \frac{Ag}{C_D} (1 - E)\psi \right)^{1/2} h^{1/2}$$

$$\Rightarrow h = \frac{Ag}{C_D} (1 - E)\psi (t - t_0)^2 = \alpha Ag(t - t_0)^2$$  

For $C_D = 10$ and $\psi = 1$, $\alpha = 0.025$.

[Full Layzer growth with $\psi = 1$ gives $\alpha = 0.023$.]

Growth rate maximised with $\psi \sim 10$ giving $\alpha \sim 0.103$
Where do the modes begin? How do they interact?

◊ Nonlinear interaction?

◊ Initial perturbation?

If modes independent and equal amplitude:

\[
\tau = \left( \frac{Ag}{H} \right)^{1/2} t
\]

\[
\delta = \alpha \tau^2 \text{ with } \alpha = 0.06
\]

\[
\frac{\lambda}{H} = \begin{align*}
0.002 & \quad 
0.004 & \quad 
0.008 & \quad 
0.016 & \quad 
0.032 & \quad 
0.064 & \quad 
0.128 & \quad 
0.256 & \quad 
0.512 & \quad 
1.024 & 
\end{align*}
\]

Instantaneous nonlinear mode halving interaction when \( h = \lambda \):

\[
\tau = \left( \frac{Ag}{H} \right)^{1/2} t
\]

\[
\delta = h/H
\]

Which is it?
**Mixing**

*See talk by Joanne Holford*

**Energy budget**

Can decompose $PE$ into **Background PE** and **Available PE**.

$PE_{back}$ is the minimum energy state that is achieved by adiabatic rearrangement of fluid parcels.

Mixing increases $PE_{back}$ – it cannot decrease it!

$PE_{avail}$ is the component of $PE$ that can be converted into $KE$, heat (through dissipation) and, if mixing occurs, into $PE_{back}$.

In the absence of external work:

$$PE = PE_{back} + PE_{avail}$$
Mixing efficiency

\[ \eta_{\text{integral}} = - \frac{\Delta E_{\text{back}}}{\Delta E_{\text{avail}}} = \frac{\Delta PE_{\text{Back}}}{\Delta PE_{\text{Back}} + \int \varepsilon \; dt} = -\frac{\Delta PE_{\text{Back}}}{\varepsilon + \Delta PE_{\text{Avail}}} \]

Overall mixing efficiency \( \eta_0 \)

\[ \eta_{\text{instantaneous}} = \frac{\delta PE_{\text{Back}}}{-\delta E_{\text{Avail}}} = \frac{\delta PE_{\text{Back}}}{\delta KE + \delta PE_{\text{Avail}}} \]
Thermal

Entrainment into a thermal

\[ \frac{dV}{dt} = \beta w A \]

\[ \beta = 0.18. \]

Energetics of a thermal

Mixing efficiency not well defined: depends on size of domain!

Rayleigh-Taylor
\[ h = \alpha Agt^2 \]  \[ \delta = \alpha \tau^2 \]  
\[ w = 2\alpha Agt \]  \[ \omega = 2\alpha \tau \]  
\[ V = 2L^2h \]  

Total potential energy

\[ PE_{Total}^* = \frac{PE_{Total}}{PE_0} = 1 - 4\alpha^2\tau^4 \]  

Background potential energy

Changes due to entrainment between counter-flowing streams.
Invoke entrainment hypothesis: \( u_e = \beta w \)
Area of entrainment independent of \( h \)
\[ \Leftrightarrow \text{depth of entrainment comparable with } \lambda \]  
\[ \Rightarrow \text{entraining area} = \varphi \times \text{plan area}. \]  
\[ PE_{Back}^* = -\left(1 - \varphi \beta \alpha^2 \tau^4\right) \]
Available potential energy

\[ PE_{\text{Avail}}^* = PE_{\text{Tot}}^* - PE_{\text{Back}}^* = 2 - (4 + \varphi \beta) \alpha^2 \tau^4 \]
Kinetic energy

\[ KE^* = 16\sigma^3 \tau^4 \]

Available energy changing

\[
\frac{dE^*_{\text{Avail}}}{d\tau} = \frac{dKE^*}{d\tau} + \frac{dPE^*_{\text{Avail}}}{d\tau} = -4(4 + \varphi \beta - 16\sigma \alpha)\alpha^2 \tau^3
\]

Hence, energy is lost whenever \( \alpha < \frac{1}{4} \) (for \( \beta = 0, \sigma = 1 \)).
Instantaneous mixing efficiency

\[ \eta_{Inst} = -\frac{dPE_{Back}^*}{d\tau} - \frac{dPE_{Avail}^*}{d\tau} + \frac{dKE^*}{d\tau} \]

\[ = \frac{\varphi \beta}{4 + \varphi \beta - 16\sigma \alpha} \]

So for \( \varphi = 16, \beta = 0.18, \sigma = 1, \) and \( \alpha = 0.06, \) then \( \eta_{Inst} = 0.49. \)
**Integral mixing efficiency**

If there no mixing after reaching the bottom...

\[ \eta_{\text{Integral}} = \frac{PE_{\text{Back}}^{(\text{bot})} - PE_{\text{Back}}^{(0)}}{PE_{\text{Avail}}^{(0)}} \]

For \( \varphi = 16 \) and \( \beta = 0.18 \), then \( \eta_{\text{Integral}} = 0.36 \).
If there is mixing after reaching the bottom...

\[ E^{*\text{(bot)}}_{\text{Avail}} = \left( 1 + 4\sigma \alpha - \frac{1}{4} \phi \beta \right) \]

If \( \Delta E^{(\text{After bot)}}_{\text{Back}} = \eta_{\text{stab}} E^{(\text{bot)}}_{\text{Avail}}, \) then

\[ \eta_{\text{Integral}} = \frac{1}{8} \phi \beta + \frac{1}{2} \eta_{\text{stab}} \left( 1 + 4\sigma \alpha - \frac{1}{4} \phi \beta \right) \]

For \( \eta_{\text{stab}} = 0.2, \) then \( \eta_{\text{Integral}} = 0.41. \)

Joanne Holford’s experiments
Let $\Delta_c$ be the fractional displacement of the centroid of the bubble from $z = 0$.

\[ \eta_{Inst} = -\frac{\delta P E_{Back}^*}{\delta P E_{Avail}^* + \delta K E^*} = \frac{\phi \beta}{4 + \phi \beta - 4(4 - \phi \beta) \Delta_c - 16 \sigma \alpha} \]

Pyramid ($\Delta_c = 1/4$): $\eta_{Inst} = 0.6$.

Parabolic ($\Delta_c = 1/6$): $\eta_{Inst} = 0.56$.

(gives linear mean concentration)
How can we avoid having to specify $C_D$?

**Shell model**

GOY model (Gledzer–Ohkitani–Yamada):

$$
\frac{dU_n}{dt} = \left( a k_n U_{n+1}^* U_{n+2}^* + b k_{n-1} U_{n-1}^* U_{n+1}^* + c k_{n-2} U_{n-1}^* U_{n-2}^* \right) - v k_n^2 U_n + F_n
$$

with $k_n = \beta^n k_0$, $a = 1$, $b = -\varepsilon$ and $c = -1 + \varepsilon$.

In Rayleigh-Taylor instability, energy input at all scales.

$$
\frac{dU_n}{dt} = \left( k_n U_{n+1} U_{n+2} - \varepsilon k_{n-1} U_{n-1} U_{n+1} - (1 - \varepsilon) k_{n-2} U_{n-1} U_{n-2} \right) - v k_n^2 U_n + F_n
$$

Recall Layzer model: $(2 + E) \frac{dw}{dt} = A g (1 - E) - C_D \frac{w^2}{\lambda}$

Hence $F_n = A_n g \frac{1 - E_n}{2 + E_n}$, where

$$
E_n = \exp \left( - \frac{6\pi h_n}{\lambda_n} \right) \quad \text{and} \quad A_n = A \frac{h_n}{h}.
$$

The mode penetrations $h_n$ and total penetration $h$ are obtained from

$$
\frac{d h_n}{dt} = U_n \quad \text{and} \quad h = \max_n h_n.
$$
◊ Approximate quadratic growth
◊ Coefficient depends on initial spectrum
◊ Possible to replicate $\alpha \sim 0.06$
Conclusions

**General**

- Initial conditions are important for gross features
- Internal details relatively insensitive to initial conditions
- Appropriate modelling of initial conditions gives close agreement

**Thermals model**

- Single-mode growth rate consistent with isolated thermal
- Simple model for transfer between modes replicates $t^2$ growth
- Mixing efficiency consistent with thermal entrainment

**Shell model**

- Baroclinic input at all scales
- Very simple model replicates $t^2$ growth
- Growth rate sensitive to initial spectrum

*An explanation?*

No, but it helps.