Combined Shear
and
Buoyancy Instabilities

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Spectral Turbulence Model for Variable Density Turbulence

- Steinkamp, Clark and Harlow, Int. J. of Multiphase Flow (1999)

\[ \hat{R}_{ij}(\bar{x}, \bar{k}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ij}(\bar{x}_1, \bar{x}_2, t) e^{-i\bar{k}\bar{r}} \, d\bar{r} \]

where \( \bar{x} = \frac{\bar{x}_1 + \bar{x}_2}{2} \), \( \bar{r} = \bar{x}_2 - \bar{x}_1 \)

- \( R_{ij}(\bar{x}, k, t) = \int_{\Omega_k} \hat{R}_{ij}(\bar{x}, \bar{k}, t) \frac{k^2 d\Omega_k}{(2\pi)^3} \)

- \( R_{ij}(\bar{x}, t) = \int_0^{\infty} R_{ij}(\bar{x}, k, t) dk \)
Advantages of Spectral Transport Models Over Single-Point Models (i.e., k-ε):

- More generality, such as in the case of non-equilibrium transients
- Does not require a length scale or dissipation equation
- Greater flexibility with modeling, such as with non-local interactions in both physical and wavenumber space

Disadvantages:

- Greater Complexity
- More computationally expensive
- More Flexibility!
Governing Equations

\[
\frac{D\tilde{u}_i}{Dt} = -\frac{1}{\bar{\rho}} \left[ \frac{\partial R_{in}}{\partial x_n} + \frac{\partial \bar{p}}{\partial x_i} \right] + \nu_m \frac{\partial^2 \tilde{u}_i}{\partial x_n^2} + g_i
\]

\[
\tilde{u}_i = \frac{\rho u_i}{\bar{\rho}}, \quad R_{in} = \rho u_i u''_n \quad \text{where} \quad u_i = \tilde{u}_i + u''_i
\]

\[
\frac{\partial \bar{p}}{\partial t} + \frac{\partial \rho \tilde{u}_n}{\partial x_n} = 0
\]

\[
\frac{DR_{ij}(k)}{Dt} = \iiint_{-\infty}^{\infty} \left[ a_i(k) \frac{\partial \bar{p}}{\partial x_j} + a_j(k) \frac{\partial \bar{p}}{\partial x_i} \right] \hat{f}(\bar{x}, \bar{x}')d\bar{x}' - R_{in} \frac{\partial \tilde{u}_j}{\partial x_n} - R_{jn} \frac{\partial \tilde{u}_i}{\partial x_n}
\]

\[
+ c_d \frac{\partial}{\partial x_n} \left[ \nu_t \frac{\partial R_{ij}}{\partial x_n} \right] + c_m \left( \frac{1}{3} \delta_{ij} R_{nn} - R_{ij} \right) \int_0^\infty \sqrt{\frac{kR_{nn}}{\bar{\rho}}} dk
\]

\[
+ \frac{\partial}{\partial k} \left\{ k^2 \sqrt{\frac{kR_{nn}}{\bar{\rho}}} \left[ -c_1 R_{ij} + c_2 k \frac{\partial R_{ij}}{\partial k} \right] \right\}
\]

+ Non - Local(T(k))

where

\[
a_i(\bar{x}_1, \bar{x}_2, t) = -u_i'(\bar{x}_1) \rho(\bar{x}_1) v(\bar{x}_2) \quad \text{and} \quad \nu_t = \int_0^\infty \sqrt{\frac{kR_{nn}}{\bar{\rho}}} \frac{dk}{k^2}
\]
Governing Equations (cont.)

$$\frac{Da_i(k)}{Dt} = \frac{b(k)}{\rho} \frac{\partial \rho}{\partial x_i} - \frac{R_{in}}{\rho^2} \frac{\partial \rho}{\partial x_n} + c_d \frac{\partial}{\partial x_n} \left[ \nu_t \frac{\partial a_i}{\partial x_n} \right]$$

$$- \left[ c_{r1} k^2 \sqrt{a_n a_n} + c_{r2} k \sqrt{\frac{kR_{nn}}{\rho}} \right] a_i$$

$$+ \frac{\partial}{\partial k} \left\{ k^2 \sqrt{\frac{kR_{nn}}{\rho}} \left[ -c_1 a_i + c_2 k \frac{\partial a_i}{\partial k} \right] \right\}$$

where $b(\bar{x}_1, \bar{x}_2, t) = -\rho'(\bar{x}_1)v'(\bar{x}_2)$

$$\frac{Db(k)}{Dt} = u_n \frac{\partial b}{\partial x_n} + \frac{2\rho - \rho_1 - \rho_2}{\rho_1 \rho_2} \frac{\partial \rho a_n}{\partial x_n} + c_d \frac{\partial}{\partial x_n} \left[ \nu_t \frac{\partial b}{\partial x_n} \right]$$

$$- c_{fb} \left[ \sqrt{\frac{\rho}{\nu}} \frac{\partial \rho}{\partial x_n} \right] \frac{\partial k a_n}{\partial k} + \frac{\partial}{\partial k} \left\{ k^2 \sqrt{\frac{kR_{nn}}{\rho}} \left[ -c_1 b + c_2 k \frac{\partial b}{\partial k} \right] \right\}$$

$$- 2\nu_t k^2 b$$
Non-Local Behavior in Physical and Wavenumber Space

Physical Space:

\[
\frac{\text{D}R_{nn}}{\text{D}t} = 2 \int_{-\infty}^{\infty} a_{\varphi}(y') \frac{\partial \hat{f}}{\partial y'} f(y, y') dy' + \ldots
\]

\[
\int_{-\infty}^{\infty} \hat{f}(y, y') dy = 1
\]

\[
\hat{f}(y, y') = \frac{\exp(-2k|y - y'|)}{\int_{-\infty}^{\infty} \exp(-2k|y - y''|) dy''}
\]

Wavenumber Space: Kraichnan and Spiegel Model for Energy Transfer (1962)

\[
T(k) = \int [S_a(k/p) - S_e(k/p)] dp
\]

\[
S_e(p/k) = \eta \sqrt{pE(p)}E(p)(\frac{p}{k})^m g(\frac{p}{k})
\]

\[
S_a(k/p) = S_e(p/k)
\]

\[
T(k) = -\Sigma(k) + \int \Sigma(p) \tilde{f}(k, p) dp \text{ where } \Sigma(k) = \eta [kE(k)]^3
\]

and \[
\tilde{f}(k, p) = p^{-1}(\frac{k}{p})^m g(\frac{k}{p})
\]
Modeling Constants

$c_1, c_2, c_m, c_{r1}, c_{r2}, c_d$ also $\eta$ and $g(x)$

$c_1/c_2 = 2$ for equipartition of energy ($c_1 = .5$)

$c_{r1} = c_{r2} = 5.0$ to match buoyancy experiments

$c_m = 1.0$ original SCH model constant

Given that $\int f(k, p)dk = 1 \Rightarrow \int x^m g(x) dx = 1$ where $x = k/p$

Let $g(x) = \exp\left[-\alpha(x + x^{-1})\right]/N$

and for 2k coupling ($m = -2, \alpha = 2/3$)

Lastly,

$$\varepsilon = \int \int S(q/p) dp dq$$

$$\eta = C_k^{-3/2} \left[ \int_1^\infty \ln(x)(x^m - x^{-m-2}) g(x) dx \right]^{-1} \Rightarrow \eta = 1.36$$
\[
\frac{\left(u-U_{\text{slow}}\right)}{U_{\text{fast}}-U_{\text{slow}}}
\]

\[
\begin{align*}
\triangle & \quad 5 \text{ cm} \\
\square & \quad 10 \text{ cm} \\
\square & \quad 30 \text{ cm} \\
x & \quad 45 \text{ cm} \\
x & \quad 50 \text{ cm} \\
\bullet & \quad 65 \text{ cm} \\
+ & \quad 70 \text{ cm}
\end{align*}
\]

\[
dW/dx = 2C(U_f-U_s)/(U_f+U_s) \\
dW/dt = C\Delta U
\]

Brown & Roshko: \( C = 0.181 \) \\
Patel: \( C = 0.18 \) \\
Bell & Mehta: \( C = 0.152 \) \\
Present Exp: \( C = 0.158 \) \\
Present Calc.: \( C = 0.160 \)
Buoyancy Driven Mixing Layer: Atwood # = \(2.5 \times 10^{-4}\)
b profiles in a buoyancy-driven mixing layer

X = 30 cm
X = 50 cm