Rayleigh–Taylor instability: experiments with image analysis

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ABSTRACT

Rayleigh–Taylor instability is investigated in the laboratory using a simple apparatus of novel design to set up the unstable initial conditions. Visualisation techniques give a qualitative view of the development of the instability. Quantitative measurements are obtained through digital image analysis. Of primary interest is the evolution of the velocity field. Measurements are made over a two dimensional slice of the flow using an efficient, high-resolution particle tracking technique. This technique is described and its strengths and limitations are discussed in comparison with traditional measurement techniques. The visualisations and velocity measurements are compared with the experimental and numerical results of previous workers.

1. INTRODUCTION

The Rayleigh–Taylor instability is relevant in fluid flows ranging from inertially confined fusion to adding milk to a cup of coffee. In the geophysical context there are a wide variety of mechanisms which may lead to the formation of an unstable density stratification, leading in turn to the development of the Rayleigh–Taylor instability, redistribution of the density, mixing, and ultimately stable stratification. An overview of the subject has been given by Sharp (1984).

Analytical and numerical models of Rayleigh–Taylor (RT) instability generally utilise initial conditions which are in some sense ideal. Typically the fluid is at rest for times \( t < 0 \). At \( t = 0 \) the instability is switched on by imposing a destabilising acceleration such that the pressure and density gradients are in opposite directions. In the laboratory similar conditions may be obtained by accelerating an initially stable stratification downward at a rate exceeding the gravitational acceleration \( g \). However, difficulties in instrumenting such experiments, the high cost of the experiments and their relatively short duration prevent such techniques being used in most
laboratories. Although in the geophysical context the initial conditions are seldom close to the ideal, there remains a strong argument for attempting to model and understand the ideal situation in the laboratory as this more readily lends insight to the basic physics of the flow.

A variety of other methods have been employed to set up the unstable stratification. A common method (e.g. Voropayev et al., 1993) involves setting up a stable stratification (either layered or continuous) within a tank, then rapidly inverting the tank. This works reasonably well for very viscous fluids, but with inviscid fluids the body of the fluid remains irrotational during the inversion process. In a cylindrical tank rotated about its axis, the fluid would remain at rest with a stable stratification, whereas in a thin rectangular tank the density contours would finish at approximately 45° from the horizontal before the growth of the instability.

Other methods rely on some form of barrier separating two homogeneous layers of different densities. The instability is then initiated by removing the barrier by rupturing a membrane, advecting the flow past it (splitter plate in stratified flume, e.g. Lawrence, personal communication, 1992), or simply pulling the barrier out (e.g. Linden and Redondo, 1991). The difficulty with these techniques arises primarily because it is not possible to remove the barrier without affecting the flow in some way: pieces of the ruptured membrane may be advected around by the flow, and the wake generated by the splitter plate or removal of the barrier may dominate the initial growth of the instability.

For the present investigation a barrier is used to separate two homogeneous layers of (salt) water of equal depth but different density. The barrier is of a novel design originally suggested by Lane-Serff (1989, p. A7), which greatly reduces the wake and its effect on the development of the instability. Details of the barrier and experimental set-up may be found in Section 2. In Section 3 we present a number of visualisations of the developing instability, comparing them with analogous visualisations from a numerical code by Youngs (1991). The velocity measurements reported in this paper have been obtained by a particle tracking system developed by the author. The basic principles employed by this system are outlined in Section 4. A more comprehensive description has been given by Dalziel (1993). Although the system is capable of simultaneous density measurements (employing fluorescent dye as an indicator of density), this feature has not yet been used for the investigation of RT instability.

2. EXPERIMENTAL SETUP

The experiments described in this paper were undertaken in a Perspex tank of 200 mm width, 400 mm length and 500 mm depth. This tank is
illustrated in Fig. 1. Initially, the tank is divided into two layers of equal depth by a horizontal barrier 250 mm above the bottom. The top of the tank is left open to the air. The barrier is constructed from two sheets of stainless steel, approximately 0.6 mm in thickness, separated from each other by 0.6 mm, using a thin strip of the same material down each side of the barrier. The stainless steel thus takes the form of a very thin tube of rectangular cross-section (shown in dark grey in Fig. 1). Two pieces of nylon fabric (light grey in Fig. 1) are fed down the centre of the stainless steel tube, one piece then being passed back along the top of the tube, inside the tank, and fastened to the tank wall just above the barrier where the barrier enters the tank through one end. The other piece of fabric passes below the stainless steel tube and is also attached to the wall of the tank.

The purpose of the nylon fabric is to eliminate the shear between the barrier and the fluid adjacent to it, thus significantly reducing the wake of the plate as it is withdrawn. During the withdrawal of the stainless steel tube the nylon fabric is pulled through the tube at twice the speed of the tube, relative to the tank. The nylon fabric immediately above and below the tube at a given point within the tank remains at rest until the end of the tube reaches that point. Once the end of the tube passes that point, the fabric is removed by being pulled out through the interior of the tube.

Except as the end of the tube passes the point, there is no shear between the fabric and the fluid; all the shear is between the fabric and the stainless steel tube. Some vorticity is introduced by the fabric passing around the end of the tube, but this is relatively small compared with that generated by
the shear layers which would be adjacent to a simple solid plate. This
vorticity has an effect of the same order as the vertical velocity imparted to
the upper layer as it is pulled down to replace the volume of the retreating
barrier. The overall thickness of the barrier is kept as small as possible
(approximately 2.4 mm) to minimise the removed volume and hence the
vertical motion of the upper layer (the upper layer must move downwards
to fill the gap left by the barrier). The small gap between the stainless steel
sheets is approximately equal to the thickness of the two pieces of nylon
fabric passing through it, thus preventing a significant volume of water
escaping from the tank through the centre of the barrier. Unfortunately,
this tight fit combined with the sharp radius the fabric must pass around at
the end of the tube leads to a number of experimental difficulties. The
most serious problem is the barrier withdrawing unevenly or jamming as a
result of a mixture of silicone grease (used to seal the barrier to the tank)
and the particles (used for the image processing) clogging the gap. With the
present manual withdrawal, only approximately 10% of runs may be
considered successful. A further problem is that this composite barrier
produces a two-dimensional jet, directed away from the barrier, behind it
as it is removed. Although detailed discussion of this jet is beyond the
scope of this paper, we note that the perturbation it imposes on the flow is
much smaller than that caused by the boundary layers on a simple solid
plate.

Before the start of an experiment, the lower layer is filled with a solution
of water and isopropyl alcohol to give a density \( \rho_2 \). After the barrier is
positioned the upper layer is filled with salt water, giving a density \( \rho_1 \)
greater than that of the lower layer. The alcohol is added to the lower layer
to match the refractive indices of the two layers. Although the alcohol also
contributes to the stratification, the major stratifying element is the salt.
When tracking particles, additional salt is added to both layers to keep the
particles in suspension for longer periods (the density of the particles is
approximately 1.03 times that of pure water).

The key parameter for the instability is the Atwood number:

\[
A = \frac{(\rho_1 - \rho_2)}{(\rho_1 + \rho_2)}
\]

This characterises both the reduced gravity nature of the present salt
stratified experiments and the importance of non-Boussinesq effects. The
experiments reported in this paper are performed with \( A \) typically \( 2.1 \times 10^{-3} \), indicating Boussinesq behaviour. Although larger Atwood numbers
may be obtained with salt–water systems, at present the sampling rate of
the measurement techniques places a limit on \( A \) of approximately \( 10^{-2} \). In
the next section we present a number of flow visualisations which give a
qualitative picture of the initial evolution of the flow.
3. VISUALISATIONS

Figure 2 shows a sequence of perspective views of the early stages of the development of the instability. The aim of this visualisation is to give a three-dimensional impression of the position of the interface between the upper and lower layers. The (video) camera is located along the axis of the barrier, approximately 30° above the horizontal. The viewed region represents the central 75% of the tank width at the opposite end of the tank to that through which the barrier is withdrawn. Illumination is provided by a slide projector separated from the camera by 60°, also 30° above the horizontal. The lower layer contains a cocktail of food colouring, sodium fluorescein and artificial pearlescence. The food colouring is present to make the lower layer opaque, and the fluorescein makes the surface of the lower layer bright. The pearlescence (TiO$_2$-coated mica flakes approximately 30 μm in diameter and 5 μm in thickness) aligns itself with the stream surfaces of the flow. The interface between the layers is such a stream surface, with the pearlescence producing a shiny image when the normal to the interface bisects the angle between the camera and light source. The overall effect is to make the interface appear as a solid, shiny surface.

The experiment in Fig. 2 represents the best images from a series of approximately 60 runs. In the successful runs, the video camera used was not always in the ideal position for recording the flow. Unfortunately, in the run illustrated the upper layer was contaminated by some residual pearlescence from the previous run, which reduced the contrast and overall quality of the image. The Atwood number for this run was not measured, but is estimated at approximately $10^{-3}$.

In Fig. 2(a) the end of the plate can be seen near the bottom of the picture. Immediately behind the plate a superposition of wave modes can be distinguished. These waves predominantly have their crests aligned with the end of the plate. The shortest waves (which were clearer on some of the other realisations) have a wavelength of approximately 2 mm. This is comparable with the most unstable linear mode for a viscous fluid (Chandrasekhar, 1961, p. 447). It should be noted that the barrier introduces perturbations at a range of scales so that this mode has probably not developed from infinitesimal disturbances, and may have significant nonlinearity already. The largest wavelength is approximately 20 mm. Recent measurements suggest that this scale is set by an instability forming on the two-dimensional jet produced by the barrier as it is withdrawn. Detailed discussion of the structure of the jet and associated instability are beyond the scope of this paper.

Even at this early stage there is some three dimensionality in the
Fig. 2. Perspective view of initial stages of instability. The central 150 mm of the tank is visible near the end opposite the wall through which the barrier is withdrawn. (a) $t = 0.4$ s after starting to withdraw the barrier. The trailing edge of the barrier is visible near the bottom of the picture. (b) $t = 1.2$ s. (c) $t = 2$ s.
instability, although this is not clear in these figures. Figure 2(b) is the instability at $t = 1.2$ s. The interface is now fully three-dimensional. Its very smooth appearance is due partly to the visualisation method not showing clearly very thin regions of fluid, and partly to the smaller scales being obscured by the relatively smooth faces of the rising bubbles (the accelerating flow on these rising faces will tend to smooth the surface). At $t = 2$ s Fig. 2(c) shows the continued growth of the length scales with a few isolated bubbles of lower layer fluid dominating the flow. The formation of secondary instabilities on the sides of these may be seen.

Youngs (1991) presented results from an inviscid numerical model of the instability in the Boussinesq limit. A comparison between Fig. 2 and his plots shows very good qualitative agreement, suggesting that the initial perturbation applied to the flow by the withdrawal of the plate has comparatively little effect later in the development of the instability.

Figure 3 presents images obtained from two experiments using light-induced fluorescence (LIF). Sodium fluorescein dye was added to the upper layer and the tank was illuminated by a sheet of light approximately 2 mm thick. The light source was a slide projector from which the light was focused into a sheet using a curved mirror. The Atwood number for both
experiments was approximately $2.1 \times 10^{-3}$. The initial conditions and position of the light sheet were nominally identical. The viewed region is approximately 150 mm in width extending from the end of the tank opposite the side through which the barrier is withdrawn. The experiment shown in Fig. 3(a) shows the formation of mushroom vortices penetrating up and down from the initial position of the interface. The size of these structures is greater at the right-hand side of the image, as this was the point where the two layers were first in contact. The remarkable regularity in the structures suggests that the instability for this experiment was essentially two-dimensional at this stage and is dominated by the approximately 20 mm wave length initial perturbation. Mixing between the two layers is confined to the intense vortex roll-up behind the heads of the mushrooms.

A more typical experiment showing evidence of the more complex structure of the three-dimensional instability is shown in Fig. 3(b). In this image we see a hierarchy of scales. The dominant length scale is similar to that of Fig. 3(a), but superimposed on this are smaller length scales. Earlier pictures in the sequence for this experiment suggest the combining of even...
smaller-scale structures to produce the small scales visible at this stage, whereas the dominant scale grew from the initial perturbation to the flow. At later times both experiments show the development of secondary instabilities on the sides of the penetrating vortices as the flow accelerates and becomes more fully three-dimensional.

Dimensional analysis of the inviscid problem may be used to estimate the growth of the mixing region. The only length scale in the problem is the depth of the tank \( H = 500 \) mm. Combined with the apparent gravitational acceleration \( 2Ag \) (= \( g' \), the reduced gravity) the time-scale is \( (Ag/H)^{-1/2} \approx 4.9 \) s and velocity scale \( (AgH)^{1/2} \approx 101 \) mm s\(^{-1}\). Assuming that the fluid near the interface accelerates at some fraction of the apparent gravitational acceleration, the velocity \( U \) and half thickness \( d \) of the mixed region are then estimated by

\[
U = 2cAg(t - t_0) \\
d = cAg(t - t_0)^2 + d_0
\]

where \( c \) is the constant of proportionality. The constant \( t_0 \) acts as a virtual origin for time. For an ideal experiment with no initial perturbation \( t_0 = 0 \). However, for experiments with a barrier being withdrawn \( t_0 < 0 \). For our present experiments the initial perturbation is principally in the fluid velocity, as a result of the combination of the vertical drop of the upper layer and the two-dimensional jet produced by the barrier. The constant \( d_0 = -cAg \tau_0 \) is hence required so that \( d = 0 \) at \( t = 0 \). Rescaling these expressions yields the dimensionless velocity \( \theta = U/(AgH)^{1/2} \) and thickness \( \delta = d/H \) scales in terms of the scale time \( \tau = (Ag/H)^{1/2}t \), i.e.

\[
\theta = 2c(\tau - \tau_0) \\
\delta = c\tau(\tau - \tau_0)
\]

Figure 4 plots a comparison of the growth of the mixing region for the experiment shown in Fig. 3(b) and that based on dimensional arguments. A very good agreement can be seen, with \( c \approx 0.07 \) when \( t_0 = -6 \) s (\( \tau_0 = -1.2 \)). This value of \( c \) is comparable with that found in the high Atwood number acceleration experiments of Read and Youngs (1984) and Smeeton and Youngs (1988), although somewhat higher than the \( c \approx 0.05 \) in the numerical work of Youngs (1991).

Linden and Redondo (1991) utilised the same tank as in the present experiment but with a simple solid plate instead of the more complex stainless steel–nylon fabric barrier. They found \( c \) to range from 0.04 to 0.16 depending on the Atwood number. For Atwood numbers smaller than approximately 0.03 (the density difference 10 times greater than in the
4. PARTICLE TRACKING

In this section we outline briefly the particle tracking system used to obtain velocity measurements in the developing instability. Details of an earlier version of the particle tracking system used to obtain the velocities from the videotape of the experiments have been given by Dalziel (1993). A number of refinements have been made primarily connected with increasing the number of particles which may be tracked from 511 to 4095 and improving the consistency of the results. Here we present a brief comparison with other velocity measurement techniques, outline the experimental set up and summarise the basic principles employed in tracking the present experiments) they found that the wake produced by the simple barrier dominated the flow and produced very scattered values of $c$. For higher Atwood numbers, $c = 0.007$ was found. Clearly, the novel barrier used in the present experiments is a significant improvement for very small Atwood numbers.
particles. Results and discussion of these measurements are presented in the next section.

4.1. Comparison with other techniques

Traditional measurement techniques such as hot wire/film and laser Doppler velocimetry (LDV) offer excellent velocity and temporal resolution. However, these techniques provide only point measurements, which limits their usefulness when considering the spatial and temporal structure of an evolving flow. Hot wire/film could not be used for the present RT experiments because of the need for a probe in the flow (which disturbs the flow) and more particularly the requirement for a mean velocity past the probe to prevent significant convection caused by the heating effect of the probe. LDV could be used to obtain measurements at one or more points (by scanning the system or having multiple systems) but with current technology it cannot achieve a spatial resolution comparable with particle image velocimetry (PIV) techniques. Although of little use for the present experiments, Lagrangian measurements are not possible with LDV or hot wire/film velocimetry.

PIV techniques fall into two broad categories: Eulerian techniques based on pattern matching (often by optimising a correlation function), and Lagrangian techniques which follow individual particles or clusters of particles. The fundamental feature of both categories of PIV is the seeding of the flow with small particles. In general, it is assumed that the concentration of the particles is sufficiently small not to affect the scales of the flow that is to be measured, and that the particles accurately follow fluid parcels. We shall not analyse these assumptions here, but simply note that for the present experiments the particles do not adversely affect the flow, but nor do they strictly follow the flow. The particles are sufficiently good markers of fluid particles for reasonable velocity measurements to be obtained, but they are not suitable for measurements of concentration.

The two strengths of particle tracking compared with pattern matching PIV techniques (such as autocorrelation or cross-correlation methods) are the direct access to a Lagrangian description of the flow and the much higher processing rates without requiring very expensive computer hardware. Correlation techniques rely on both spatial and temporal correlation of the particle images between one time step and the next. In contrast, the present particle tracking system requires only a degree of temporal correlation for individual particles. The particles may be tracked from the two-dimensional projection of a region of the flow, which is much thicker than the integral length scale of the flow, allowing long Lagrangian paths to be measured. For the present RT experiments, Lagrangian statistics are of
little use as a result of difficulties in interpreting them; however, they have been employed extensively in other flows for which this system has been used (e.g. Drayton, 1993).

To obtain adequate spatial and velocity resolution, pattern matching PIV techniques require the flow to be recorded on conventional photographic film and the image subsequently to be divided into a large number of separate (possibly overlapping) interrogation cells. Each interrogation region is then processed in turn, normally requiring the film to be traversed mechanically from one region to the next before digitisation. Although the mechanical nature of this process allows the full resolution of the photographic film to be employed by digitising one cell at a time, it is also very time consuming. Each of the interrogation regions is normally analysed using a combination of optical and digital processing (although completely digital processing is also possible). In the absence of a powerful array processor this analysis is also substantially slower than that for particle tracking. The low processing speed means that pattern matching PIV techniques are not suitable, using present hardware, for analysis of evolving flows where ensemble descriptions are required.

At present, pattern matching PIV techniques offer potentially higher spatial resolution than can be achieved with particle tracking. However, with the introduction of high-definition videotape, suitable frame grabbers and software, particle tracking will be able to offer spatial resolution comparable with that of pattern matching PIV techniques. As the processing time for particle tracking is found to increase approximately linearly with the number of particles, it is feasible to track very large numbers of particles with only modest computer hardware.

The primary limitation with particle tracking is the frame rate of the media used to record the flow (using autocorrelation PIV techniques this limitation is overcome by multiple exposures of a single frame). This limitation affects both the frequencies which may be resolved and the maximum velocities measurable (the particles must not move too far between each frame compared with the mean spacing between the particles and must reside in the light sheet for a sufficient number of frames). The 25 Hz frame rate of PAL video systems proves adequate for most laboratory scale experiments relating to geophysical fluid dynamics. When higher frame rates are required, high-speed videotape or cine film may be employed.

4.2. Experimental considerations

For the experiments presented in this paper, particles are added to both layers before the withdrawal of the barrier. The particles used were Pliolite VT (a white resin used in the manufacture of paint) ground and sieved to
obtain the desired size range. Approximately 3 wt.% salt was added to the two layers (in addition to the salt added to the upper layer to provide stratification) to adjust the density of the water so that the particles were approximately neutrally buoyant. In practice, the density of the particles was intermediate between the density of the two layers.

Illumination generally took the form of a sheet of light approximately 5 mm in thickness. Broader light sheets may be used to obtain Lagrangian statistics (Drayton, 1993); such statistics are beyond the scope of this paper. The light sheet was located at the centre of the tank and oriented normal to the trailing edge of the plate. Illumination was provided by a slide projector reflected off a curved mirror. A 250 mW argon-ion laser has also been used to produce the light sheet, but offers no significant advantages over the slide projector.

A standard monochrome charge coupled device (CCD) video camera with a 1/125 s electronic shutter was used to record the flow on a PAL super VHS video-recorder. Because of the time lag between the two interlaced video fields produced by this camera (the electronic shutter opens once per field), only the odd video field were used during the subsequent particle tracking, effectively reducing the vertical resolution by a factor of two.

To locate the particles sufficiently accurately (a mass centroid technique is used to determine the positions of the particles from digitised images of the flow) it is necessary for the particles to extend over at least four video scan lines (i.e. two lines of each field) and two pixels horizontally in a digitised image. This places a lower limit on the size of the particles which could be used of between 100 μm and 300 μm, depending on the size of the viewed region and the available light (with more light it is possible to defocus the camera so that a small particle would produce a response in the camera over an area larger than it would if it were in focus).

Before each run a sheet of Perspex engraved with a grid of lines was placed within the light sheet. This grid was subsequently used to calibrate the imaging system to obtain two-dimensional (world) coordinates for individual particles. It should be noted that we are considering a two-dimensional projection of some region of the flow rather than a strictly two-dimensional slice. In addition, a series of reference points were positioned permanently down one side of the tank. These reference points were later used by the particle tracking system to correct for jitter introduced by the video and digitising hardware.

4.3. Matching particles

Particle tracking is essentially a Lagrangian technique. The basic principle is to try to follow each and every particle visible in the image at any one
time from one time step to the next. Between each time step some of the
departicles present will leave the viewed region (e.g. by passing out of the
light sheet) or be obscured by other particles (not necessarily visible
themselves). Other particles will enter the light sheet, or move out from
behind some obscuring particle.

Let us suppose at some time $t = t_P$ the particles visible in the light sheet
belong to the set $P = \{ p_i, i = 1, n_p \}$. At a later time $t = t_Q$ those particles
visible belong to $Q = \{ q_j, j = 1, n_Q \}$. In general, some of the particles
visible at $t_P$ will also be visible at $t_Q$ so that $P \cap Q \neq \{ \}$. If the time step
$\Delta t = t_Q - t_P$ is too large, then $P \cap Q$ may be very small compared with
either $P$ or $Q$, and the particles in $P \cap Q$ may have moved a large distance
compared with the spacing between the particles in $P$ or $Q$. Particle
tracking is unlikely to be successful in such a situation unless every particle
has some unique distinguishing feature; for the type of particles and
resolution we are using, the individual particles all appear much the same.

At the other limit if $\Delta t$ is relatively small, $P \cap Q$ will contain most of the
particles in $P$ and $Q$, and the distance travelled by any particle will not be
too large compared with the mean spacing of the particles. In this situation,
it would be possible to follow the particles by eye, and hence it should be
possible to follow them automatically with a computer system.

A number of automated and semi-automated particle tracking systems
have been developed in recent years (see Adrian, 1991), using a variety of
strategies to determine which particle image in $P$ is caused by the same
physical particle as a particular particle image in $Q$. These strategies vary
in the assumptions they make concerning the flow, their efficiency and
accuracy, and the velocity and spatial scales which may be measured. The
simplest strategy of taking the nearest neighbours (e.g. Frieden and Zoltani,
1989) in $P$ and $Q$ is satisfactory if the particles move only a small fraction
of the distance to their neighbours during $\Delta t$ and particles do not enter or
leave the viewing region. The method developed by Perkins and Hunt
(1989) relies on spatial correlation of groups of particles, the assumption
being that each group has a unique signature which changes only slowly
with time. Although this technique works well for the two-dimensional
flows for which it was developed, it does not work as effectively for flows
where particles enter and leave the light sheet, or when considering a
two-dimensional projection of a three-dimensional flow.

In this paper we use the method developed by Dalziel (1993), which
utilises some elements of each of these approaches. We define an associa-
tion vector $\alpha = \alpha_{ij}$ such that

$$
\alpha_{ij} = \begin{cases} 
1 & \text{if } p_i \text{ and } q_j \text{ are considered the same physical particle} \\
0 & \text{otherwise}
\end{cases}
$$

(4)
In addition to the associations \( \alpha_{ij} (i = 1, n_P, j = 1, n_Q) \) between \( P \) and \( Q \), we define \( \alpha_{i0} = 1 \) to indicate that the particle \( p_i \) is not in the set \( Q \) (\( \alpha_{i0} = 0 \) implies that \( p_i \) is in \( Q \)) and \( \alpha_{0j} = 1 \) to indicate that \( q_j \) is not in \( P \) (\( \alpha_{0j} = 0 \) implies \( q_j \) is in \( P \)). The constraints on the association vector arise from the fact that a given physical particle may only be in one place at one time, and so any particle image \( p_k \) may be associated with at most one \( q_j \). Thus for every \( i = k > 0 \), \( \alpha_{k,j} = 1 \) for one and only one \( j \geq 0 \). Similarly, for every \( j = l > 0 \), \( \alpha_{i,l} = 1 \) for one and only one \( i \geq 0 \). It should be noted that \( \alpha_{i0} \) and \( \alpha_{0i} \) may be unity for more than one value of \( i \) or \( j \). Further, \( \alpha_{00} \), the association between particles outside both sets \( P \) and \( Q \), is not defined (and is not important) for our present analysis.

The aim of the tracking method is to determine the actual association vector \( \alpha = \hat{\alpha} \) (say) for each time step. Except in trivial examples, it is not possible to determine \( \hat{\alpha} \) from \( P \) and \( Q \) without further knowledge of the flow or introducing some additional assumptions on how \( P \) and \( Q \) are related. The first assumption we introduce here is that \( \hat{\alpha} \) is in some sense the optimal association vector out of the set of all possible association vectors \( \alpha \). We also assume that the optimality is linear in the sense that \( \hat{\alpha} \) yields the smallest value of \( \zeta \), where \( \zeta \) is given by the linear combination

\[
\zeta = \sum_i \sum_j \alpha_{ij} C_{ij}
\]

and \( C = C_{ij} \) \( (i = 0, n_P, j = 0, n_Q) \) are functions of the particles in \( P \) and \( Q \) (but not of the association vector \( \alpha \)).

The functions \( C \) may be considered as the costs of making the associations \( \alpha \). Ideally, the costs \( C \) should be chosen so that the true association vector \( \hat{\alpha} \) minimises \( \zeta \) in such a way that minimum is clearly defined for all time steps and represents the minimisation of some physically relevant quantity. However, as \( \hat{\alpha} \) is not known in advance, we must choose \( C \) based on physical arguments alone, and then test that the association \( \alpha = \hat{\alpha} \) (say) which minimises \( \zeta \) is equal to \( \hat{\alpha} \) in the vast majority of pairs (preferably all pairs) of \( P \) and \( Q \).

The first physical assumption we make is that particle accelerations, projected onto the viewing plane, are relatively small. An alternative is to assume that the rate of change of this acceleration is small; as these two strategies are handled in the same way and produce much the same results we shall confine our attention to the former. In particular, let us suppose particle \( p_i \) is found at \( x_i \) with a known velocity \( u_i \) in the viewing plane at \( t = t_p \). We may then predict the position of the particle at \( t = t_Q \) as

\[
x_i^{(p)} = x_i + u_i \Delta t
\]

Our key assumption here is that \( |x_i^{(p)} - x_i^{(a)}| \) is much smaller than the
mean separation between particles in \( P \) or \( Q \) \( (x_i^{(o)} \) is the actual position of the particle at \( t = t_Q \)). More specifically, we require \( |x_i^{(p)} - x_i^{(o)}| \leq D \), where \( D \) is chosen on the basis of the expected accelerations and the mean particle spacing. In addition, we may wish to utilise some measure of the geometry and intensity of the particles (or the fluid surrounding them) in \( P \) and/or \( Q \) when evaluating \( C \). Experience has shown that reliable results are obtained with \( C_{ij} \) of the form

\[
C_{ij} = \begin{cases} 
N_i + B_{ij} \eta_i \epsilon_j \mathcal{O}_j \tau_j & \text{if } C_{\text{max}} \\
\infty & \text{otherwise}
\end{cases} 
\]

(7)

where the basic cost \( B_{ij} \) is given by

\[
B_{ij} = |x_i + u_i \Delta t - x_j|^m
\]

(8)

Typically, the exponent \( m = 2 \) is chosen so that the basic cost increases as the square of the error in the predicted position for the particle \( p_j \). For boundary associations \( a_{iq} \) or \( a_{oj} \), the basic cost is the distance between the predicted position of the particle and the nearest boundary.

The functions \( \epsilon_j, \mathcal{O}_j \) and \( \tau_j \) are used to modify the basic cost in light of the geometry of the particle image \( q_j \) and allow for the possibility of two particles appearing as a single image at \( t = t_Q \). As the tracking procedure is not very sensitive to these functions (details of which have been given by Dalziel (1992)), they may be taken as unity and ignored for the present discussion. An implicit assumption in forming \( B_{ij} \) is that we know the projected velocity \( u_i \) for particle \( p_i \). This assumption is justified if we tracked \( p_i \) from the previous time step, in which case \( u_i \) may be obtained from the change of the position of \( p_i \). However, if we have no previous history for \( p_i \) then we must estimate \( u_i \) in some manner. For thin light sheets or in two-dimensional flows we may use the velocities obtained at the previous time step for other particles in the neighbourhood of \( p_i \) to estimate the velocity \( u_i = u_i^{(e)} \) (say). If we have no information on which to base our estimate, we may simply guess \( u_i^{(e)} = 0 \). In general, an estimated velocity will give a much poorer prediction \( x_i^{(p)} \) than when \( u_i \) is known. Rather than imposing the very high basic cost \( B_{ij} \) such a particle is likely to suffer, we offer associations with particles \( p_i \) with no velocity history a discount \( 0 < \eta_i \leq 1 \), which is related to some measure of the error in the estimated velocity (see below). However, to prevent such new particles scavenging \( q_j \) from particles with a velocity history falling equally close to \( q_j \) (the particle discount would mean that an association with the new particle would be cheaper), the new particle is charged a supplementary joining fee \( 0 \leq N_i < C_{\text{max}} \) (typically \( N_i \approx \frac{1}{2} C_{\text{max}} \)). When \( u_i \) is known from the velocity history we set \( \eta_i = 1 \) and \( N_i = 0 \).
Costs $N_i + B_{ij} \eta_i \epsilon_j \sigma_j \tau_j$ which evaluate to more than $C_{\text{max}}$ are assumed to be unreasonable and the particle images involved have a very low probability of forming an association. Except for boundary associations ($\alpha_{i0}$ and $\alpha_{0j}$), the cost for such an association is set to infinity. The infinite cost effectively eliminates the corresponding association from the matching problem. The value of $C_{\text{max}}$ is equal to the basic cost for a pair of particles separated by distance $D$; thus $C_{\text{max}}$ is a measure of the maximum allowable error in the prediction of the particle’s position when the velocity history for the particle is known. For boundary associations with a cost exceeding $C_{\text{max}}$, $C_{ij}$ is set equal to $C_{\text{max}}$, ensuring that at least one finite cost always exists for each $p_i$ and $q_j$.

The new particle discount $\eta_i$ is evaluated from $C_{\text{max}}$ and the expected error in the estimated velocities. For such particles the cost range $N_i$ to $C_{\text{max}}$ represents errors in the estimated velocity $u_i^{(e)}$ of between zero and the maximum allowable error in the estimate. Associations with a larger cost are set to infinity (particle-particle associations) or $C_{\text{max}}$ (boundary associations).

As a result of the necessity of estimating the velocity for a new particle, there is a reasonably large probability of a mismatch between $t_P$ and $t_Q$ occurring for new particles. However, the probability that the velocity determined from such a mismatch will predict the position of a particle at the next time step sufficiently accurately for a matching to be made (i.e. within $D$ of the position of the particle and there being no more favourable particle to match with) is very small. It is estimated that with well set-up experiments and appropriate values set for $C_{\text{max}}$ ($D$ typically one pixel), $N_i$ and $\eta_i$, fewer than one 5000 paths extending two or more time steps contain an invalid matching of this type. Thus any mismatches caused by an inappropriate velocity estimate for a new particle may be eliminated effectively by requiring paths to extend over a number of time steps. (We shall not discuss the bias this introduces in the velocity samples, other than to note that the effect is very small for the present experiments).

The optimisation problem to minimise $\zeta$ has linear constraints and a linear objective function. As such it may be written as an integer linear program. Alternatively, and more efficiently, Dalziel (1993) showed that the structure of this particle matching problem is very similar to the structure of the Transportation Problem, originally considered by Hichcock (e.g. Carré, 1979, p. 221), and may be solved in a manner very similar to the graph theory approach applied to this earlier problem. We shall not discuss the solution process here, though we note that the implementation used in the present study is more efficient in terms of both memory and computation than that described by Dalziel (1993).

The technique we have described in this section has been implemented
on a Belton (Cheltenham, UK) PC/AT compatible computer utilising a Data Translation (Marlboro, NJ) DT-2862 frame grabber and a Panasonic (Osaka, Japan) AG-7300 series Super VHS video-recorder. Processing speeds in the region of 4–20 frames per min\(^{-1}\) are normally achieved. The time per image is a function of both the number of particles and the image quality. The effectiveness of this system has been proven over a range of laboratory-scale geophysical flows during the past 3 years.

5. VELOCITY MEASUREMENTS

In this section we present the velocity measurements obtained from the particle tracking system introduced in the previous section. Once the particles have been tracked, their velocities may be determined from the particle path data. Typically, \(x, t\) and \(z, t\) data from a number of time steps are fitted with a linear or quadratic function of time using a least-squares process. The number of time steps used depends on the Lagrangian time-scales of the flow and the expected residence time in the light sheet. Typically, data from four to six video frames are used to improve the accuracy of the velocity measurements, producing a standard error for velocities of approximately 0.5 mm s\(^{-1}\) (0.5\% of the maximum velocity).

Preliminary measurements employing a 1/100 s mechanical shutter on a 25 Hz duty cycle with a more sophisticated CCD video camera (with a separate 1/25 s electronic shutter for each of the two video fields) gives significantly improved velocity resolution for such experiments. This improvement is the result of eliminating the 1/50 s time lag between the two video fields, allowing the full 512 pixel vertical resolution to be employed and so improving the accuracy with which the particles may be located. The new camera is also somewhat more sensitive, allowing smaller particles to be used by defocusing the optics to smear the particle image over more than one pixel (the requirement for locating the particle position with subpixel accuracy). The overall improvement in the velocity measurements approaches a factor of four.

Figure 5 shows the evolution of the basic velocity statistics for an experiment with \(A = 2.1 \times 10^{-3}\). Initially, there were an approximately equal number of particles in the two fluid layers. The statistics were calculated by averaging the individual particle paths falling within the tracking window which represents the central 33\% of the tank depth. The sheet of light illuminating the flow was approximately 5 mm thick.

From Fig. 5(a) we may see that for this experiment the region we are considering contains predominantly buoyant fluid, exhibiting a mean upward velocity during the early stages of the experiment. The balancing
downflow would have occurred elsewhere in the tank. The region-averaged r.m.s. velocity is plotted in Fig. 5(b). The buoyancy driven vertical motion increases in magnitude much more rapidly than the horizontal component of the velocity which arises from a combination of continuity in the early stages and strong nonlinearities once the flow becomes established. The vertical velocity peaks at \( t = 12 \text{s} \) \((\tau - \tau_0 \approx 3.7)\), corresponding to the time at which the buoyancy-driven flow first reaches the top and bottom of the tank. A combination of continued nonlinearities and the conversion of vertical to horizontal kinetic energy at the boundaries allows the magnitude of the horizontal component of the velocity to continue growing, rapidly exceeding the now decreasing vertical component. The horizontal velocity reaches its peak approximately 4 s after the vertical component. The two velocity components then decay in a similar fashion with the formation of a breaking internal wave field.

Using (2), we may estimate the window-averaged r.m.s. velocity as

\[
\bar{w} = \left[ (2cA g)^3 t(t - 2t_0)(t - t_0)^2 \right]^{1/2} / h
\]

where \( h \) is the height of the tracking window. This expression is valid only while \( d \leq h \). This estimate is plotted in Fig. 5(b) as a dotted line and shows close agreement with the experimental measurements using the value \( c \approx 0.07 \) suggested in Section 3.

Figure 5(b) also plots the r.m.s. velocities from the numerical work of D.L. Youngs (personal communication, 1992) as dot–dash (horizontal) and dot–dot–dash (vertical) lines. These curves have been shifted to allow for the virtual origin \( t_0 = -6 \text{s} \). The time axis for the numerical results has been rescaled by \((c_e/c_n)^{1/2}\) and the velocity axis by \((c_e/c_n)^{-1/2}\), where \( c_e = 0.07 \) is the growth rate for the experiments and \( c = 0.05 \) is the growth rate for the numerical solutions. This rescaling means that the mixed region in the experimental and numerical flows takes the same rescaled time to penetrate the entire depth of the tank, thus allowing direct comparison of the evolution despite the different growth rates.

After allowing for the nonideal initial conditions and difference in growth rates we can see a reasonable qualitative agreement between experimental and numerical evolutions, although the experimental peaks occur somewhat earlier in the evolution. The differences may be explained in terms of the initial perturbation to the flow and the different measurement domains. The experiments tracked the flow in a window of height \( h \approx H/3 \) whereas the numerical curves are for \( h = H \). Thus once \( d > h \) the experimental curves no longer have a contribution from the flow in the neighbourhood of the front of the mixed region, whereas the numerical results include contributions from the entire flow. This may account for
Fig. 5. Evolution of basic velocity statistics averaged over the tracking region. Horizontal components shown as solid lines, and vertical components shown as dashed lines. (a) Mean velocity. (b) Root mean square horizontal (solid line) and vertical (dashed line) velocity. The estimate from dimensional analysis (dotted line), and the numerical simulations of D.L. Youngs (personal communication, 1992; dot–dash and dot–dot–dash lines) are also shown.
Fig. 6. Map of evolving velocity (lines) and vorticity (grey scale). Scale is shown at left-hand side of each figure. The maps represent the central 60% of the tank length and 33% of the depth. (a) $t = 2\ s$, $\tau - \tau_0 = 1.6$. (b) $t = 5\ s$, $\tau - \tau_0 = 2.2$. (c) $t = 10\ s$, $\tau - \tau_0 = 3.3$. (d) $t = 15\ s$, $\tau - \tau_0 = 4.3$. (e) $t = 20\ s$, $\tau - \tau_0 = 5.3$. (f) $t = 40\ s$, $\tau - \tau_0 = 9.4$. No velocity arrows or vorticity field are plotted in regions where there were insufficient particle paths for a reliable velocity fit to be made.
some of the difference in the heights of the peaks for the vertical component of the velocity.

The higher horizontal velocities in the experiments appear to be due to the development of strong motions on the scale of the horizontal dimensions of the tank (a feature not found in the numerical study). Preliminary
work based on an ensemble of experiments suggests this is repeatable feature introduced by asymmetry in the barrier removal.

The effect of the initial perturbation on the evolution of the flow is more difficult to determine. Preliminary experiments with a modified barrier producing a weaker perturbation suggest a very similar growth rate to that
in the experiment described here, but it is not known whether this is due to
the similar spatial structure of the perturbation or some other reason.

In Fig. 6 a regular grid has been fitted to the randomly distributed
particle paths. The velocity at the grid points is indicated by the length and
direction of the corresponding arrows issuing from the grid point, and the
grey scale indicates the vorticity for this velocity field. A velocity and
vorticity scale is included in the figure.

The mapping of the velocity field to the grid was achieved using a
weighted least-squares fit of a bilinear function at each grid point. The
contribution from the particles in the neighbourhood of the grid point is
determined by a Gaussian weighting based on the distance of the particle
from the grid point. Rignot and Spedding (1988) have considered a number
of methods for transferring random particle velocities onto a regular grid.
For the present flow their analysis suggested that fitting a constant term
with a Gaussian window would give errors of approximately 5% for velocity
and 10% for vorticity, where the vorticity is calculated by finite differencing
the velocity grid. The present method of fitting a bilinear function \( u = u_0 + u_1 \text{x} + u_2 \text{y} + u_3 \text{xy} \) should produce a smaller degree of smoothing of the
velocity field. Moreover, the velocity gradients may be obtained directly
from the fitted surface (e.g. \( \partial u / \partial x = u_1 \); \( \partial u / \partial y = u_2 \) ) further reducing the
associated errors. As yet, however, a full analysis of this method has not
been undertaken.

In the early stages of the development of the instability (Fig. 6(a); \( t = 2 \text{ s}, \tau - \tau_0 = 1.6 \) ) we see the formation of regular regions of positive and
negative vorticity adjacent to the unstable interface. As the instability
grows these regions interact and distort as they are advected by the
accelerating flow (Fig. 6(b); \( t = 5 \text{ s}, \tau - \tau_0 = 2.2 \) ). Regions of streaming flow
become established, rapidly rearranging the density structure (Fig. 6(c);
\( t = 10 \text{ s}, \tau - \tau_0 = 3.3 \) ). The mixing is largely confined to the swirling regions
being advected around by the flow, and the shear layers forming between
regions of upflow and downflow.

The structure of the flow changes substantially once the top and bottom
boundaries are felt. The large vertical component of the velocity is replaced
by strong horizontal motions (Fig. 6(d); \( t = 15 \text{ s}, \tau - \tau_0 = 4.3 \) ). A larger
proportion of the energy is contained in the vorticity field and overturning
of the internal wave field. Locally unstable stratification is rapidly mixed
(Fig. 5(e); \( t = 20 \text{ s}, \tau - \tau_0 = 5.3 \) ), leading to a stable density structure as the
motion decays (Fig. 5(f); \( t = 40 \text{ s}, \tau - \tau_0 = 9.4 \) ).

The velocity measurements presented here are obtained from tracking
approximately 1800 particles in a sheet of light of 5 mm thickness to obtain
an average of approximately 1400 valid paths at each time step. At this
particle density the spatial resolution is adequate to determine accurately
the Eulerian velocities on a regular grid in addition to the fundamentally Lagrangian description provided by the particle paths. Employing the improved camera–shutter system (mentioned at the start of this section) particle densities of approximately 2500 per frame are possible, offering improved spatial resolution for the Eulerian descriptions.

Other characteristics of the flow (e.g. two-point Eulerian velocity correlations, power spectra, Lagrangian velocity autocorrelations and velocity probability density functions) have been analysed, but are beyond the scope of this paper. A more detailed ensemble of experiments is currently being performed using improved video equipment which promises further improvements in the quality of the measurements.

6. CONCLUSIONS

In this paper we have shown that near ideal initial conditions for Rayleigh–Taylor instability may be produced using a simple apparatus of novel design. Despite an initial nonlinear perturbation to the flow by the removal of the barrier separating the two fluid layers, very good qualitative agreement and adequate quantitative agreement between the present experiments and previous experimental studies have been achieved. These experiments have been performed using smaller density ratios than has been possible previously. The ability to consider small Atwood numbers facilitates laboratory measurements of the flow by increasing the time-scale over which the instability develops. The lower Reynolds numbers associated with these slower flows should also allow for a more direct comparison with numerical simulations.

Youngs (1991) noted that numerical simulations of miscible fluids produced consistently smaller rates of growth \( c \) than acceleration experiments with immiscible fluids, suggesting the difference is due in part to the dissipation associated with mixing. The present miscible experiments suggest a very similar value of \( c \) compared with experiments performed in immiscible fluids. It is not clear whether the difference between the present experiments and the numerical simulations of Youngs (1991) is due to the (relatively small) initial perturbation, the relatively low Reynolds number leading to less mixing between the layers, the finite resolution of the numerical model, or simply random variation between the small number of successful experiments considered so far. An extensive ensemble of experiments is currently in progress to address the last possibility.

The suggestion that the weak initial perturbation strongly influences the development of the flow is difficult to test, as there are limits to how clean the initial conditions of the laboratory experiments may be made. Preliminary results with a modified barrier producing a weaker initial perturbation
(but with a similar spatial structure) suggest only weak dependence on the strength of the initial perturbation. The question then is whether the initial coherence introduced by the spatial structure of the weak perturbation persists as the instability grows. If it does, then the increased coherence of the regions of upflow and downflow may transport density more rapidly in the vertical direction than for a flow starting from rest. Thus we would expect to see a larger growth rate and decreased mixing efficiency.

Particle tracking has been shown to provide accurate velocity measurements with a relatively high spatial resolution. The efficiency of this method of particle image velocimetry allows thorough analysis of evolving flows. The particle tracking system is capable of simultaneous velocity and concentration measurements (using a fluorescent tracer). This feature will be employed in a further series of experiments to allow details of the mixing process to be explored. Image processing techniques such as particle tracking offer access to a wealth of information not previously available from experiments. What were previously considered simply flow visualisations may now be used as the basis of quantitative measurement techniques. Continued development in video and computer technology offers even greater possibilities in the future.

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