Technical Abstract

Stability of a Rotating Boundary Layer

Megan Davies Wykes, Jesus College, Group A, 2009/2010

This project investigated the unsteady flow around an impulsively-started rotating cylinder. An instability in this flow causes Taylor vortices to form from the initially laminar boundary layer. Previous studies of this phenomenon have assumed the flow to be quasi-steady in order to analyse this flow. It is unclear how valid this assumption is, as the length scales of the instability and the flow are similar.

Experiments were carried out on a cylinder with a diameter of either 32.09 or 9.95 mm, rotating in a tank measuring 0.449 m x 0.447 m x 0.596 m. In order to capture the behaviour of the instability, Reynolds numbers in the range $440 > Re > 30$ were investigated.

Comparisons were made between the size of the Taylor vortices found in these experiments and predictions from linear stability theory, which assumed that the flow was quasi-steady and that the boundary layers were thin; the results are shown in Figure 1. Quasi-steady theory correctly predicts the form of the relation between the size of the vortices and the Reynolds number, which follows equation (see 0.1), but it incorrectly predicts the precise size of the vortices.

$$\frac{d}{R} = B(Ta_{crit})^{1/3}(Re)^{-2/3} \quad (0.1)$$

When the time at which the instability occurred (shown in Figure 2) was compared to the quasi-steady predictions for $Re > 50$, the same result was obtained: the theory predicts the general form of the equation 0.2, but not the time at which the vortices become visible. When $Re < 50$ the thin boundary layer approximation is no longer valid and results deviate from this model.

$$\tau = \frac{\nu Ta_{crit}}{R^2} = A(Ta_{crit})^{1/2}(Re)^{-4/3} \quad (0.2)$$

The centrifugal instability in this flow is analogous to the thermal- or buoyancy-driven instability from a flat heated plate. The extent of this analogy was investigated and evidence of plume-like behaviour was found. However, further quantitative investigations are needed before a definitive statement can be made about the outward velocity of the plumes.
Figure 1: $d/R$ vs Reynolds number

Figure 2: $\tau$ vs Reynolds number
Stability of a Rotating Boundary Layer
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I hereby declare that, except where specifically indicated, the work
submitted herein is my own original work.

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Abstract

This project investigates the unsteady flow around an impulsively-started rotating cylinder. An instability in the flow causes Taylor vortices to form from the initially laminar boundary layer. Experiments are carried out to compare the size of these vortices with the predictions from quasi-steady theory. We find that quasi-steady theory correctly predicts the form of the relation between the size of the vortices and the Reynolds number, but it incorrectly predicts the precise size of the vortices. We then compare the time at which the instability occurs to the quasi-steady predictions, and for $Re > 50$ we obtain the same result: the theory predicts the general form of the equation, but not the time at which the vortices become visible. When $20 < Re < 50$ the thin boundary layer approximation is no longer valid and results deviate from this model. The centrifugal instability in this flow is analogous to the thermal- or buoyancy-driven instability from a flat heated plate. The extent of this analogy is investigated and evidence of plume-like behaviour is found. However, further quantitative investigations are needed before a definitive statement can be made about the outward velocity of the plumes.
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1 Introduction

Unsteady flows are abundant in nature, and their stability characteristics are therefore of interest to engineers. It is useful to have some idea of when flows transition to turbulence or between flow regimes. For example, it is important to know when a gust over an aeroplane wing will become turbulent. This project examines one such flow regime transition: the formation of toroidal vortices in the flow around an impulsively-started rotating cylinder. A lack of understanding of the instability in this flow was suggested as the reason for a Russian stiletto missile veering off course recently (Tan 2010, personal communication). We can compare this with the steady case of Couette flow - flow between concentric cylinders separated by a narrow gap - which is well understood with clear criteria for stability. Both flows are subject to an instability that results in the formation of toroidal vortices around the cylinder (known as Taylor vortices). Linear stability theory accurately predicts the onset of instability in Couette flow (see [12]). In the unsteady case, however, the stability characteristics are much less well understood; the main aim of this thesis is to examine the onset of these Taylor vortices.

1.1 Previous Studies

The instability of a boundary layer on an impulsively started rotating cylinder has been of interest to fluid dynamicists for many years. It was first studied by C. F. Chen and David K. Christensen in 1967 [2]. Chen and Christensen noted the appearance of instability discs on an impulsively started rotating cylinder. A marginal stability curve was plotted, but no further analysis was performed.

A famous picture of this flow was taken in 1977 by Taneda, Honji and Tatsuno (Figure 1, taken from [10]). Taneda also briefly examined this flow when investigating unsteady separated flows around bodies in [11].

Figure 1: Taneda, Honji and Tatsuno [10]

The first serious analysis was carried out by C. F. Chen and R. P. Kirchner in 1970 (see [3] and [4]) where this unsteady flow was considered a form of wide gap Couette flow. The apparatus consisted of two concentric cylinders with an inner to outer radius ratio of 1 : 10.

The unsteady flow was analysed by Chen and Kirchner in two ways: firstly, they modelled it as an initial value problem; secondly, they assumed the flow was quasi-
steady and applied linear stability methods. The initial value model worked well, but could not provide a criterion for stability. On the other hand, the quasi-steady approach overestimated the size of the vortices that formed on the cylinder. Their theoretical predictions consistently underestimated the time at which vortices appeared.

Chen and Kirchner attempted to reconcile this time delay with the theory, but their explanation was criticised in a subsequent paper by Tan and Thorpe in 2003 [9]. This paper challenged the wide gap Couette flow approach. Instead, they used a method known as transient instability theory. However, this theory used a form of the Taylor number which is commonly associated with Couette flow, calling into question the validity of their analysis. The quasi-steady assumption is questionable, as the timescales of the perturbation and the flow will be similar.

It is clear, therefore, that the existing literature has identified a gap between theoretical predictions and experimental results. One of the aims of this project is to determine the extent to which the quasi-steady assumption is applicable.

There is a well-known analogy between the centrifugal instability in Couette flow and the thermal instability present in flow between flat plates when the lower plate is heated [1]. There is likely to be a similar analogy for the unsteady cases for both these flows; a second aim of this project is to examine this analogy.

1.2 Motivation

As noted above, there is a considerable mismatch between theoretical predictions and experimental results. As well as being of academic interest a full analysis of this flow could have interesting consequences for the study of other unsteady flows. Furthermore, a fuller understanding of the analogy between centrifugal and thermal instabilities could enable a considerable crossover of theoretical and practical results.

Previous experiments have been in a wide gap Couette apparatus and using visualisation methods which do not give an impression of the overall flow field. This project will look at the flow around an impulsively-started rotating cylinder in a much larger tank, and will use a method of visualisation that allows us to observe the evolution of the whole flow field over time.

1.3 Aims

1. To give a qualitative description of the flow around an impulsively-started rotating cylinder.

2. To test the assumption of quasi-steady for this flow by comparing theoretical predictions against experimental results.

3. To investigate whether the behaviour of this flow is similar to that of a heated plate.
1.4 Notation

\( \Omega = \) angular velocity of the cylinder (measured in \( \text{rad.s}^{-1} \)).

\( R = \) the radius of the cylinder (measured in \( m \)).

\( R_1 = \) the radius of the inner cylinder in Couette flow (measured in \( m \)).

\( R_2 = \) the radius of the outer cylinder in Couette flow (measured in \( m \)).

\( \nu = \) the kinematic viscosity of the fluid (measured in \( m^2\text{s}^{-1} \)).

\( d = \) the diameter of a vortex (measured in \( m \)).

\( \delta = \) the boundary layer thickness (measured in \( m \)).

\( t_{\text{crit}} = \) the critical time, i.e. the time an instability first occurs (measured in \( s \)).
2 Theoretical Analysis

When the cylinder is impulsively started, the boundary layer initially remains laminar since the viscosity damps out any disturbances. At some critical time $t_{\text{crit}}$ an instability occurs, and Taylor vortices begin to form. Taylor vortices are pairs of toroidal vortices which form along the length of the cylinder.

2.1 Dimensional Analysis

Dimensional analysis should be able to identify the important dimensionless groups for this problem. The dependent variables are:

- $d =$ size of the vortices (units: length)
- $t_{\text{crit}} =$ critical time i.e. the time the instability occurs (units: time)

The independent variables in this problem are:

- $R =$ radius of the cylinder (units: length)
- $\Omega =$ angular velocity of the cylinder (units: time$^{-1}$)
- $\nu =$ kinematic viscosity of the fluid (units: length$^2$ time$^{-1}$)

Dimensional analysis gives us a Reynolds number (equation 2.1) and a time scale of the instability (equation 2.2). In addition a third dimensionless group is the ratio between the size of the vortices and a length scale of the flow (equation 2.3).

\[
\text{Reynolds number} \quad Re = \frac{\Omega R^2}{\nu} \quad (2.1)
\]

\[
\text{Time scale of the instability} \quad \tau = \frac{\nu t_{\text{crit}}}{R^2} \quad (2.2)
\]

\[
\text{Ratio of length scales} \quad \frac{d}{R} \quad (2.3)
\]

If these are the only important variables in the problem we would expect to have the following relations:

\[\tau = f(Re) \quad (2.4)\]

\[\frac{d}{R} = g(Re) \quad (2.5)\]

This is the limit of dimensional analysis. In order to derive the forms of $f$ and $g$ we need to look at the specifics of the problem. As this is a swirling flow, an obvious starting point is the Rayleigh stability criterion.
2.2 Rayleigh Stability Criterion

The Rayleigh stability criterion was first suggested by Rayleigh [7]. It states that a necessary and sufficient condition for stability of a swirling flow to axisymmetric disturbances is that the square of the circulation must not decrease with radius [5]. If the square of the circulation decreases with radius at any point in the flow field, then it would become possible to swap two elemental rings of fluid and release kinetic energy. Any disturbance that resulted in this exchange would therefore grow, and the flow would become unstable.

Rayleigh’s criterion can also be stated as follows:

\[
\frac{d(\Gamma^2)}{dr} < 0 \text{ for instability.} \quad (2.6)
\]

where \( \Gamma = \text{circulation} = 2\pi r V(r) \). This is clearly true for our flow, as next to the cylinder we have

\[
\Gamma^2 = (2\pi \Omega r^2)^2
\]

and far away from the cylinder we have \( \Gamma^2 = 0 \). Therefore Rayleigh’s criterion tells us that our flow is unstable.

2.3 Couette Flow

The flow around a cylinder could be considered as a special case of Couette flow for which the distance between the cylinders is equal to the boundary layer thickness.

The viscous flow between two concentric cylinders was first studied by Taylor in [12]. For the case of two concentric rotating cylinders, with the outer cylinder stationary and the inner cylinder rotating at angular velocity \( \Omega \), the velocity profile can be derived once the flow reaches a steady state, as shown in Figure 2.

![Figure 2: Couette Flow](image)

The velocity profile for this flow is shown in equation 2.7. It is derived by Acheson in [1].
\[ U_\theta(r) = \Omega(Ar + \frac{B}{r}) \quad (2.7) \]

where

\[ A = -\frac{R_1^2}{R_2^2 - R_1^2}, \quad B = \frac{R_1^2R_2^2}{R_2^2 - R_1^2} \]

At first it appears that there are two dimensionless groups in this problem. These are the Taylor number and a ratio of length scales:

\[ Ta = \frac{\Omega^2 R_1 (R_2 - R_1)^3}{\nu^2} \quad (2.8) \]

Ratio of length scales = \( \frac{R_2 - R_1}{R_1} \) \quad (2.9)

Using linear stability theory it is possible to derive a condition for instability in terms of the Taylor number alone \([1]\). This condition is:

\[ Ta > 1708 \quad (2.10) \]

This analysis involves a few assumptions. Namely,

- That there is a narrow gap between the cylinders. This is true when \((R_2 - R_1) \ll R_1\).
- That the angular velocity \(\Omega\) is small (or if both cylinders are rotating, that the angular velocity of both cylinders is almost equal). In practice, however, this is still a good approximation as long as the angular velocities of both cylinders are of the same sign.
- That the instability occurs in a non-oscillatory manner, i.e. that one eigenvalue goes from negative to positive and one wavelength begins to grow.

### 2.4 Assumption of Quasi-Steady Flow

There is an important difference between the case of an impulsively started cylinder and Couette flow. The analysis of Couette flow involved adding a perturbation to a steady base flow. For an impulsively started cylinder the angular momentum is constantly diffusing outwards and there is no steady base flow on which to base our stability analysis.

If the flow is assumed to be quasi-steady we can take a velocity profile at an instant in time and analyse this for instability. This has been the one of the traditional approaches taken for analysing this flow (see \([3]\) and \([4]\)).

One method of approaching the quasi-steady assumption would be to assume that we can equate the edge of the boundary layer with the outer cylinder in Couette flow.
2.4.1 Assuming Similarity to Couette Flow

If we assume the flow is quasi-steady then we can use linear stability theory to analyse the growth and decay of perturbations in the flow. A similar analysis to that used for Couette flow will reveal, that so long as the boundary layer is thin at the time of instability, the only important dimensionless group for stability is the Taylor number.

The Taylor number for this flow is given by:

\[ Ta = \frac{\Omega^2 \delta^3 R}{\nu^2} \]  \hspace{1cm} (2.11)

Here \( \delta \) is the boundary layer thickness (in Couette flow this term was equal to \( R_2 - R_1 \)). As this is a diffusion problem, we expect \( \delta \) to scale with \( \sqrt{\nu t} \).

\[ \delta \sim \sqrt{\nu t} \]  \hspace{1cm} (2.12)

If we can assume the flow is quasi-steady then there is a critical Taylor number at which the flow becomes unstable:

\[ Ta_{\text{crit}} = \frac{\Omega^2 \delta_{\text{crit}}^3 R}{\nu^2} \]  \hspace{1cm} (2.13)

where \( \delta_{\text{crit}} \) is the size of the boundary layer when the flow becomes unstable:

\[ \delta_{\text{crit}} \sim \sqrt{\nu t_{\text{crit}}} \]  \hspace{1cm} (2.14)

This implies that:

\[ Ta_{\text{crit}} \sim \frac{\Omega^2 (\nu t_{\text{crit}})^{3/2} R}{\nu^2} \]  \hspace{1cm} (2.15)

We can rearrange this in terms of \( \tau \) and the Reynolds number:

\[ \tau \sim (Ta_{\text{crit}})^{1/2} (Re)^{-4/3} \]  \hspace{1cm} (2.16)

If we assume that the size of the vortices \( d \) scales with the thickness of the boundary layer when the instability occurs then we can derive a similar equation for \( d \) in terms of the critical Taylor number and the Reynolds number.

Firstly we assume \( d \) scales with \( \delta_{\text{crit}} \).

\[ d \sim \delta_{\text{crit}} \sim \sqrt{\nu t_{\text{crit}}} \]  \hspace{1cm} (2.17)

This allows us to derive the following relation:

\[ Ta_{\text{crit}} \sim \frac{\Omega^2 d^{3/2} R}{\nu^2} \]  \hspace{1cm} (2.18)

which can be rearranged to be in terms of our ratio of length scales and the Reynolds number.
\[
\frac{d}{R} \sim (Ta_{\text{crit}})^{1/3}(Re)^{-2/3} \quad (2.19)
\]

We have now derived the possible forms of \(f\) and \(g\) from equations 2.4 and 2.5 using the quasi-steady assumption and assuming a thin boundary layer.

\[
\tau = A(Ta_{\text{crit}})^{1/2}(Re)^{-4/3} \quad (2.20)
\]

\[
\frac{d}{R} = B(Ta_{\text{crit}})^{1/3}(Re)^{-2/3} \quad (2.21)
\]

where \(A\) and \(B\) are constants to be determined by experiment.

The critical Taylor number, \(Ta_{\text{crit}}\) will not be equal to 1709 (as it was for Couette flow) because the velocity profile in the diffusing problem will be different. Very small changes in the velocity profile can have large impacts on stability criteria.

2.5 Previous Models

2.5.1 Predictions Using the Quasi-Steady Assumption

Chen and Kirchner [4] carried out experiments on apparatus with concentric cylinders separated by a wide gap (the radius ratio was 1 : 10). They used two methods to analyse this flow. Their first method was numerical, while their second used the quasi-steady approximation. The quasi-steady method used had two possible stability criteria.

- The first criterion considered the flow to be marginally stable when the instantaneous velocity profile was such that the growth rate of perturbations was zero.
- The momentary stability criterion compares the growth rate of a perturbation to the rate of change of the basic flow. The second criterion deemed the flow to be unstable when perturbations grew fast enough to overtake the basic flow.

The detailed analysis of this flow is beyond the scope of this project (but can be found in [4]). The graph of the Reynolds number against the size of vortex over the radius of the cylinder \((d/R)\) derived during this analysis is shown in Figure 3.

The experimental results from this project will be compared to these models. The same quasi-steady theory was used to predict critical times. These will also be compared to experimental values (although there are problems with this, to be discussed later).
2.5.2 Predictions Using Transient Stability

Tan and Thorpe put forward an alternative theory based on transient instability theory [9]. The full analysis can be found in appendix B. Briefly, a new type of Taylor number is defined using the velocity gradient of the flow around the cylinder (as it is this that drives the instability). For any instant in time the transient Taylor number varies with radius, \( r \) (as it now depends on the velocity gradient which varies with \( r \)). An instability is most likely to occur where the transient Taylor number is at a maximum. This, along with the analogy with the thermal instability (a similarly defined transient \( Ra \) at instability for a flat plate = 1100), is used to calculate the critical time and most unstable wavelengths.

The theory put forward by Tan and Thorpe regarding the sizes of the vortices is fairly simple. It predicts that

\[
\frac{d}{R} = 47.7 \ Re^{-2/3}
\]

where \( R \) is the radius of the cylinder. A plot of this model can be found in figure 3. It follows the same form as equation 2.21 which was found using quasi-steady theory, suggesting that there is in fact no difference between transient instability theory and quasi-steady theory. The derivation of 2.22 included a value that may have been obtained empirically (see Section B).
A model for $\tau$ was also derived in [9]:

$$\tau = 82.9 \, Re^{-4/3}$$  \hspace{1cm} (2.23)

This will also be compared against experimental results.

### 2.6 Analogy with Thermal Instability

There is an analogy between Couette flow and the thermal instability between two plates. The instability occurs in the thermal case at a Rayleigh number of 1709 (see [6]) and in Couette flow at a Taylor number of 1709.

This is due to an equivalence between the two problems [1]. The equations derived for each problem are mathematically equivalent.

For the thermal case the following equation can be derived,

$$(\frac{d^2}{dz^2} - a^2)^3 \kappa W = \alpha g \frac{dT_0}{dz} Wa^2$$  \hspace{1cm} (2.24)

For the centrifugal instability the following equation can be derived,

$$(\frac{d^2}{dx^2} - a^2)^3 u_r = -Taa^2 u_r$$  \hspace{1cm} (2.25)

These equations were taken from Acheson [1].

In the unsteady case of a flat heated plate in a semi-infinite fluid we see the formation of thermal plumes, unlike in the steady case where convection cells form. If the same analogy is valid for the thermal case then we would expect to see plume-like structures in the flow.
3 Experiment

This experiment is carried out in a larger tank than has previously been used to study this type of flow. The Reynolds number is varied by changing both the size of the cylinder and its angular velocity. The visualisation technique used allows observation of the entire flow field whereas previous studies have generally used techniques which show only one cylindrical streamline in the boundary layer. This will make it possible to investigate the thermal analogy by looking for plume-like behaviour.

3.1 Apparatus

The cylinder is placed vertically in a tank measuring 449 mm x 447 mm x 596 mm. Two cylinders are used, one measuring 32.086 ± 0.34 mm in diameter, the other 9.954 ± 0.18 mm in diameter (measurements in Section C). They are rotated by an electric motor attached to one of two gearboxes, giving two ranges of speeds. One gearbox is roughly ten times faster than the other, to give similar Reynolds numbers when it is attached to the smaller cylinder.

The speed ranges of the motor are:

\[ 1.7 > \Omega > 0.7 \text{ and } 15.1 > \Omega > 3.1 \text{ rad.s}^{-1} \]

The range of Reynolds numbers is thus:

\[ 440 > Re > 30 \]

A photo of the setup can be seen in Figure 4, while a diagram can be seen in Figure 5.

Figure 4: A photo of the apparatus

Figure 5: A diagram of the apparatus
3.2 Pearlescence

The experiment is visualised using pearlescence and a light sheet. Pearlescence is composed of many tiny platelets which act like mirrors. When these are placed in a shear flow they align with the shear (more precisely, they are rotating, but spend more time aligned with the shear than in any other orientation). In the flow under consideration, circular streamlines are expected, at least at early times. This means that if the tank is illuminated from one side then the flow on the side of the cylinder nearest the light source will be bright when the pearlescence is aligned with circular streamlines. Similarly, the side of the cylinder furthest from the light source will be dark when the pearlescence is aligned with circular streamlines there. Both of these can be used to estimate the extent of the boundary layer.

![Figure 6: Tank with Pearlescence](image)

An example of this can be seen in Figure 7, where it is clear that on the left side of the cylinder all the pearlescence is orientated to reflect light away from the camera.

![Figure 7: Pearlescence](image)
3.3 Procedure

3.3.1 Pilot work

Pilot work was carried out to detect any factors which may affect the results of the experiment. Twenty seven pilot experiments were performed to perfect the techniques involved and to decide upon a final configuration.

Disturbances present in the tank  In order to obtain accurate results the tank should be as free from disturbances as possible. To this aim the tank should be left to settle for as long as possible. However, compromises must be made as the pearlescence starts to lose its plate-like structure when mixed with water. It no longer aligns with shear in the flow and the images obtained are significantly less clear. The tank appeared uniform one hour after the pearlescence had been added. It was decided to carry out experiments two hours after adding pearlescence (the extra hour added as a precaution). This was within the time scale required for the pearlescence to remain plate-like.

End effects  The tank was almost completely filled to a height of 0.49 m to increase the aspect ratio and reduce end effects.

Heat from lamp  Heat from the lamp produced a secondary flow in the tank. For this reason the lamp was placed as far away from the tank as space allowed (1.5 m) and was turned on 2 minutes before each experiment. It was observed that a boundary layer formed on the side of the tank nearest the lamp after the lamp was on for 15 minutes which was approximately 10 mm thick. This was less than the 5% of the total distance between the tank and the outer edge of the cylinder. The boundary layer on the cylinder was typically less than 15% of the distance between the tank wall and the outer edge of the cylinder.

The speed-up time of the cylinder  This was only considered to be a possible problem for the faster experiments where the critical time (time to instability) is small and the time taken for the cylinder to reach a steady speed would be larger. From observation the cylinder reached its final velocity within 0.15 s of when it first started rotating. This means that some of the results achieved at high velocities should be treated with caution if the critical times were of a similar magnitude.

Configuration  Many different configurations were attempted to get a good picture. The best results were obtained with the light sheet tangent to the cylinder as in figure 8. Too little pearlescence was found to make the picture too dark, too much made the tank foggy. The size of the aperture of the camera also had a large affect on the quality of the results. Most of the pilot experiments were performed to perfect the visualisation technique.
**Time and speed of recording**  For the larger cylinder it was found that a frame rate of 10 frames per second was sufficient and three minutes of recording time captured all of the initial behaviour. For the narrow cylinder higher speeds were used to obtain similar Reynolds numbers and so critical times were shorter. This meant a higher frame rate was required but also that only two minutes of recording was needed to capture the necessary data.

**Number of experiments**  As Chen and Kirchner performed 30 experiments, this is considered the minimum number needed to show a trend. Three runs will be performed for each cylinder, consisting a series of experiments spaced over the range of Reynolds numbers achievable with that cylinder up to a maximum Reynolds number of 440. Gaps of 50 to 60 have been chosen. This means performing 15 experiments with the larger cylinder and 21 with the smaller cylinder (as this can achieve lower Reynolds numbers and so has a wider range). Further experiments can be performed, if necessary, if results appear to be anomalous (i.e. differing by more than 2 standard deviations).

### 3.3.2 Final Procedure

- The water tank was filled and left for ten hours (or overnight) until the gas in the water condensed out.

- The bubbles collected on the tank walls were knocked out and pearlescence was added and mixed in.

- After leaving the tank to settle for two hours the speed of the motor would be set, the camera set up and finally the arc lamp switched on.

- The motor would then be switched on and the camera would record the flow (at a speed of 10 fps for the large cylinder and 15 fps for the narrow cylinder).

- After a certain time (3 minutes for the large cylinder and 2 for the narrow cylinder), recording would stop and the motor and arc lamp switched off.
3.4 Measurement of Size of Vortices

The size of the vortices will be measured by analysing the images taken by the camera. This is a three step process:

Select region  The first step is to select the region close to the cylinder where the vortices form. This can be seen in Figure 9.

Average  The second step is to average the intensity of this region in the radial direction (the x direction in Figure 9) and create a time series. An example of a time series can be seen in Figure 10. Note that the x-axis in this diagram is time. Each column of this image is the average intensity over the radial direction for each y-position.

Fourier Transform  The third and final step is to perform a Fourier transform on each column of the time series (each of which is one pixel wide) in Figure 10. This gives us the energy content of each wavelength present in that column, effectively finding the most common length scales. An energy spectrum from an experiment carried as part of this project can be seen in the results section, Figure 12. In this particular spectrum there is a clear peak at around $k = 55$.

By this method we can find the average vortex size and also for later times the approximate size (in the y direction) of eddies in the flow.

3.5 Measuring $t_{crit}$

It is not possible to measure the critical time from an experiment. This is for two reasons:
• A perturbation has to grow by a large amount before it becomes visible.

• Most criteria for deciding when vortices are visible are arbitrary.

One method for measuring the time of the instability was considered (other than watching each film individually and deciding when vortices appeared to have formed). In this method a value for the maximum amplitude of a wavelength is set. If the energy of any wavelength within a specified window exceeds this value then vortices would be said to have formed. Pilot work using this method showed that is was extremely difficult to set a value for this maximum wavelength (even if the images were normalised by average intensity). This was because it was very difficult to control the amount of pearlescence quantitatively. Once the pearlescence was added it began to dissolve and the overall intensity of the image decreased. To counteract this effect the aperture of the camera had to be adjusted. The overall intensity therefore varied from experiment to experiment.

The method for measuring the time of the instability that was eventually used was as follows. Three times were recorded for each experiment.

\[ \tau_1, \text{When the boundary layer on the cylinder appeared to be breaking down.} \]

\[ \tau_2, \text{When there was evidence of a regular structure in the flow (looking like vortices).} \]

\[ \tau_2, \text{When the vortices were fully formed and began to move away from the wall of the cylinder.} \]

The first of these times was the most difficult to judge due to a lack of clearly identifiable features in the flow at this stage, while the second two were a little easier. The measurement of these critical times is more susceptible to inaccuracies than the measurement of the vortex sizes due to the more subjective nature of the method used.

### 3.6 Accuracy

The Fourier transform method used is one of maximum entropy. This fits rational functions to the energy spectrum. It was decided that a rational function with 50 terms was sufficient. The images analysed had 920 pixels in each column so the maximum wavelength it was possible to find was 460 pixels. The shortest possible wavelength would theoretically be 2 pixels long. The scale of the images was that 2.2 pixels = 1 mm. This makes the range of wavelengths that are possible to identify as follows:

\[ 0.91 \text{mm} < \lambda < 209.09 \text{mm} \quad (3.1) \]

Typical wavelengths were between 4.7 mm and 24.66 mm, well within these limitations.

To work out the error in the measurement of the wavelength first consider how many wavelengths are visible in an image.
If the wavelength = \( \lambda \) and the number of pixels in each image is \( L \) then

\[
\text{The number of wavelengths } = \frac{L}{\lambda} \quad (3.2)
\]

This will need to change by one pixel before we can detect a change in the wavelength. Therefore

\[
(\pm \epsilon) \frac{L}{\lambda} = 1 \text{ pixel} \quad (3.3)
\]

The error therefore is as follows,

\[
\pm \epsilon = \frac{\lambda}{L} \text{ pixels} \quad (3.4)
\]

\( L = 920 \) pixels and the largest wavelength was around 70 pixels.

\[
\pm \epsilon_{\text{max}} = 0.077 \text{ pixels} = 0.035 \text{ mm} \quad (3.5)
\]

This analysis ignores any noise present in the signal.

The speed of the camera (as has been previously stated) was 10 fps for the large cylinder and 15 fps for the narrow cylinder. This meant that critical times could be measured to within \( \pm 0.05 \) s when filming at 10 fps and \( \pm 0.033 \) s when filming at 15 fps.
4 Results and Discussion

Following the pilot experiments, forty eight main experiments were carried out. Each experiment had a time extract created of the results (see Section 3.4). An example of a time extract can be seen in Figure 11. When a Fourier transform was applied to each column of this image a power spectrum like the one in Figure 12 was created. The value of the initial peak was recorded for each experiment.

Few problems were encountered except when using the narrow cylinder at high Reynolds number. Under these conditions it was difficult to measure the size of the vortices accurately.

To increase the Reynolds number high enough to match the larger cylinder, the velocity of the cylinder, Ω had to be increased. This meant that the time scales were dramatically shortened. This can be understood by considering that time can be non-dimensionalised by Ω. If Ωt_{crit} is constant then as Ω is increased for constant Reynolds number, t_{crit} will fall. The instability occurred and the flow became turbulent over a short time scale. This made it much more difficult to get a good estimate for the size of the vortices. If these experiments are excluded this is mentioned in the text.

4.1 Description of the Flow

There are a three main stages to the flow’s development over time.

4.1.1 Stage 1

At first the flow remains laminar (Figure 13). Momentum is slowly diffusing outwards. In this regime viscosity is damping out disturbances. The Taylor number of the flow is still low but is gradually increasing as the thickness of the boundary layer (δ) increases.

4.1.2 Stage 2

At some critical point an instability occurs and the boundary layer rolls up into a series of regular, toroidal vortices as shown in Figure 14. The vortices start out axisymmetric then become irregular as they diffuse outwards. Pairs of vortices interact to form plume-like structures.

4.1.3 Stage 3

As the boundary layer diffuses out and the Taylor number increases further the flow becomes turbulent. This can be seen in Figure 15.

This progression can also be clearly seen on a time series. In figure 11 it is clear that the boundary layer is initially laminar, breaks down into regular vortices then becomes turbulent.
Figure 11: Time series

Figure 12: Energy Spectrum, R = 15 mm, Ω = 9.2 rpm
Figure 13: Stage 1

Figure 14: Stage 2

Figure 15: Stage 3
4.2 Size of the Vortices

Assuming the flow is quasi-steady and thin boundary layers we derived the following equation for $d/R$:

$$\frac{d}{R} = B(Ta_{crit})^{1/3}(Re)^{-2/3} \quad (4.1)$$

The sizes of the vortices at different Reynolds numbers has been measured and a graph of the results can be seen in Figure 16.

![d/R vs Re](image)

Figure 16: $d/R$ vs Reynolds number (1: Transient Stability, 2: Zero Growth Rate Criterion, 3: Momentary Stability Criterion)

There appears to be a good correlation between Reynolds number and size of vortex, with the exception of measurements for the small cylinder at high Reynolds numbers. If the experiments with the narrow cylinder at high Reynolds numbers are removed and the same graph is plotted on a log-log scale we get Figure 17.

The results appear to follow a power law as was predicted by quasi-steady theory. If a line of the form of equation 4.1 is fitted though the data (using a least squares method and excluding the high Reynolds number, narrow cylinder results) then an $R^2$ value of 0.9099 is obtained, with $B(Ta_{crit})^{1/3} = 42.12$.

This is strong evidence that the quasi-steady approach is valid for predicting the form of the relationship between the size of the vortices and the Reynolds number.
4.2.1 Predictions of Transient Instability

Tan and Thorpe’s model of the vortex sizes has also been plotted in figure 16 (the model is shown in equation 4.2).

\[
\frac{\lambda_c}{R} = 47.7 \, Re^{-2/3}
\]  

(4.2)

It appears to fit with my results \((R^2 = 0.8643\) when the high Reynolds number, small cylinder experiments are excluded).

4.2.2 Predictions of Quasi-Steady Theory

Chen and Kirchner predicted the exact vortex sizes that would be expected. Effectively they predicted the constant that this project left to be found from experiments. The predicted vortex sizes of Chen and Kirchner are roughly twice what was found in this set of experiments. Their predictions can also be seen in Figure 16.

Chen and Kirchner used stability theory to calculate \(T_{a_{crit}}\). The size of the vortices that they predicted was based on the value of the most unstable wavelength. The most unstable wavelength by this definition is the wavelength that starts to grow first, not the wavelength that grows the fastest.
This can be better understood by looking at Figure 18. As the Taylor number increases eventually it will reach the critical Taylor number ($Ta_{\text{crit}}$). At this point a single wavelength will begin to grow.

![Neutral stability curve](image)

Figure 18: Neutral stability curve

In Couette flow the Taylor number is constant for a given setup (constant $R_1$, $R_2$, $\Omega$ and $\nu$). When the flow just goes unstable (for example because the velocity of the inner cylinder has been increased very slightly) it is because perturbations at one wavelength (the most unstable wavelength from stability theory) began to grow. This is the only wavelength that is able to grow so the Taylor vortices formed at this point will be of the size predicted by stability theory.

In the unsteady flow the momentum is slowly diffusing outwards. This means that the Taylor number is increasing over time, this is the reason for the time arrow in Figure 18. While initially there is only one unstable wavelength it is soon joined by other unstable wavelengths which may have faster growth rates. The wavelength that grows first no longer determines the length scale of the Taylor vortices that form.

One way of looking at this is that the wavelength that eventually dominates the flow is the one with the fastest growth rate, which happens to have a wavelength half that of the wavelength that starts to grow first. However, this is a slightly simplistic version of events. In fact as soon as more than one wavelength can grow, non-linear interactions will begin to happen. Linear theory assumes the amplitudes of all disturbances are very small so they do not affect each other. This means the evolution of each wavelength can be examined separately. Once perturbations in the flow have significant amplitude they will interact and linear theory can no longer predict what will happen.

### 4.3 Critical Times

Both my results and those of Chen and Kirchner are plotted in Figure 19. There is more scatter in $t_1$ as it was this time that was difficult to judge.

The model predicted for the critical times was as follows:
Figure 19: \( \tau \) vs Reynolds number (including results from [3])

\[
\tau = A(T_{a\text{crit}})^{1/2}(Re)^{-4/3}
\]  

4.3.1 Predictions

The critical times predicted by all models are smaller than the time the instability was observed (see Figure 19). One reason for this is because the predicted critical time is the moment that perturbations first start to grow, whereas the time recorded in an experiment is when the instability becomes visible. Chen and Kirchner theorised that perturbations become visible when their kinetic energy has increased by a factor of 24.
1000 [4], and this determines the time delay between the critical time and the time the vortices are observed.

Any critical times predicted by quasi-steady theory are the moment the first perturbation starts to grow. However, experiments from this project show that the perturbation which first begins to grow does not correspond to the wavelength which eventually becomes visible.

On way of thinking about this is that the perturbation that eventually dominates the flow is the one with the fastest growth rate, effectively ignoring any non-linear effects. This wavelength will start to grow a short time later at a slightly higher Taylor number and will have a different growth rate.

This makes it very difficult to predict when we first expect to see vortices. Setting aside non-linear effect for the moment, the time when the vortices become visible is as in equation 4.4.

\[
\tau_{\text{visible}} = \tau_{\text{crit}} + \tau_{\text{delay}} + \tau_{\text{growth}}
\]

where

\(\tau_{\text{visible}}\) is the time at which the vortices become visible;

\(\tau_{\text{crit}}\) is the critical time from instability theory;

\(\tau_{\text{delay}}\) is the time delay between the first perturbation starting to grow and the perturbation that eventually dominates the flow beginning to grow;

\(\tau_{\text{growth}}\) is the time it takes the perturbation that eventually dominates the flow to grow to a size such that it can be seen.

We have a possible equation for \(\tau_{\text{crit}}\) from quasi-steady theory in equation 4.3. However, it is quite possible that \(\tau_{\text{delay}}\) and \(\tau_{\text{growth}}\) will also be functions of Reynolds number (this is not known from the present theory). Also \(\tau_{\text{growth}}\) will depend on the experiment as how much a wavelength has to grow to become visible will depend on the definition of visible being used. This explains why the times observed do not follow exactly the same form as equation 4.3.

In reality when multiple wavelengths begin to grow they will interact with each other and so it will not necessarily be as simple as the above analysis suggests. However, the fastest growing wavelength will tend to dominate.

4.4 Analogy with Thermal Instability

An example of thermal plumes can be seen in Figure 20. This photo was taken from a study by Sparrow, Husar and Goldstein in 1970 (see [8]). This study found that thermals were shaped like mushroom clouds. They were generated continuously at fixed sites, spaced periodically along the heated surface and the spacing between them was dependent on the heating rate.
The same was found to be true for centrifugal plumes. They were spaced regularly along the cylinder and were generated continuously at a fixed spacing. A photo of these plumes can be seen in Figure 21. The spacing between the plumes was much larger than the spacing between the initial Taylor vortices. Similar to the thermal plumes, when the rotation rate was increased the plumes moved outwards at a faster rate. There was also some evidence of entrainment as the plumes increased in size as they moved outwards.
Figure 20: Thermals from a heated plate, Sparrow, Husar and Goldstein [8]

Figure 21: Plumes from a rotating cylinder
5 Conclusions

The Reynolds number of this experiment was limited, but the critical times found for \( Re \sim 440 \) suggest that it would be very difficult to measure critical times accurately for higher Reynolds number experiments (this would require at least a high speed camera). While the images generated by the visualisation method used here were not as clear as some previous methods, it allowed the entire flow field to be observed. The method used for measuring the size of the vortices (using a Fourier transform to find the wavelength) was theoretically more accurate than previous methods, this was possible due to computer processing, which was not widely available when this flow was first studied.

5.1 Quasi-Steady Assumption

The quasi-steady assumption is valid for finding the form of the relation between the Reynolds number and the size of the Taylor vortices. This relation is of the form:

\[
\frac{d}{R} = B'(Re)^{-2/3} \quad (5.1)
\]

However, if a value for \( B' \) is predicted using quasi-steady theory then this calculated value will be determined by the wavelength that begins to grow first. The final size of the vortices is dependent on the wavelength which grows at the fastest rate and on any non-linear effects.

A model was also found for the critical time of the instability using the assumption of quasi-steady:

\[
\tau = A'(Re)^{-4/3} \quad (5.2)
\]

This fits well with results at Reynolds numbers larger than 50. It is possible that the fit was less good at very low Reynolds numbers because the linear theory used to derive equation 5.2 assumed thin boundary layers.

It is difficult to compare quasi-steady predictions of the critical time to experimental results, because the predicted critical time is the time at which the first wavelength begins to grow while the time recorded from experiments is the time at which the perturbation becomes visible. This time difference will depend on many factors such as: when the dominant perturbation begins to grow; the growth rate of this perturbation; how we determine visible for the purposes of this experiment; and any non-linear effects.

In conclusion, the quasi-steady approximation can be used to derive the dependence on the Reynolds number of both the critical time and the size of the vortices. However, it cannot be used to derive the constants in equations 5.1 and 5.2, as these will be affected by factors not taken into account by linear theory.
5.2 Thermal Analogy

The plumes released from the cylinder appear to be similar to thermal plumes. They are released periodically and at a fixed spacing along the cylinder. There was evidence of entrainment: the plumes increased in size as they travelled radially outwards.

5.3 Future Work

It might be possible to provide a more systematic measure of the time at which the instability occurs by measuring the torque on the cylinder. If the torque on the cylinder can be accurately measured, the moment of formation can be determined by a deviation from the predicted torque due to laminar flow.

Further quantitative investigation into the extent of the thermal analogy could be carried out. The outward velocity of plumes and how this varies with Reynolds number might be compared to thermal plumes, as could the entrainment velocity if this could be measured.

6 Acknowledgements

Many thanks to my supervisor, Dr Peter Davidson and to Dr Stuart Dalziel (DAMTP) for all their help and support.
References


Appendix

A Risk Assessment Retrospective

There were a few dangers associated with the project. These were in three main areas:

**Use of electronic equipment around water** The was not found to be a serious risk so long as sensible precautions were taken.

**Use of chemical substances** The most dangerous substance to be used was the pearlescence. Care was taken to avoid contact with the skin but beyond this no other precautions needed to be taken.

**Use of strong light sources** This was easily the most problematic part of the experiment. A strong arc lamp was necessary for visualisation. The danger of looking into the beam was reduced by placing it behind a screen with a blackout curtain hung on it. However there was unavoidably a gap where the tank was (so that the lamp lit up the tank). Care had to be taken not to look into the beam accidentally and the lamp was turned off when not in use. Regular breaks were also taken as it was considered unwise to stay sitting in a dark room with a bright light in the corner (though it was shielded as much as possible).

Overall the risk assessment performed at the start of the project was correct though if performing this project again more thought should go into the dangers of bright light sources and the experiment designed accordingly.

B Tan and Thorpe

Tan and Thorpe used the similarity of the flow being considered with unsteady thermal instability. They argued that the instability of the flow around a rotating cylinder can be explained by the use of the theory of transient instability. In this theory the critical transient Taylor number at instability would be expected to be equal to around 1100. The transient Taylor number is defined in equation B.1, with the normal definition of the Taylor number for a stationary outer cylinder shown (for comparison) in equation B.2. $\Omega R_1$ has been replaced by $r(\delta u/\delta r)$, where $r$ is a characteristic distance in the flow (e.g. $r = (R_2 - R_1)$).

$$
Ta_t = \frac{r^5(\delta u/\delta r)^2}{\nu^2 R_1} \quad \text{(B.1)}
$$

$$
Ta = \frac{\Omega^2 R_1(R_2 - R_1)^3}{\nu^2} \quad \text{(B.2)}
$$
$du/dr$ can be found by considering the transient velocity profile of the boundary layer in a stationary semi-infinite fluid.

$$u = \Omega R_1 \text{erfc}(\frac{y}{\sqrt{4\nu t}}) \quad (B.3)$$

where \text{erfc}(z) is the complementary error function which is defined as,

$$\text{erfc}(z) \equiv 1 - \text{erf}(z) \quad (B.4)$$

$$\text{erfc}(z) = 2 \frac{1}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2} dt \quad (B.5)$$

This gives us $du/dr$,

$$\frac{du}{dr} = -\Omega R_1 e^{-y^2/4\nu t} \quad \sqrt{\pi\nu t} \quad (B.6)$$

This can be substituted into the transient Taylor number.

$$Ta_t = \frac{r^5\Omega^2 R_1^2}{\nu^2 \pi \nu t} e^{-r^2/2\nu t} \quad (B.7)$$

The maximum transient Taylor number at any instant can be found by differentiating equation B.7 with respect to $r$ and setting it to zero. This gives the position of the maximum value of transient Taylor number.

$$r_{max} = 2.24\sqrt{\nu t} \quad (B.8)$$

If this is substituted back into the maximum transient Taylor number we get equation B.9.

$$Ta_{max} = \frac{\Omega^2 R_1 (1.14\sqrt{\nu t})^3}{\nu^2} \quad (B.9)$$

If it is assumed that the critical transient Taylor number at instability is 1100 (for the reasons stated above) then the time of instability can be calculated.

$$t_c = 82.9(\frac{\nu^{1.5}}{\Omega^2 R_1})^{2/3} \quad (B.10)$$

This can also gives us the theoretical critical dimension of the vortices.

$$\lambda_c = \frac{2\pi r_{max}}{a_c} = 5.24\sqrt{\nu t_c} \quad (B.11)$$

where $\lambda_c$ is the wavelength of disturbances and $a_c = 2.68$ is the theoretical critical dimensionless wavenumber. The value for the latter is not fully explained in the paper and it is possible that its value was empirical.

Substituting in the critical time gives,
\[
\lambda_c = 47.7 (\frac{\nu^2}{\Omega^2 R_1})^{1/3}
\]  

(B.12)

Non-dimensionalising this by \( R_1 \) and expressing in terms of the Reynolds number \( (Re = \Omega R_1^2/\nu) \) gives,

\[
\frac{\lambda_c}{R_1} = 47.7 \, Re^{-2/3}
\]

(B.13)

\( \lambda_c \) should be equal to the size of the vortices \( d \).
C Results

Experiments 54 - 56 and 61 had problems with the files and so are not listed here. The cylinder was measured at several points along its length.

Larger cylinder measurements These were 32.00, 32.00, 32.03, 32.40, 32.00. Mean = 32.086 mm, SD = 0.176, 1.96 x SD = 0.345 mm

Smaller cylinder measurements These were 9.80, 9.90, 10.00, 10.00, 9.90, 9.98, 10.10. Mean = 9.954 mm, SD = 0.0920, 1.96 x SD = 0.180 mm

Table 1: Results for cylinder with R = 16.0 mm

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