1 Basic calculus

1.1 Differentiation as a limit

1.1.1 Definition

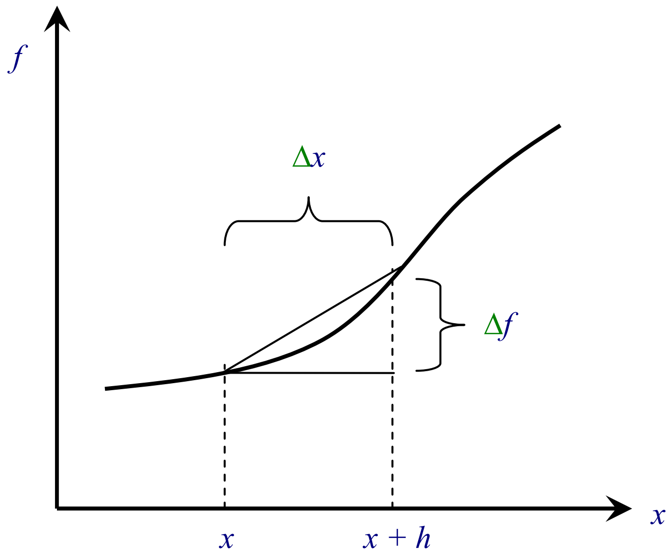


Figure 1: Approximation of slope over finite range of *x*.

Let
$$\Delta x \equiv x+h - x = h$$
$$\Delta f \equiv f(x+h) - f(x)$$

Define the derivative as

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0}\left(\frac{\Delta f}{h}\right)\\ &= \lim_{\Delta x\to 0}\left(\frac{\Delta f}{\Delta x}\right)\\ &= \lim_{\Delta x\to 0}\left(\tan\theta\right)\\ &= \text{"slope" of curve } f(x)\\ &= \text{"gradient" of curve } f(x)\end{aligned}$$

Other notations

$$\frac{df}{dx},\ \frac{d}{dx}f,\ f',\ f_x,\ f_{,x},\ Df$$

Additionally, especially when $f = f(t)$, the notation $\dot{f}$ is often used to represent df/dt.

1.1.2 DIFFERENTIABILITY

The derivative f' exists if it is finite and defined.

Must have left- and right-hand limits the same:

$$\lim_{h\to 0}\left(\frac{f(x)-f(x-h)}{h}\right)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

If f' exists, then f is said to be **differentiable**.

If f' exists, then $\Delta f \to 0$ as $\Delta x \to 0 \Rightarrow f$ is continous.

Note: Converse is not necessarily true, *i.e.* continuous f does not necessarily mean f is differentiable.

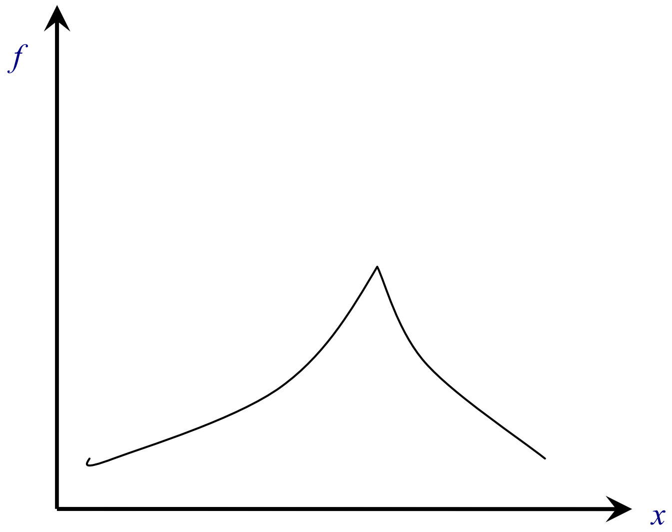


Figure 2: A function with a cusp is continuous, but not differentiable at the cusp as left- and right-hand derivatives are not the same.

1.1.3 EXAMPLES

- $f(x) = ax$, for $a = const$

$$\frac{df}{dx}=\frac{d(ax)}{dx}=\lim_{h\to 0}\frac{a(x+h)-ax}{h}=\lim_{h\to 0}\frac{ah}{h}=\lim_{h\to 0}a=a$$

Can check that left-hand derivative is equal to right-hand derivative.

- $f(x) = 1/x^2$

$$\frac{df}{dx} = \frac{d\left(x^{-2}\right)}{dx} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x^2 - (x+h)^2}{x^2 (x+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{-2xh - h^2}{hx^2 (x+h)^2}$$

$$= \lim_{h\to 0} \frac{-2x - h}{x^2 (x+h)^2}$$

$$= \frac{-2x}{x^4}$$

$$= -2x^{-3}$$

➡ $f(x) = x^n$

$$\frac{df}{dx} = \frac{d\left(x^n\right)}{dx} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

Use binomial expansion

$$= \lim_{h\to 0} \frac{x^n + nhx^{n-1} + \frac{n(n-1)}{2!} h^2 x^{n-2} + \frac{n(n-1)(n-2)}{3!} h^3 x^{n-3} + \cdots + h^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{nhx^{n-1} + \frac{n(n-1)}{2!} h^2 x^{n-2} + \frac{n(n-1)(n-2)}{3!} h^3 x^{n-3} + \cdots + h^n}{h}$$

$$= \lim_{h\to 0} \left(nx^{n-1} + \frac{n(n-1)}{2!} hx^{n-2} + \frac{n(n-1)(n-2)}{3!} h^2 x^{n-3} + \cdots \right)$$

$$= nx^{n-1}$$

Also works for $n < 0$.

1.2 Rules for differentiating

1.2.1 Sums and differences

Consider $y = f(x) + g(x)$

$$\frac{dy}{dx} = \frac{d}{dx}(f+g) = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$
$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)$$
$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$
$$= \frac{df}{dx}+\frac{dg}{dx}$$

provided *f* and *g* are differentiable.

Similarly $\frac{d}{dx}(f-g) = \frac{df}{dx}-\frac{dg}{dx}$.

➡ $f(x) = e^x$

Exponential function can be defined in a number of ways:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Can use binomial expansion

$$e^x = \lim_{n\to\infty}\left(1+n\frac{x}{n}+\frac{n(n-1)}{2!}\frac{x^2}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{x^3}{n^3}+\cdots\right)$$
$$= \lim_{n\to\infty}\left(1+x+\frac{x^2}{2!}\left(1-\frac{1}{n}\right)+\frac{x^3}{3!}\left(1-\frac{3}{n}+\frac{2}{n^2}\right)+\cdots\right)$$
$$= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$
$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

[Should really look at convergence, *etc.*, but not part of this course.]

Differentiate term-by-term:

$$\frac{d}{dx}(e^x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^n}{n!}\right)$$
$$= \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}$$
$$= \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}$$
$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$= e^x$$

since $d1/dx = 0$

1.2.2 THE CHAIN RULE

Consider $y = f(g(x))$:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \frac{g(x+\Delta x) - g(x)}{\Delta x} \right) \\
&= \lim_{\Delta x \to 0} \left(\frac{f(g+\Delta g) - f(g)}{\Delta g} \frac{\Delta g}{\Delta x} \right) \\
&= \lim_{\Delta g \to 0} \left(\frac{f(g+\Delta g) - f(g)}{\Delta g} \right) \lim_{\Delta x \to 0} \left(\frac{\Delta g}{\Delta x} \right) \\
&= \frac{df}{dg} \frac{dg}{dx}
\end{aligned}$$

since $\Delta g \to 0$ as $\Delta x \to 0$ for $g(x)$ (and hence y) to be continuous.

➧ $y = e^{\lambda x}$

Write as $y = f(g(x))$ where $f(g) = \exp(g) = e^g$ and $g(x) = \lambda x$.

Now $f'(g) = e^g$ and $g'(x) = \lambda$,

so
$$\begin{aligned}
\frac{dy}{dx} = \frac{df}{dg}\frac{dg}{dx} &= e^g \lambda \\
&= \lambda e^{\lambda x}
\end{aligned}$$

➧ $y = f(\lambda x)$

Similar to above example, yielding $\frac{d}{dx} f(\lambda x) = \lambda \frac{df(\lambda x)}{d(\lambda x)} = \lambda f'(\lambda x)$.

➧ $y = e^{ix}$ $(i^2 = -1)$

$$\Rightarrow \frac{d}{dx} e^{ix} = i e^{ix}$$

So
$$\begin{aligned}
\frac{d}{dx} \cos x &= \frac{d}{dx} \frac{1}{2} \left(e^{ix} + e^{-ix} \right) \\
&= \frac{1}{2} \left(i e^{ix} - i e^{-ix} \right) \\
&= \frac{-1}{2i} \left(e^{ix} - e^{-ix} \right) \\
&= -\sin x
\end{aligned}$$

$$\frac{d}{dx}\sin x = \frac{d}{dx}\frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

Similarly $$= \frac{1}{2i}\left(ie^{ix} + ie^{-ix}\right) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$= \cos x$$

End of Lecture 1

1.2.3 Products and Quotients

Consider $y = f{\cdot}g$:

$$\frac{dy}{dx} = \frac{d}{dx}(f \cdot g) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left(f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\right)$$

$$= f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f\frac{dg}{dx} + g\frac{df}{dx}$$

$$= f'g + fg'$$

The **product rule**

- $y = e^x \cos x$

  Let $f(x) = e^x \Rightarrow f' = e^x$

  and $g(x) = \cos x \Rightarrow g' = -\sin x$

  so $dy/dx = f'g + fg' = e^x(\cos x - \sin x)$.

Similarly $y = f/g$:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{f}{g}\right) = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x+h)g(x)}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) - f(x)g(x+h) + f(x)g(x)}{h\,g(x+h)g(x)}$$

$$= \lim_{h\to 0}\left(g(x)\frac{f(x+h)-f(x)}{h\,g(x+h)g(x)}\right) - \lim_{h\to 0}\left(f(x)\frac{g(x+h)-g(x)}{h\,g(x+h)g(x)}\right)$$

$$= \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

$$= \frac{f'g - fg'}{g^2}$$

The quotient rule

Clearly need $g \neq 0$!

1.2.4 Higher derivatives

First derivative $$\frac{df}{dx} = f' = f_x = f_{,x} = Df$$

Second derivative $$\frac{d}{dx}\frac{df}{dx} \equiv \frac{d}{dx}\frac{d}{dx}f \equiv \frac{d^2 f}{dx^2} = f'' = f_{xx} = f_{,x,x} = D^2 f$$

Third derivative $$\frac{d}{dx}\frac{d^2 f}{dx^2} \equiv \frac{d}{dx}\frac{d}{dx}\frac{d}{dx}f \equiv \frac{d^3 f}{dx^3} = f''' = f_{xxx} = f_{,x,x,x} = D^3 f$$

n^{th} derivative $$\frac{d}{dx}\frac{d^{n-1} f}{dx^{n-1}} \equiv \frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}f \equiv \frac{d^n f}{dx^n} = f^{(n)} = D^n f$$

➡ Derivatives of x^n:

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d^2}{dx^2}x^n = n\frac{d}{dx}x^{n-1} = n(n-1)x^{n-2}$$

$$\frac{d^3}{dx^3}x^n = n(n-1)\frac{d}{dx}x^{n-2} = n(n-1)(n-2)x^{n-3}$$

$$\frac{d^m}{dx^m}x^n = n(n-1)\cdots(n-m+1)x^{n-m} = \frac{n!}{(n-m)!}x^{n-m}$$

$$\frac{d^n}{dx^n}x^n = n!$$ for integer n

$$\frac{d^{n+1}}{dx^{n+1}}x^n = 0$$

1.2.5 LEIBNITZ' THEOREM

For the n^{th} derivative of a product

Let $f = f(x)$ and $g = g(x)$.

$$\frac{d}{dx}(fg) = f'g + fg'$$

$$\frac{d^2}{dx^2}(fg) = \frac{d}{dx}(f'g) + \frac{d}{dx}(fg') = f''g + f'g' + f'g' + fg''$$
$$= f''g + 2f'g' + fg''$$

$$\frac{d^3}{dx^3}(fg) = \frac{d}{dx}(f''g) + 2\frac{d}{dx}(f'g') + \frac{d}{dx}(fg'')$$
$$= f'''g + f''g' + 2(f''g' + f'g'') + f'g'' + fg'''$$
$$= f'''g + 3f''g' + 3f'g'' + fg'''$$

which suggests

$$\frac{d^n}{dx^n}(fg) = f^{(n)}g + nf^{(n-1)}g' + \frac{n(n-1)}{2!}f^{(n-2)}g'' + \cdots + \binom{n}{m}f^{(n-m)}g^{(m)} + \cdots + nf'g^{(n-1)} + fg^{(n)}$$

Can prove by induction. Exercise

- Establish true for $n = 1$
- Assume true for arbitrary n
- Show true for $n+1$

In Cambridge we like Newton, but…

Calculus was invented by Sir Isaac Newton and Gottfried Wilhelm Leibniz at around the same time, each claiming to have been first. Leibniz published in 1686 whereas Newton published in 1687, but it appears that Newton actually made the breakthrough some 20 years earlier (1665/66). This led to animosity between them (at least Newton hated Leibniz), and there is/was some suggestion of plagiarism on the part of Leibniz.

Newton's approach was based on the ideas of limits whereas Leibniz used geometric arguments and developed a much simpler notation, including the d/dx and integral symbols we still use. Newton's notation was almost incomprehensible, changing it depending on the context (find example), so was difficult to use and understand. (Was this Newton trying to show off?) British mathematicians used Newton's notation during the 18th century, whereas the rest of the world adopted Leibniz's notation and made more progress. Some of Newton's notation, for example $\dot{x}$ to represent dx/dt, is still used.

http://www.angelfire.com/md/byme/mathsample.html

➨ Use Leibniz rule for $y = x^{m+n} = x^m \cdot x^n$:

$$\frac{d^2}{dx^2}x^{m+n} = \frac{d^2}{dx^2}\left(x^m x^n\right)$$
$$= m(m-1)x^{m-2}x^n + 2mx^{m-1}nx^{n-1} + x^m n(n-1)x^{n-2}$$
$$= \left(m^2 - m + 2mn + n^2 - n\right)x^{m+n-2}$$
$$= (m+n)(m+n-1)x^{m+n-2}$$

as expected.

- Second derivative of $y = 2\sin x \cos x$:

$$\frac{d^2}{dx^2}2\sin x\cos x = 2\left[(-\sin x)\cos x + 2\cos x(-\sin x) + \sin x(-\cos x)\right]$$
$$= -8\sin x\cos x$$
$$= -4\sin 2x$$

As expected from $d^2/dx^2(\sin 2x)$.

There are a few somewhat more complicated functions on the examples sheet.

1.2.6 Partial differentiation

Frequently functions may depend on more than one variable. How do we differentiate these?

Consider $f(x,y)$. This might, for example, represent the height of a hill. In general, the *slope* will depend on the direction in which we are looking.

Define $\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$, the partial derivative of f with respect to x. This is the slope, at a given x,y in the x direction (*i.e.* holding y constant).

Similarly $\frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$. Note the use of ∂ in place of d to represent the derivative.

Other common notations: $\frac{\partial f}{\partial x}$ f_x $f_{,x}$ $\partial_x f$ and $\frac{\partial f}{\partial y}$ f_y $f_{,y}$ $\partial_y f$.

- $f(x,y) = 1 - x^2 - x\sin y + y^3$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{\left(1-(x+h)^2-(x+h)\sin y+y^3\right)-\left(1-x^2-x\sin y+y^3\right)}{h}$$

Then
$$= -\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} - \sin y\lim_{h\to 0}\frac{(x+h)-x}{h}$$
$$= -\lim_{h\to 0}(2x-h) - \sin y\lim_{h\to 0}(1)$$
$$= -2x - \sin y$$

which is the same as df/dx if we treat y as a constant.

$$\frac{\partial f}{\partial y}=\lim_{h\to 0}\frac{\left(1-x^2-x\sin\left(y+h\right)+\left(y+h\right)^3\right)-\left(1-x^2-x\sin y+y^3\right)}{h}$$

Similarly
$$=-x\lim_{h\to 0}\frac{\sin\left(y+h\right)-\sin y}{h}+\lim_{h\to 0}\frac{\left(y+h\right)^3-y}{h}$$

$$=-x\cos y+3y^2$$

What happens if we wish to move in another direction, say at an angle θ to the x axis? Let $\boldsymbol{s}=s(\cos\theta, \sin\theta)$ be a unit vector in this direction, then the slope will be

$$\frac{df}{ds}=\lim_{h\to 0}\frac{f\left(x+h\cos\theta,y+h\sin\theta\right)-f\left(x,y\right)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(x+h\cos\theta,y+h\sin\theta\right)-f\left(x,y+h\sin\theta\right)+f\left(x,y+h\sin\theta\right)-f\left(x,y\right)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(x+h\cos\theta,y+h\sin\theta\right)-f\left(x,y+h\sin\theta\right)}{h\cos\theta}\cos\theta+\lim_{h\to 0}\frac{f\left(x,y+h\sin\theta\right)-f\left(x,y\right)}{h\sin\theta}\sin\theta$$

$$=\lim_{h\cos\theta\to 0}\frac{f\left(x+h\cos\theta,y\right)-f\left(x,y\right)}{h\cos\theta}\cos\theta+\lim_{h\sin\theta\to 0}\frac{f\left(x,y+h\sin\theta\right)-f\left(x,y\right)}{h\sin\theta}\sin\theta$$

$$=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta$$

Now since $x=s\cos\theta\Rightarrow dx/ds=\cos\theta$ and $y=s\sin\theta\Rightarrow dy/ds=\sin\theta$, we may rewrite this as

$$\frac{df}{ds}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta=\frac{\partial f}{\partial x}\frac{dx}{ds}+\frac{\partial f}{\partial y}\frac{dy}{ds}$$

We shall not be using the above in this course, although you will require it in other courses.

More generally, we may have a function f that depends on the variables x_0, x_1, $\ldots x_{n-1}$, from which we may form the derivatives $\partial f/\partial x_i$ for $i=0,1,\ldots n-1$.

Higher order derivatives are handled in a similar way to functions of a single variable

$$\frac{\partial^2 f}{\partial x^2}=\lim_{h\to 0}\frac{\left.\frac{\partial f}{\partial x}\right|_{x+h,y}-\left.\frac{\partial f}{\partial x}\right|_{x,y}}{h}$$

$$=\lim_{h\to 0}\frac{\lim_{a\to 0}\frac{f\left(x+h+a,y\right)-f\left(x+h,y\right)}{a}-\lim_{a\to 0}\frac{f\left(x+a,y\right)-f\left(x,y\right)}{a}}{h},$$

but of course we have the possibility of taking the derivatives in different directions.

So long as the limits are well behaved, then

$$\frac{\partial^2 f}{\partial x \partial y} = \lim_{h\to 0} \frac{\left.\frac{\partial f}{\partial y}\right|_{x+h,y} - \left.\frac{\partial f}{\partial y}\right|_{x,y}}{h}$$

$$= \lim_{h\to 0} \frac{\lim_{a\to 0} \frac{f(x+h,y+a)-f(x+h,y)}{a} - \lim_{a\to 0} \frac{f(x,y+a)-f(x,y)}{a}}{h}$$

$$= \lim_{h\to 0}\left(\lim_{a\to 0}\left[\frac{f(x+h,y+a)-f(x+h,y)-f(x,y+a)+f(x,y)}{ah} \right]\right)$$

$$= \lim_{a\to 0}\left(\frac{1}{a}\lim_{h\to 0}\left[\frac{f(x+h,y+a)-f(x,y+a)}{h} - \frac{f(x+h,y)-f(x,y)}{h} \right]\right)$$

$$= \lim_{a\to 0} \frac{\left.\frac{\partial f}{\partial x}\right|_{x,y+a} - \left.\frac{\partial f}{\partial x}\right|_{x,y}}{a}$$

$$= \frac{\partial^2 f}{\partial y \partial x}$$

➡ $f(x,y) = 1 - x^2 - x \sin y + y^3$

$$\frac{\partial f}{\partial x} = -2x - \sin y \Rightarrow \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(-2x - \sin y\right) = -\cos y$$

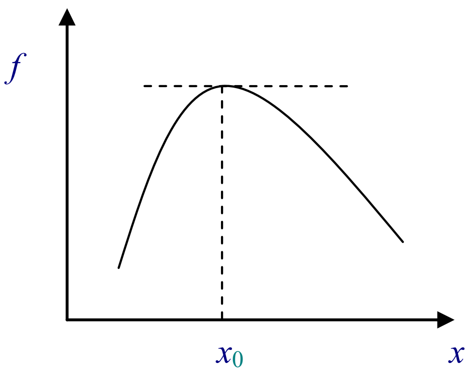
$$\frac{\partial f}{\partial y} = -x\cos y + 3y^2 \Rightarrow \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-x\cos y + 3y^2\right) = -\cos y$$

Other courses will require partial differentiation more than is the case for this course.

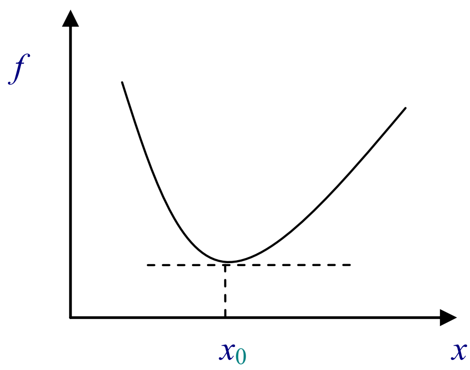
1.3 Curve sketching

Knowing the *stationary* points of a function $f(x)$ can help you sketch the function. Stationary points are where $df/dx = 0$ and the curve is horizontal.

Suppose $df/dx = 0$ at $x = x_0$. Four possible cases:



(a)



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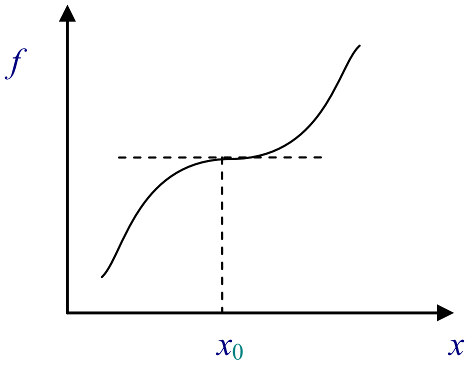

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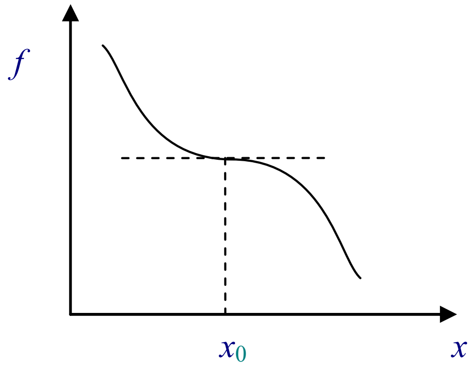


(d)

(a) $df/dx > 0$ for $x < x_0$ and $df/dx < 0$ for $x > x_0 \Rightarrow$ local maximum; $d^2f/dx^2 < 0$.

(b) $df/dx < 0$ for $x < x_0$ and $df/dx > 0$ for $x > x_0 \Rightarrow$ local minimum; $d^2f/dx^2 > 0$.

(c) $df/dx > 0$ for $x < x_0$ and $df/dx > 0$ for $x > x_0 \Rightarrow$ point of inflection; $d^2f/dx^2 = 0$.

(d) $df/dx < 0$ for $x < x_0$ and $df/dx < 0$ for $x > x_0 \Rightarrow$ point of inflection; $d^2f/dx^2 = 0$.

Note: A *point of inflection* is where $d^2f/dx^2 = 0$ irrespective of the value of df/dx (provided $d^3f/dx^3 \neq 0$).

End of Lecture 2

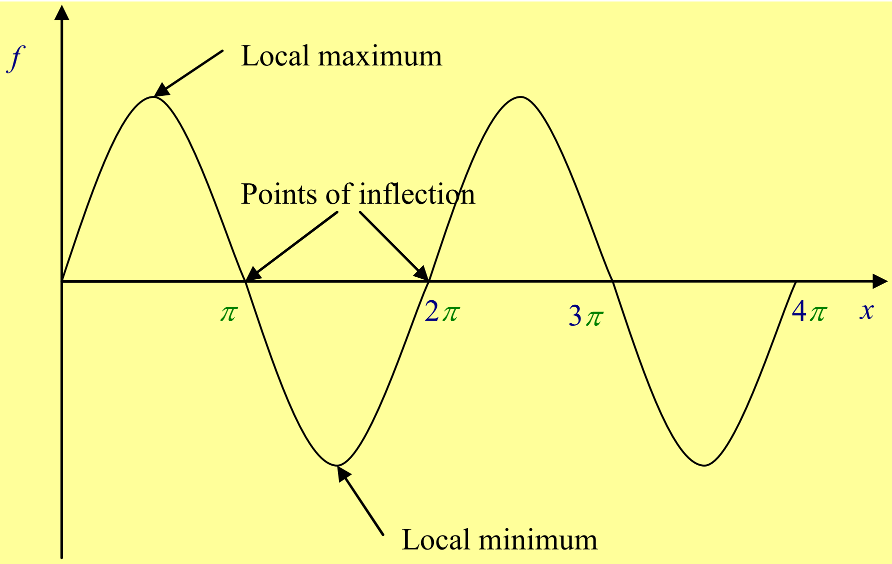
More generally, if $d^{(n)}f/dx^{(n)} = 0$ at $x = x_0$ $\forall$ $n < m$ and $d^{(m)}f/dx^{(m)} \neq 0$ then x_0 is a turning point if m is even, or a point of inflection if m is odd. Further, if m is even, then x_0 is a local maximum if $d^{(m)}f/dx^{(m)} < 0$ or a local minimum if $d^{(m)}f/dx^{(m)} > 0$.

➡ $f(x) = \sin x$

$f(x) = 0$ at $x = n\pi$

$df/dx = \cos x = 0$ at $x = (n + ½)\pi \rightarrow$ turning points

$d^2f/dx^2 = -\sin x = 0$ at $x = n\pi \rightarrow$ inflection points where $f = 0$.



$f(x) = x^3 + 2x^2 + x + 1$

As $x \to \pm\infty$, have $f \to x^3$

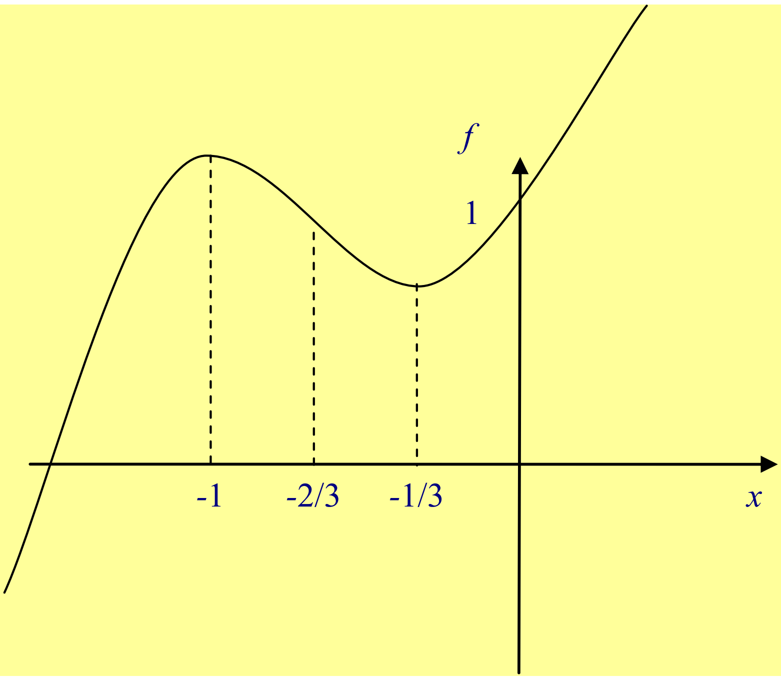
At $x = 0$, $f = 1$

$df/dx = 3x^2 + 4x + 1 = (3x + 1)(x + 1) = 0$ at $x = -1$ and $x = -1/3$.

$d^2f/dx^2 = 6x+4$. At $x = -1$, $d^2f/dx^2 = -2 \Rightarrow$ maximum, $f = 1$.

At $x = -1/3$, $d^2f/dx^2 = 2 \Rightarrow$ minimum, $f = 23/27$.

Point of inflection at $d^2f/dx^2 = 6x+4 = 0 \Rightarrow x = -2/3 \Rightarrow f = 25/27$



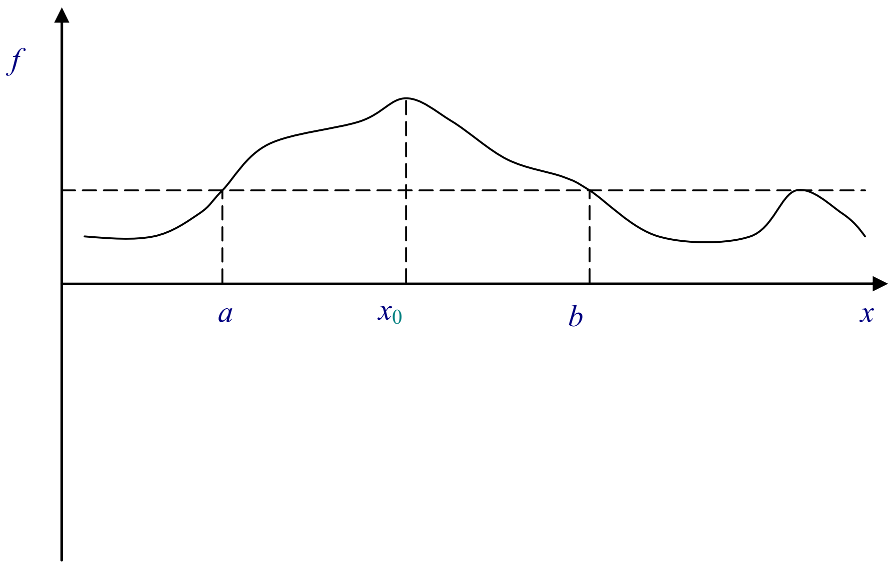
1.3.1 ROLLE'S THEOREM

Let $f(x)$ be a function such that

- $f(x)$ is continuous in $x \in [a,b]$
- $f(x)$ is differentiable in $x \in (a,b)$
- $f(a) = f(b)$

then there exists at least one point $x = x_0 \in (a,b)$ such that $f'(x_0) = 0$.

Proved in Part IB Analysis course, but intuitively obvious:



Consider $\varphi = f(x) - f(a)$ [giving $\varphi = 0$ at $x=a,b$]

Either $\varphi = 0$ everywhere in $[a,b]$, in which case $\varphi' = 0$ in (a,b)

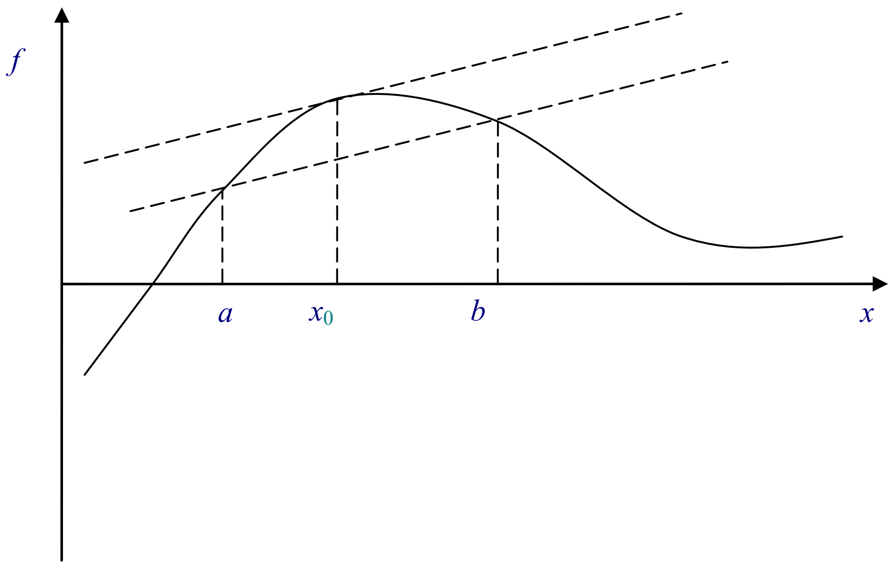
or $\varphi > 0$ somewhere or $\varphi < 0$ somewhere.

If $\varphi > 0$ somewhere then $\varphi' > 0$ in some places and $\varphi' < 0$ in some places as φ must be continuous and differentiable. Hence φ' passes through zero at some point and φ must have a (locally) greatest value at the point.

Note that if $f(x)$ is continuous but not differentiable, then it must still have either a local maximum, a local minimum or be constant. The extremum may occur at a point where $f' = 0$ or at a point where f' is not defined (*e.g.* at a cusp).

1.3.2 Mean value theorem

Let $f(x)$ be continuous in $[a,b]$ and differentiable in (a,b), then $\exists$ x_0 in (a,b) for which $f'(x_0) = \dfrac{f(b) - f(a)}{b-a}$.



This is simply Rolle's theorem relative to the line $y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$.

Define new function $\varphi(x) = f(x) - \frac{f(a)(b-x)+f(b)(x-a)}{b-a} = f - y$ which has $\varphi(a) = \varphi(b) = 0$ and apply Rolle's theorem.

1.3.3 CAUCHY'S FORMULA

Consider two functions $f(x)$ and $g(x)$ which satisfy the conditions of the mean value theorem (*i.e.* continuous in $[a,b]$ and differentiable in (a,b)), and suppose $g'(x) \neq 0$ everywhere in (a,b).

Then, $\exists\ x_0$ in (a,b) such that $\frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

[Note that since $g' \neq 0$, then $g(b) \neq g(a)$.]

Proof: Define $\varphi(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x)$ (so that $\varphi(a) = \varphi(b) = \frac{f(a)g(b)-f(b)g(a)}{g(b)-g(a)}$), and apply Rolle's theorem shows that $\varphi'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x)$ must vanish for some $x = x_0$, thus proving the result.

Note that cannot apply mean value theorem separately to f and g then divide as the corresponding x_0 will not generally be the same.

Corollary 1

Suppose $f(0) = 0$ and $g(0) = 0$,

then $\frac{f(x)}{g(x)} = \frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(\alpha x)}{g'(\alpha x)}$ for some $0 < \alpha < 1$.

Corollary 2

If $f(x_0) = 0$ and $g(x_0) = 0$, then $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$. l'Hôpital's rule. (Sometimes spelt as l'Hospital's rule; the two are equivalent in Old French.)

1.4 Taylor series

Useful to represent some function $f(x)$ as a linear combination (sum) of simpler functions. Particularly important in computing, but also when dealing with more complex functions.

In some cases we might be happy to approximate the function in $x \in [a,b]$ rather than represent it accurately.

- $f(x) \sim \psi_0(x) \equiv c_0$
  - ◊ Can select c_0 so that $f(a) = \psi_0(a)$.
- $f(x) \sim \psi_1(x) \equiv c_0 + c_1(x\text{-}a)$
  - ◊ Could use linear interpolation to have $\psi_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$, which will give $f(x) = \psi_1(x)$ at $x = a$ and $x = b$.
  - ◊ Using the ideas from the mean value theorem (§1.3.2) we may rewrite this as $\psi_1(x) = f(a) + f'(x_0)(x-a)$, for some x_0.
  - ◊ Often more convenient to have function and derivative at same location, using $\psi_1^*(x) = f(a) + f'(a)(x-a)$. This will give $f(a) = \psi_1(a)$, but in general $f(b) \neq \psi_1(b)$. We will, however, have $f'(a) = \psi'_1(a)$.
- $f(x) \sim \psi_2(x) \equiv c_0 + c_1(x-a) + c_2(x-a)^2$
  - ◊ We can apply the mean value theorem to the first derivative, $f''(x_1) = \frac{f'(b)-f'(a)}{b-a}$ for some x_1, and thus $\psi_2(x) = f(a) + f'(x_0)(x-a) + f''(x_1)(x-a)^2$.

More convenient to have all derivatives at the same place, with the approximation improving with the more derivatives that are matched.

Recall from §1.2.4 that

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d^2}{dx^2}x^n = n\frac{d}{dx}x^{n-1} = n(n-1)x^{n-2}$$

$$\frac{d^3}{dx^3}x^n = n(n-1)\frac{d}{dx}x^{n-2} = n(n-1)(n-2)x^{n-3}$$

$$\frac{d^m}{dx^m}x^n = n(n-1)\cdots(n-m+1)x^{n-m} = \frac{n!}{(n-m)!}x^{n-m}$$

$$\frac{d^n}{dx^n}x^n = n! \qquad \text{for integer } n$$

and that x^n, dx^n/dx, d^2x^n/dx^2, … $d^{(n-1)}x^n/dx^{(n-1)}$ all vanish at $x = 0$.

Similarly, $(x-x_0)^n$ and its first $n-1$ derivatives all vanish at $x = x_0$.

1.4.1 TAYLOR'S THEOREM

Consider $\varphi(x) \equiv f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$

[we assume $f^{(n)}$ exists]

Note that for any $m \le n$, this definition gives $\varphi^{(m)}(x_0) = f^{(m)}(x_0)$ since $\frac{1}{m!}\frac{d^m}{dx^m}(x-x_0)^m = \frac{m!}{m!} = 1$

Define the *remainder* $R_n(x;x_0)$ such that $f(x) = \varphi(x) + R_n(x;x_0)$.

This is generally called Taylor's theorem, but is actually a tautology. Its usefulness is in ignoring R_n if R_n is *small*, and thus approximating $f(x)$ with just $\varphi(x)$.

The remainder $R_n(x;x_0)$ may be considered the error in the approximation $f(x) \approx \varphi(x)$, when $\varphi(x)$ contains all derivatives up to and including the n^{th} at the point x_0.

For *smooth* functions, R_n is small if $x - x_0$ is sufficiently small.

1.4.2 LAGRANGE ESTIMATE OF REMAINDER

There are several ways of estimating R_n. One of the more useful is Lagrange's estimate:

$$R_n(x;x_0) = \frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}\left(x+\theta(x_0-x)\right),$$

where $\theta \in [0,1]$. The residual is equal to the next term in the series with the $f^{(n+1)}$ derivative evaluated at some point between x and x_0.

This estimate comes from repeated application of Corollary 1 of Cauchy's formula to $\frac{R_n(x;x_0)}{(x-x_0)^{n+1}/(n+1)!} = \frac{f(x)-\varphi(x)}{(x-x_0)^{n+1}/(n+1)!}$, noting that $\varphi^{(n+1)} = 0$.

Let $h = x - x_0$ so that $\frac{f(x)-\varphi(x)}{(x-x_0)^{n+1}/(n+1)!} = \frac{f(x_0+h)-\varphi(x_0+h)}{h^{n+1}/(n+1)!}$

and define $R_n(x,x_0) = \psi(h) = f(x_0+h) - \varphi(x_0+h)$ and $\chi(h) = h^{n+1}/(n+1)!$

Now

$$\psi(0) = f(x_0) - \varphi(x_0) = 0 \qquad \chi(0) = 0$$
$$\psi'(0) = f'(x_0) - \varphi'(x_0) = 0 \qquad \chi'(0) = 0$$
$$\ldots \qquad \ldots$$
$$\psi^{(n)}(0) = 0 \qquad \chi^{(n)}(0)$$
$$\psi^{(n+1)}(0) = f^{(n+1)}(x_0) \qquad \text{and } \chi^{(n+1)}(h) = 1\ \forall\ h.$$

since $\varphi^{(n+1)}(x) = 0$.

Cauchy's formula gives

$$\frac{\psi(h)}{\chi(h)} = \frac{\psi(h)-\psi(0)}{\chi(h)-\chi(0)} = \frac{\psi'(h_1)}{\chi'(h_1)} \qquad \text{where } 0 < h_1 < h$$

but

$$\frac{\psi'(h_1)}{\chi'(h_1)} = \frac{\psi'(h_1)-\psi(0)}{\chi'(h_1)-\chi(0)} = \frac{\psi''(h_2)}{\chi''(h_2)} \qquad \text{where } 0 < h_2 < h_1 < h$$

$$\cdots$$

$$\Rightarrow \frac{\psi(h)}{\chi(h)} = \frac{\psi'(h_1)}{\chi'(h_1)} = \frac{\psi''(h_2)}{\chi''(h_2)} = \cdots = \frac{\psi^{(n+1)}(h_{n+1})}{\chi^{(n+1)}(h_{n+1})} = \frac{f^{(n+1)}(x_0+h_{n+1})}{1} \qquad 0 < h_{n+1} < h_n < \cdots < h_2 < h_1 < h$$

whence $R_n(x;x_0) = \psi(h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(x_0+h_n) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0+\theta h)$ with $0 < \theta < 1$.

QED.

1.4.3 Taylor series examples

If **all** the derivatives of f exist and are finite, then one can take the limit $n \to \infty$.

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$$

This will provide an accurate representation of $f(x)$ in the neighbourhood of x_0 provided $R_n(x;x_0) \to 0$ as $n \to \infty$. [This is a necessary but not sufficient condition for the summation to converge.] For some functions, convergence is achieved for any value of x, while for others, x may need to be very close to x_0.

➡ $f(x) = e^x$

$f' = e^x, \cdots, f^{(n)} = e^x$.

Expand about $x = 0$ [f and all its derivatives are unity at $x = 0$].

$$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^{\theta x}$$

Note that for e^x the series converges for all x in limit $n \to \infty$

- Mclaren series

➡ $f(x) = \sin x$

$f = \sin x = 0$ at $x = 0$,

$f' = \cos x = 1$ at $x = 0$,

$f'' = -\sin x = 0$ at $x = 0$,

$f''' = -\cos x = -1$ at $x = 0$,

…

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

[Can also derive from $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$ using expansion for e^x]

- $f(x) = \cos x$

…

$$\Rightarrow \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

- $f(x) = e^{-1/x^2}$

$f(0) = 0$

$$f(0) = \lim_{x\to 0} e^{-1/x^2} = \lim_{y\to\pm\infty} e^{-y^2} = 0$$

$f'(0) = 0$

$$f'(0) = \lim_{x\to 0}\frac{2e^{-1/x^2}}{x^3} = \lim_{x\to 0}\frac{4e^{-1/x^2}}{3x^5} = \lim_{x\to 0}\frac{8e^{-1/x^2}}{15x^7}$$
$$= \lim_{x\to 0}\frac{2^n e^{-1/x^2}}{1\times 3\times\cdots\times(2n-1)x^{2n+1}}$$

$f''(0) = 0$

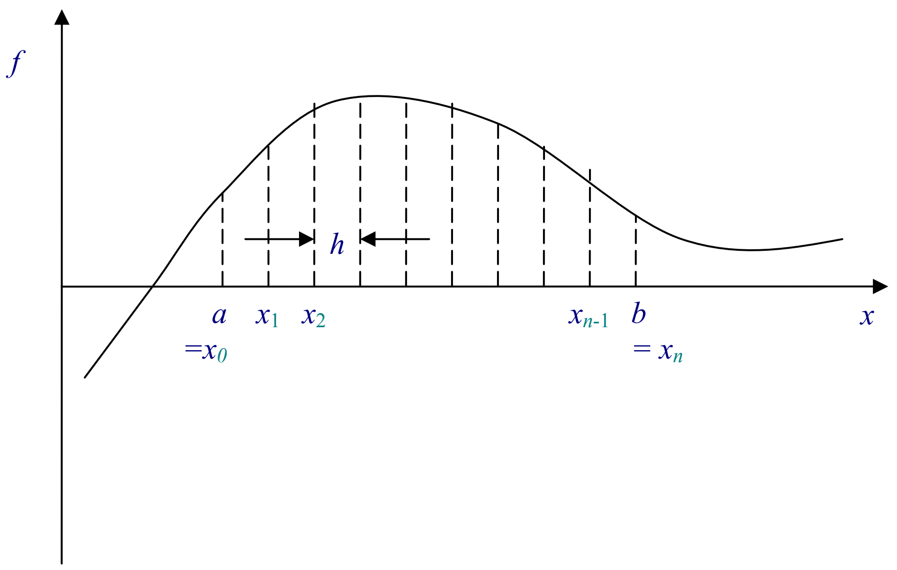
…

which suggests $f(x) = \Sigma\, 0\times x^n = 0$

This only works asymptotically close to $x = 0$.

1.5 Integration: fundamentals

1.5.1 Integration as sum of areas



"dx_r" $= x_r - x_{r-1} = h$, and $nh = b - a$.

Can approximate area as trapezoids passing through $f_n = f(x_n)$:

$$\int_a^b f(x)\,dx \approx \sum_{r=1}^{n} \tfrac{1}{2}(f_r + f_{r+1})h = \left(\tfrac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-1} + \tfrac{1}{2}f_n\right)h$$

This approximation is frequently referred to as the Trapezium Rule and may be used to estimate the integral, for example in a computer code. There are, however, better ways.

If we let $h \to 0$ ($n \to \infty$)

$$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=1}^{n} \tfrac{1}{2}(f_r + f_{r+1})h$$
$$= \lim_{h\to 0}\left(\tfrac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-1} + \tfrac{1}{2}f_n\right)h$$

Since $\lim_{h\to 0} \tfrac{1}{2}(f_0 + f_n)h = 0$, then can rewrite

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\left(f_0 + f_1 + f_2 + \cdots + f_{n-1} + f_n\right)h = \lim_{h\to 0}\sum_i f_i dx_i \equiv \int f\,dx$$

It is obvious that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and $$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b \left[f(x) + g(x)\right]dx$$

➡ $f(x) = x$

$$\int_0^X x\,dx = \lim_{h\to 0} h\left(0 + x_1 + x_2 + \cdots + \underbrace{x_n - \tfrac{1}{2}x_n}_{=+\frac{1}{2}x_n}\right)$$, where $x_i = ih$ and $n = X/h$.

$$= \lim_{h\to 0} h^2\left(0 + 1 + 2 + \cdots + (n-1) + n - \tfrac{1}{2}n\right)$$
$$= \lim_{h\to 0} h^2\left(\frac{n(n+1)}{2} - \tfrac{1}{2}n\right)$$
$$= \lim_{h\to 0} \tfrac{1}{2}n^2h^2$$
$$= \tfrac{1}{2}X^2$$

Note that $\frac{d}{dX}\left(\tfrac{1}{2}X^2\right) = X$, which anticipates that integration is the inverse of differentiation.

➡ $f(x) = e^x$

$$\int_0^X e^x\,dx = \lim_{h\to 0} h\left(e^0 + e^{x_1} + e^{x_2} + \cdots + e^{x_n}\right)$$
$$= \lim_{h\to 0} h\left(1 + e^h + \left(e^h\right)^2 + \cdots + \left(e^h\right)^n\right)$$

Recall that for a geometric series $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$; $r \neq 1$, so

$$\int_0^X e^x \, dx = \lim_{h\to 0} h \frac{1-(e^h)^{n+1}}{1-e^h}$$

$$= \lim_{h\to 0} h \frac{e^X e^h - 1}{e^h - 1}$$ noting $e^X e^h \to e^X$ and $e^h - 1 \to h$ as $h \to 0$

$$= (e^X - 1) \lim_{h\to 0} \frac{h}{e^h - e^0}$$

$$= e^X - 1$$

Further on examples sheet.

End of Lecture 3

1.5.2 First mean-value theorem for integrals

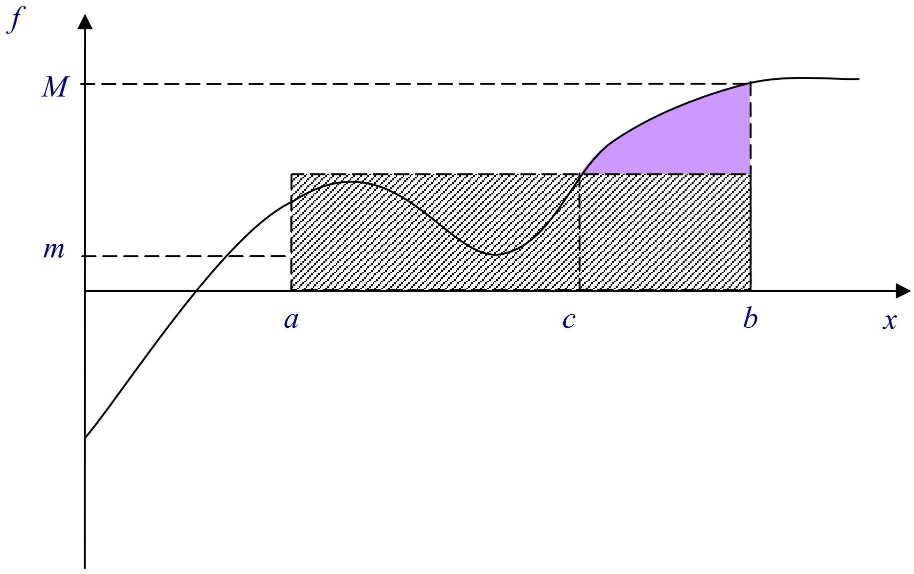
If $m \leq f(x) \leq M$ in (a,b)

then $$(b-a)m \leq \int_a^b f(x) \, dx \leq (b-a)M$$

Corollary

If $f(x)$ is continuous in (a,b) then $\exists\, c$: $a < c < b$ such that

$$\int_a^b f(x) \, dx = (b-a) f(c)$$



1.6 Fundamental theorem of calculus

If $f(x)$ is continuous in (a,x), and if $F(x) = \int_a^x f(t)\, dt$, then $\frac{dF}{dx} = f(x)$.

Proof: $\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt$

More rigorously:

First mean value theorem for integrals gives

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\, dt = hf(x+\theta h), \text{ where } 0 < \theta < 1$$

giving $\int_x^{x+h} f(t)\, dt \to hf(x)$ as $h \to 0$.

Thus $\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\, dt = f(x+\theta h) \to f(x)$, QED.

The function F whose derivative is $f(x)$ is called a *primitive* of f. Note that f will have more than one primitive: if F is a primitive, so is F + *const*.

The primitive is often called the *indefinite integral* and is written as

$$\int^x f(t)\, dt \quad \text{or more simply as} \quad \int f(x)\, dx$$

because changing the lower limit of integration simply adds a constant.

The definite integral specifies both limits:

$$\int_a^b f\ dx = \int_a^b \frac{dF}{dx}\ dx = F(b) - F(a) \equiv F(x)\Big|_a^b \equiv \left[F(x)\right]_a^b$$

$$\begin{aligned}\int_a^b f\ dx &= \lim_{h\to 0} h\left(f_0 + f_1 + f_2 + \cdots f_n\right) \\ &= \lim_{h\to 0} h \sum_{k=1}^{n} \frac{F_k - F_{k-1}}{h} \\ &= \sum_{k=1}^{n}\left(F_k - F_{k-1}\right) \\ &= F_n - F_0 \\ &= F(b) - F(a)\end{aligned}$$

1.7 Toolkit for basic integration

Differentiation	Integration
$\frac{d}{dx} x^n = nx^{n-1}$	$\int x^n\ dx = \frac{1}{n+1} x^{n+1}$; $n \neq -1$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\ dx = \sin x$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\ dx = -\cos x$
$\frac{d}{dx} e^x = e^x$	$\int e^x\ dx = e^x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\ dx = \tan x$
$\frac{d}{dx}\sinh x = \cosh x$	$\int \cosh x\ dx = \sinh x$

What about integral of x^{-1}?

Consider binomial expansion, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ (only converges for $x \in (-1,1]$)

$$\int \frac{1}{1+x}\ dx = \int 1 - x + x^2 - x^3 + \cdots\ dx$$

Integrate this term-by-term $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ (only converges for $x \in (-1,1]$)

$$= \ln(1+x) = \log_e(1+x)$$

More generally, the definition of the natural logarithm gives $\ln x = \int_0^x \frac{1}{t}\ dt$ for $x > 0$, hence $\frac{d}{dx}\ln x = \frac{1}{x}$.

1.8 Integration of more complex functions

In general we shall encounter more complex functions than those listed in §1.7, so we need to know how to deal with them. The main approach is to transform an integral into a form you can recognise and know the integral of.

There are some standard techniques you should know, and often more than one way of solving the problem. Knowing the best approach is largely a matter of practice.

1.8.1 Integration by parts

Useful approach when the integrand is the product of recognisable functions.

$$\int_a^b f\frac{dg}{dx}\,dx = [fg]_a^b - \int_a^b \frac{df}{dx}g\,dx \quad \text{or} \quad \int fg'\,dx = fg - \int f'g\,dx + const$$

Proof (obvious from differentiation $(fg)' = f'g + fg'$):

$$\int_a^b \left(f\frac{dg}{dx} + g\frac{df}{dx}\right)dx = \int_a^b \frac{d}{dx}(fg)\,dx = [fg]_a^b, \text{ by fundamental theorem.}$$

➡ $\int x\cos x\,dx$

Let $f = x$ and $g' = \cos x \Rightarrow f' = 1$ and $g = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + const$$

Note: if we had selected $f = \cos x$ and $g' = x$, then integration by parts would have given

$\int x\cos x\,dx = \frac{1}{2}x^2\cos x + \int \frac{1}{2}x^2\sin x\,dx$, which is a more complex form.

➡ $I_n \equiv \int x^n\cos x\,dx$

Need to apply integration by parts repeatedly.

$$I_n \equiv \int x^n\cos x\,dx = x^n\sin x - \int nx^{n-1}\sin x\,dx$$

$$= x^n\sin x + n(n-1)x^{n-2}\cos x - \int n(n-1)x^{n-2}\cos x\,dx$$

$$= x^n\sin x + n(n-1)x^{n-2}\cos x - n(n-1)I_{n-2}$$

Have recurrence relation for I_n. Need to know $I_0 \equiv \int \cos x\,dx = \sin x$ (for I_n with n even) and $I_1 = \int x\cos x\,dx = x\sin x + \cos x$ (for n odd) from earlier example.

Note that as I_n is an indefinite integral, then there is also an arbitrary constant.

1.8.2 Integration by substitution

When integrating $\int f(x)\,dx$, suppose we choose to express x as a function of some other variable s, *i.e.*, $x = x(s)$ and consider $\int f(x(s))\,dx$. Now $dx = \frac{dx}{ds}ds$, so

$$\int f(x(s))\,dx = \int f\frac{dx}{ds}\,ds$$

This is desirable if the new integrand, $f\dfrac{dx}{ds}$ is a simpler function of s which we know how to integrate.

➧ $I = \int (x+a)^n \, dx$

Suppose $x + a = s \Leftrightarrow x = s - a$

Then $dx = (dx/ds)\, ds = ds$

$$I = \int (x+a)^n \, dx = \int s^n \, ds = \frac{1}{n+1} s^{n+1} + const = \frac{1}{n+1}(x+a)^{n+1} + const$$

In this (trivial) example, the substitution effectively translated the x axis.

If dealing with a definite integral, then need to remember to transform the limits:

$$\int_{x_0}^{x_1} (x+a)^n \, dx = \int_{x_0+a}^{x_1+a} s^n \, ds = \left[\frac{1}{n+1} s^{n+1}\right]_{x_0+a}^{x_1+a} = \frac{1}{n+1}\left[(x_1+a)^{n+1} - (x_0+a)^{n+1}\right]$$

➧ $I = \int \frac{1}{\sqrt{1-x^2}} \, dx$

This only makes sense (with real arithmetic) when $|x| \le 1$.

Noting that $\cos^2\theta + \sin^2\theta = 1$ suggests the substitution $x = \sin\,\theta \Rightarrow dx = \cos\,\theta\, d\theta$

hence $\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta = \int \frac{\cos\theta}{\sqrt{\cos^2\theta}} d\theta = \theta = \sin^{-1} x$

(omitting the arbitrary constant)

Common substitutions

$1 - \sin^2\theta = \cos^2\theta$	$1 - x^2$	$\to x = \sin\,\theta$ if $\lvert x\rvert < 1$
	better than using $x = \cos\,\theta$ since $dx/ds > 0$ in $(0,\pi/2)$	
$1 + \tan^2\theta = \sec^2\theta$	$1 + x^2$	$\to x = \tan\,\theta$ no limit on x
$1 - \tanh^2\theta = \text{sech}^2\theta$	$1 - x^2$	$\to x = \tanh\theta$
$\cosh^2\theta - 1 = \sinh^2\theta$	$x^2 - 1$	$\to x = \cosh\theta$
$1 + \sinh^2\theta = \cosh^2\theta$	$x^2 + 1$	$\to x = \sinh\theta$

These combinations often arise from Cartesian geometry as a consequence of the theorem of Pythagoras.

➧ $I = \int \frac{dx}{1+x^2}$

We could use $x = \tan\theta$ or $x = \sinh\varphi$.

Option 1: Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\Rightarrow \quad I=\int \frac{dx}{1+x^2}=\int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta=\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta=\theta=\tan^{-1}x$$

Option 2: Let $x = \sinh\varphi \Rightarrow dx = \cosh\varphi\, d\varphi$

$$\Rightarrow \quad I=\int \frac{dx}{1+x^2}=\int \frac{\cosh\varphi}{1+\sinh^2\varphi}\,d\varphi=\int \frac{\cosh\varphi}{\cosh^2\varphi}\,d\varphi=\int \frac{d\varphi}{\cosh\varphi}=\int\sqrt{1-\tanh^2\varphi}\,d\varphi$$

Obviously we need to do some further manipulations.

Now let $\tanh\varphi = \sin\xi \Rightarrow \operatorname{sech}^2\varphi\, d\varphi = \cos\xi\, d\xi \Rightarrow\ d\varphi = \cosh^2\varphi\cos\xi\, d\xi$

Note that $\tanh\varphi = \sin\xi \;\Rightarrow\; \sinh^2\varphi = \cosh^2\varphi\sin^2\xi \;\Rightarrow\; 1 - \cosh^2\varphi = \cosh^2\varphi\sin^2\xi \;\Rightarrow$ $\cosh^2\varphi\,(1-\sin^2\xi) = 1 \Rightarrow \cosh^2\varphi\cos^2\xi = 1 \Rightarrow\ d\varphi = \cosh^2\varphi\cos\xi\, d\xi = \dfrac{1}{\cos^2\xi}\cos\xi\, d\xi = \dfrac{d\xi}{\cos\xi}$

$$\Rightarrow \quad \begin{aligned} I&=\int \frac{dx}{1+x^2}=\int\sqrt{1-\tanh^2\varphi}\,d\varphi=\int\frac{\sqrt{1-\sin^2\xi}}{\cos\xi}\,d\xi=\int\frac{\cos\xi}{\cos\xi}\,d\xi\\ &=\xi=\sin^{-1}\tanh\varphi=\sin^{-1}\frac{\sinh\varphi}{\sqrt{1+\sinh^2\varphi}}=\sin^{-1}\frac{x}{\sqrt{1+x^2}}\end{aligned}$$

Two different answers… Or are they?

$$\xi=\sin^{-1}\frac{x}{\sqrt{1+x^2}}=\sin^{-1}\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}=\sin^{-1}\frac{\sin\theta}{\sqrt{\cos^2\theta+\sin^2\theta}}=\sin^{-1}\sin\theta=\theta=\tan^{-1}x$$

$\Rightarrow$ They are the same!

1.9 Differentiation under integrals

Consider the definite integral $F(x)=\int_a^b f(t,x)\,dt$ that depends on a parameter x.

Now, in the case when the limits a and b do not depend on x, then

$$\begin{aligned}\frac{dF}{dx}&=\lim_{h\to 0}\frac{1}{h}\left[\int_a^b f(t,x+h)\,dt-\int_a^b f(t,x)\,dt\right]\\ &=\int_a^b \lim_{h\to 0}\frac{1}{h}\left[f(t,x+h)-f(t,x)\right]dt\\ &=\int_a^b \frac{\partial f}{\partial x}\,dt\end{aligned}$$

from our definition of the partial derivative in §1.2.6.

➡ $F(x)=\int_0^\pi x\sin\frac{t}{x}\,dt$

Integrate $F(x)=\int_0^\pi x\sin\frac{t}{x}\,dt=\left[-x^2\cos\frac{t}{x}\right]_0^\pi=-x^2\left(\cos\frac{\pi}{x}-\cos\frac{0}{x}\right)=x^2\left(1-\cos\frac{\pi}{x}\right)$

then differentiate
$$\frac{dF}{dx} = \frac{d}{dx}\left[x^2\left(1-\cos\frac{\pi}{x}\right)\right] = 2x\left(1-\cos\frac{\pi}{x}\right) + x^2\left(-\frac{\pi}{x^2}\right)\sin\frac{\pi}{x}$$
$$= 2x\left(1-\cos\frac{\pi}{x}\right) - \pi\sin\frac{\pi}{x}$$

Alternatively, differentiate then integrate:

$$\frac{dF}{dx} = \int_0^\pi \frac{\partial}{\partial x}\left(x\sin\frac{t}{x}\right) dt = \int_0^\pi \sin\frac{t}{x} + x\left(-\frac{t}{x^2}\right)\cos\frac{t}{x}\, dt$$
$$= \int_0^\pi \sin\frac{t}{x} - \frac{t}{x}\cos\frac{t}{x}\, dt$$
$$= \left[-x\cos\frac{t}{x} - t\sin\frac{t}{x}\right]_0^\pi + \int_0^\pi \sin\frac{t}{x}\, dt$$
$$= \left[-x\cos\frac{t}{x} - t\sin\frac{t}{x} - x\cos\frac{t}{x}\right]_0^\pi$$
$$= 2x\left(1-\cos\frac{\pi}{x}\right) - \pi\sin\frac{\pi}{x}$$

These are the same! QED.

If $a = a(x)$ and $b = b(x)$, then let $\hat{F}(x,T) = \int_0^T f(t,x)\, dt$ so we may write

$$F(x) = \hat{F}(x,b(x)) - \hat{F}(x,a(x)) = \int_{a(x)}^{b(x)} f(t,x)\, dt$$

Applying the chain rule

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial a}\frac{da}{dx} + \frac{\partial F}{\partial b}\frac{db}{dx}$$
$$= \frac{\partial}{\partial x}\left(\hat{F}(x,b(x)) - \hat{F}(x,a(x))\right) - \frac{\partial \hat{F}(x,a)}{\partial a}\frac{da}{dx} + \frac{\partial \hat{F}(x,b)}{\partial b}\frac{db}{dx}.$$
$$= \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}\, dt - f(x,a)\frac{da}{dx} + f(x,b)\frac{db}{dx}$$

➡ $F(x) = \int_x^{x^2} (x-t)^2\, dt$

Integrate: $F(x) = \int_x^{x^2} (x-t)^2\, dt = \left[\frac{1}{3}(t-x)^3\right]_x^{x^2} = \frac{1}{3}x^3(x-1)^3$

then differentiate
$$\frac{dF}{dx} = \frac{d}{dx}\left[\frac{1}{3}x^3(x-1)^3\right] = x^2(x-1)^3 + x^3(x-1)^2$$
$$= x^2(x-1)^2(2x-1)$$

Differentiating then integrating

$$\begin{aligned}\frac{dF}{dx} &= \int_x^{x^2} \frac{\partial}{\partial x}(x-t)^2\,dt - (x-t)^2\Big|_{t=x}\frac{dx}{dx} + (x-t)^2\Big|_{t=x^2}\frac{dx^2}{dx} \\ &= \int_x^{x^2} 2(x-t)\,dt - (x-x)^2 + \left(x-x^2\right)^2 2x \\ &= \left[-(x-t)^2\right]_x^{x^2} + 2x^3(x-1)^2 \\ &= -x^2(x-1)^2 + 2x^3(x-1) \\ &= x^2(x-1)^2(2x-1)\end{aligned}$$

1.10 Multiple integrals

1.10.1 Integration over a rectangle

We frequently want to integrate functions of more than one variable. For example, if $h(x,y)$ is the height of a pile of grain in the region $a \le x \le b$, $c \le y \le d$, we may wish to know the volume V of grain.

$$V = \int_a^b \left[\int_c^d h(x,y)\,dy\right] dx$$

If c and d are independent of x then this is the same as

$$V = \int_c^d \left[\int_a^b h(x,y)\,dx\right] dy$$

This equivalence is sometimes referred to as Fubini's theorem for a rectangle. Proof can be constructed in a number of ways, but is obvious by noting that integration is the limit of summation and $\sum_i \sum_j h(x_i, y_j) = \sum_j \sum_i h(x_i, y_j)$.

We may also write the above *repeated integrals* as a *double integral*

$$V = \int_a^b \left[\int_c^d h(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b h(x,y)\,dx\right] dy = \iint_A h(x,y)\,dA$$

where A is the area $a \le x \le b$, $c \le y \le d$, and dA here is $dx\,dy$.

Note that the above integrals will often not be written with the square brackets and may be written in a number of different but equivalent ways. For example

$$\int_a^b \left[\int_c^d h(x,y)\,dy\right] dx = \int_a^b \int_c^d h(x,y)\,dy\,dx = \int_a^b dx \int_c^d dy\; h(x,y)$$

1.10.2 Integration in a general region

Of course, the region over which we wish to integrate a function will not in general be rectangular. If we know that the area A of interest falls within $a \le x \le b$ and that $c(x) \le y \le d(x)$, then

$$V = \iint_A h(x,y)\,dA = \int_a^b \left[\int_{c(x)}^{d(x)} h(x,y)\,dy \right] dx$$

Alternatively if we knew that y was in the range $[c,d]$ and in this range we had $a(y) \le x \le b(y)$, then

$$V = \iint_A h(x,y)\,dA = \int_c^d \left[\int_{a(y)}^{b(y)} h(x,y)\,dx \right] dy$$

Integrate $f(x,y) = x^2 + y^2$ in the triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

Can express area as $0 \le x \le 1$ with $0 \le y \le 1 - x$, so

$$V = \iint_A x^2 + y^2\,dA = \int_0^1 \int_0^{1-x} x^2 + y^2\,dy\,dx$$

$$= \int_0^1 \left[x^2 y + \tfrac{1}{3} y^3 \right]_0^{1-x} dx = \int_0^1 \frac{1}{3} - x + 2x^2 - \frac{4}{3}x^3\,dx$$

$$= \left[\frac{1}{3}x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 \right]_0^1 = \frac{1}{6}$$

Alternatively, can express area as $0 \le y \le 1$ with $0 \le x \le 1 - y$, which yields the same result.

1.10.3 Integration of functions of more variables

The ideas for double integration and repeated integration may, of course, be extended further to integrate over three or more variables.

1.11 Special functions

1.11.1 Heaviside step function

$$H(x) = \begin{cases} 0 & x < 0 \\ ? & x = 0 \\ 1 & x > 0 \end{cases}$$

1.11.2 Dirac delta function

Limit of off-on-off as width tends to zero.

$$\delta(x) = \begin{cases} 0 & x < 0 \\ \infty & x = 0 \\ 0 & x > 0 \end{cases}; \quad \int_{-\infty}^{\infty} \delta(x)\,dx = \int_{0_-}^{0_+} \delta(x)\,dx = 1$$

$$\delta(x) = \frac{dH}{dx}; \quad H(x) = \int_{-\infty}^{x} \delta(t)\,dt$$

End of Lecture 4