2 First order linear equations

Suppose the dependent variable y is a function of the independent variable t. Then

$f(y,y',y'',\ldots,y^{(n)};t) = 0$ is an ordinary differential equation of order n.

Order of equation $\Rightarrow$ the order of the highest derivative.

Ordinary differential equation (ODE) $\Rightarrow$ y is a function of only one independent variable. [A *partial differential equation* (PDE) has more than one independent variables.]

Linear ODE $\Rightarrow$ differential equation may be written as a linear function of y and its derivatives:

$\psi(t) + \varphi_0(t)\, y(t) + \varphi_1(t)\, y'(t) + \varphi_2(t)\, y''(t) + \ldots = 0$

Not that coefficients φ_i may be functions of the independent variable t.

Homogeneous equation $\Rightarrow$ $\psi(t) = 0$

Inhomogeneous equation $\Rightarrow$ $\psi(t) \neq 0$

Linear first order ordinary differential equation

$\psi(t) + \varphi_0(t)\, y(t) + \varphi_1(t)\, y'(t) = 0$

2.1 Equations with constant coefficients

$ay' + by + c = 0$, where $a,b,c \in const$

2.1.1 Homogeneous equation

$$c = 0 \Rightarrow \quad a\frac{dy}{dt} + by = 0$$

Can rearrange $\frac{1}{y}\frac{dy}{dt} = -\frac{b}{a}$ and integrate with respect to t

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int \frac{1}{y}\,dy = \int -\frac{b}{a}\,dt \quad \Rightarrow \quad \ln|y| = -\frac{b}{a}t + c \quad \Rightarrow \quad y = \pm e^{c} e^{-bt/a} = Ae^{-bt/a}\,.$$

2.1.2 Inhomogeneous equation

$$\frac{dy}{dt} + \kappa y = \eta \quad \Rightarrow \quad \frac{dy}{dt} + \kappa\left(y - \frac{\eta}{\kappa}\right) = 0$$

If we replace $y-\eta/\kappa$ with z, noting $dz/dt = dy/dt$, then $\frac{dz}{dt} + \kappa z = 0$

so $z = Ae^{-\alpha t} \quad \Rightarrow \quad y = \frac{\eta}{\kappa} + Ae^{-\kappa t}$. Here, A is an *unknown* constant of integration.

Note that $y = \eta/\kappa$ is the steady solution of the ode. This is referred to as the *particular integral*.

The general solution is the sum of the homogeneous solution and the particular integral. As we shall see later, the sum of the solution to the homogeneous equation and the particular integral is the *most general* solution.

➡ $\frac{dy}{dt}+\kappa y=t$

The complementary function (solution of homogeneous problem) is $A\,e^{-\kappa t}$. For the particular integral, try $y = a + bt$ and substitute:

$$b + \kappa(a + bt) = t.$$

Equating coefficients $\kappa\, b\, t = t \Rightarrow b = 1/\kappa$

$$b + \kappa\, a = 0 \Rightarrow a = -1/\kappa^2$$

The general solution $y = A\,e^{-\kappa t} - 1/\kappa^2 + t/\kappa.$.

➡ $\frac{dy}{dt}+\kappa y=e^{-\beta t}$

Try $y = a\,e^{-\beta t} \Rightarrow$ $-\beta a\,e^{-\beta t} + \kappa\, a\,e^{-\beta t} = e^{-\beta t}$ $a = 1/(\kappa-\beta)$

and the general solution $y = Ae^{-\kappa t}+\frac{e^{-\beta t}}{\kappa-\beta}=e^{-\kappa t}\left(A+\frac{e^{(\kappa-\beta)t}}{\kappa-\beta}\right).$

➡ $\frac{dy}{dt}+\kappa y=e^{-\kappa t}$

In analogy with previous example, could try $y = e^{-\kappa t}$, but this is a solution to the homogeneous equation, and so would not satisfy the right-hand side… Try instead $y = a\,t\,e^{-\kappa t}$:

$$(a\,e^{-\kappa t} - a\,\kappa\,t\,e^{-\kappa t}) + a\,\kappa\,t\,e^{-\kappa t} = e^{-\kappa t}$$

$\Rightarrow$ $a = 1$

and the general solution $y = A\,e^{-\kappa t} + t\,e^{-\kappa t} = (A + t)\,e^{-\kappa t}$

We can see why the solution has this form by considering the previous example in the limit $\beta \to \kappa$. In this limit

$$\lim_{\beta\to\kappa}\left[A+\frac{e^{(\kappa-\beta)t}}{\kappa-\beta}\right]=\lim_{\beta\to\kappa}\left[A+\frac{1+(\kappa-\beta)t+\cdots}{\kappa-\beta}\right]=\lim_{\beta\to\kappa}\left[A+\frac{1}{\kappa-\beta}\right]+t\,.$$

Since A is arbitrary, then we may absorb the $1/(\kappa-\beta)$ into the constant, and so obtain the particular integral $t\,e^{-\kappa t}$.

2.1.3 Initial conditions

Also referred to as boundary conditions. Required to determine the unknown constant of integration.

A first order equation gives one constant of integration, so one initial/boundary condition is required.

Suppose $y = y_0$ at $t = t_0$, then $y_0=\frac{\eta}{\kappa}+Ae^{-\kappa t_0} \Rightarrow A=\left(y_0-\frac{\eta}{\kappa}\right)e^{-\kappa t_0} \Rightarrow y=\frac{\eta}{\kappa}+\left(y_0-\frac{\eta}{\kappa}\right)e^{\kappa(t-t_0)}$

2.2 Difference equations

Recall that for a differential equation we sought some continuous relationship between the dependent variable y and the independent variable x that was defined by the differential equation $f(y,y',y'',\ldots,y^{(n)};t) = 0$. We could equally have written this as $y^{(n)} = g(y,y',y'',\ldots,y^{(n-1)};t)$.

For a difference equation instead of having a continuous variable y depending on a continuous variable x, we have a discrete sequence of values z_n, with the index n playing the role of the independent variable. We may then write the m^{th} order difference equation as

$$z_n = \gamma(z_{n-1},z_{n-2},\ldots,z_{n-m}).$$

The order of the difference equation indicates the number of previous values of z_n the next value depends on. A first order difference equation, $z_n = \gamma(z_{n-1})$, uses only one previous value.

The difference equation is linear if γ is linear in z_{n-1}, z_{n-2}, ….

The most general first order linear difference equation is therefore

$$z_n = \lambda_{n-1}\, z_{n-1} + \beta_{n-1}.$$

Note that the coefficients λ_{n-1} and β_{n-1} need not be constant from one n to the next, in the same way that the coefficients of a linear first order ordinary differential equation may depend on the independent variable x (although we have not yet covered how to solve this).

Suppose our initial condition for the first order difference equation is that z_0 is known. Hence

$$z_1 = \lambda_0\, z_0 + \beta_0,$$
$$z_2 = \lambda_1\, z_1 + \beta_1 = \lambda_1(\lambda_0\, z_0 + \beta_0) + \beta_1$$
$$etc.$$

Can, at least in principle, easily calculate this using a computer. Note, however, that in some circumstances the solution may be unstable due to round-off error in the calculation.

2.2.1 LINEAR DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS

Constant coefficients ⇒ λ and β are independent of n:

$$z_n = \lambda\, z_{n-1} + \beta$$

⇒

$$z_0 \text{ known}$$
$$z_1 = \lambda\, z_0 + \beta$$
$$z_2 = \lambda\, z_1 + \beta = \lambda^2 z_0 + (1+\lambda)\beta$$
$$\cdots$$
$$z_n = \lambda z_{n-1} + \beta = \lambda^n z_0 + \left(1+\lambda+\lambda^2+\cdots+\lambda^{n-1}\right)\beta$$
$$= \lambda^n z_0 + \frac{1-\lambda^n}{1-\lambda}\beta$$
$$= \frac{\beta}{1-\lambda} + \left(z_0 - \frac{\beta}{1-\lambda}\right)\lambda^n$$

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r}; \quad r \neq 1$$

2.2.2 RELATIONSHIP WITH ODE

We can see the relationship with a *first order ode with constant coefficients* by considering $\frac{dy}{dt}+\kappa y=0 \;\rightarrow y=A\,e^{-\kappa t}$. Suppose we are interested in the solution at $t_n = t + n\,\Delta t$, then

$$y_0 = y(t) = A\,e^{-\kappa t}$$

$$y_1 = y(t+\Delta t) = A\,e^{-\kappa(t+\Delta t)} = e^{-\kappa\Delta t}\,(A\,e^{-\kappa t}) = \lambda\,(A\,e^{-\kappa t})$$

where $\lambda = e^{-\kappa\Delta t}$

$$y_2 = y(t+2\Delta t) = A\,e^{-\kappa(t+2\Delta t)} = e^{-2\kappa\Delta t}\,(A\,e^{-\kappa t}) = \lambda^2\,(A\,e^{-\kappa t})$$

$$\ldots$$

$$y_n = y(t+n\Delta t) = A\,e^{-\kappa(t+n\Delta t)} = \lambda^n\,(A\,e^{-\kappa t})$$

The solution of the inhomogeneous differential equation $\frac{dy}{dt}+\kappa y=\eta$ is thus the difference equation

$$y_n = \frac{\eta}{\kappa}+\lambda^n A = \frac{\eta}{\kappa}+\lambda^n\left(y_0-\frac{\eta}{\kappa}\right)$$

which has the same form as the difference equation in §2.2.1 with $\lambda = e^{-\kappa\Delta t}$ and $\beta = (1-\lambda)\,\eta/\kappa$.

2.2.3 NUMERICAL SOLUTION OF ODEs

Relationship between ODE and difference equation suggests method for numerically solving differential equations (covered more in later courses).

- Computers are good at difference equations

Consider $\frac{dy}{dt}+\kappa y=\eta$

Mean value theorem states that $\left.\frac{dy}{dt}\right|_{t_{n-1}<t<t_n} = \frac{y_n-y_{n-1}}{t_n-t_{n-1}} = \frac{y_n-y_{n-1}}{\Delta t}$.

Try the difference equation $$\frac{y_n-y_{n-1}}{\Delta t}+\kappa y_{n-1}=\eta \qquad \text{(Euler)}$$

$\Rightarrow$ $$y_n = \left(1-\kappa\Delta t\right)y_{n-1}+\eta\Delta t$$

Comparing with §2.2.1 $z_n=\lambda z_{n-1}+\beta = \frac{\beta}{1-\lambda}+\left(z_0-\frac{\beta}{1-\lambda}\right)\lambda^n$ and noting $\lambda = (1-\kappa\Delta t)$ and $\beta = \eta\Delta t$, then

$\Rightarrow$ $$y_n = \frac{\eta\Delta t}{\kappa\Delta t}+\left(y_0-\frac{\eta\Delta t}{\kappa\Delta t}\right)\left(1-\kappa\Delta t\right)^n = \frac{\eta}{\kappa}+\left(y_0-\frac{\eta}{\kappa}\right)\left(1-\kappa\Delta t\right)^n$$

Now since $y \rightarrow 0$ as $t \rightarrow \infty$, then we must have $\kappa\Delta t < 1$ for the approximation y_n to converge.

In the previous section we showed that for an exact solution, the coefficient λ needed to be

$$\lambda = e^{-\kappa\Delta t} = 1-\kappa\Delta t+\frac{1}{2!}\kappa^2\Delta t^2-\frac{1}{3!}\kappa^3\Delta t^3+\cdots$$

so our difference equation may be seen to be just using the first two terms of the Taylor Series for $e^{-\kappa\Delta t}$.

We could alternatively used the difference equation

$$\frac{y_n - y_{n-1}}{\Delta t} + \frac{\kappa}{2}\left(y_n + y_{n-1}\right) = \eta \qquad \text{(Crank-Nicholson)}$$

$$\Rightarrow \quad y_n = \frac{1-\frac{1}{2}\kappa\Delta t}{1+\frac{1}{2}\kappa\Delta t} y_{n-1} + \eta\Delta t$$

$$= \left(\frac{1-\frac{1}{2}\kappa\Delta t}{1+\frac{1}{2}\kappa\Delta t}\right)^n \left(y_0 - \frac{\eta}{\kappa}\right) + \frac{\eta}{\kappa}$$

A binomial expansion for λ,

$$\lambda = \frac{1-\frac{1}{2}\kappa\Delta t}{1+\frac{1}{2}\kappa\Delta t}$$

$$= \left(1-\tfrac{1}{2}\kappa\Delta t\right)\left(1-\tfrac{1}{2}\kappa\Delta t + \tfrac{1}{4}\kappa^2\Delta t^2 - \tfrac{1}{8}\kappa^3\Delta t^3 + \cdots\right)$$

$$= 1 - \kappa\Delta t + \tfrac{1}{2}\kappa^2\Delta t^2 - \tfrac{1}{4}\kappa^3\Delta t^3 + \cdots$$

shows that this is a better approximation to $e^{-\kappa\Delta t}$, and so should be more accurate for a given Δt. Moreover, note that (for $\kappa > 0$) $|1-½\kappa\Delta t| < |1+½\kappa\Delta t|$ so $|\lambda| < 1$ and $y_n \to 0$ as $n \to \infty$ regardless of the choice of Δt. This means the difference equation is stable.

2.2.4 INHOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

- Homogeneous equation

$$z_{n+1} + z_n = 0$$

Try $z_n = K\,a^n$ (using similarity with differential equation giving $y = K\,e^x$ or earlier analysis of difference equation noting that $z_{n+1} = -z_n$); K is arbitrary constant.

$$\Rightarrow \quad K\,a^{n+1} + K\,a^n = K\,a^n(a+1) = 0$$

$$\Rightarrow \quad a = -1$$

$$\Rightarrow \quad z_n = K\,(-1)^n$$

- $z_{n+1} + z_n = 1$

Look for particular integral. Try $z_n = const = A$ and substitute: $A + A = 2A = 1 \Rightarrow A = ½$.

Hence general solution is sum of these, *i.e.*

$$z_n = K\,(-1)^n + ½$$

End of Lecture 5

- $z_{n+1} + z_n = n$

Might guess particular integral to be $z_n = ½\,n$, but this would give

$$z_{n+1} + z_n = ½\,(n+1+n) = n + ½$$

which leaves us with ½ too much. More generally, try $z_n = A + Bn$

$$\Rightarrow \quad z_{n+1} + z_n = A + B(n+1) + A + Bn = 2A + B + 2Bn = n$$

$\Rightarrow$ $$B = \tfrac{1}{2},$$ $$A = -\tfrac{1}{2}B = -\tfrac{1}{4}.$$

$\Rightarrow$ $$z_n = K(-1)^n + \tfrac{1}{2}\, n - \tfrac{1}{4}$$

➡ $z_{n+1} + z_n = n^2$

Try $z_n = A + Bn + Cn^2$

➡ $z_{n+1} + z_n = a^n$

Try $z_n = Aa^n$

$\Rightarrow$ $$A\, a^{n+1} + A\, a^n = A\, a^n(a + 1) = a^n$$

$\Rightarrow$ $$A = 1/(a + 1)$$

$\Rightarrow$ $$z_n = K(-1)^n + a^n/(a + 1)$$

➡ $z_{n+1} + z_n = n\, a^n$

Try $z_n = (A + Bn)a^n$

$\Rightarrow$ $$(A + Bn + B)a^{n+1} + (A + Bn)a^n = [A(a+1) + Ba]a^n + (a + 1)Bna^n = na^n$$

$\Rightarrow$ $$(a + 1)B = 1 \Rightarrow B = 1/(a + 1)$$

$$A(a+1) + Ba = 0 \Rightarrow A = -Ba/(a+1) = -a/(a+1)^2$$

$\Rightarrow$ $$z_n = K(-1)^n - a^{n+1}/(a + 1)^2 - na^n/(a+1)$$

2.2.5 Series solution

Here we explore another method for solving these first order odes with constant coefficients. This additional approach is not sensible in this case, but serves to illustrate a technique that can be applied to more complex differential equations where the solution may be less easy to obtain.

Consider again $$\frac{dy}{dt} + \kappa y = \eta$$

subject to $y = y_0$ at $t = 0$.

Seek a solution of the form $$y = \sum_{n=0}^{\infty} a_n t^n \qquad (y = a_0 \text{ at } t = 0)$$

Now $$\frac{dy}{dt} = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

[We shall assume this is OK since if y is differentiable, then dy/dt exists and is finite.]

Substitute into ode:

$$\sum_{n=1}^{\infty} n a_n t^{n-1} + \kappa \sum_{n=0}^{\infty} a_n t^n = \eta$$

$\Rightarrow$ $$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} + \kappa a_n \right] t^n - \eta = 0$$

Need to equate coefficients to zero for all powers of t (each t^n is linearly independent).

$n = 0$ $$a_1 + \kappa a_0 - \eta = 0$$

$n > 0$ $$(n+1)a_{n+1} + \kappa a_n = 0$$

$\Rightarrow$ $$a_{n+1} = -\frac{\kappa}{n+1} a_n \qquad \text{(recurrence relation)}$$

Solution:

$$a_0 = y_0$$
$$a_1 = \eta - \kappa y_0$$
$$a_2 = -\tfrac{1}{2} \kappa a_1 = -\tfrac{1}{2} \kappa(\eta - \kappa y_0)$$
$$a_3 = -(1/3) \kappa a_2 = (1/6) \kappa^2(\eta - \kappa y_0)$$
$$\cdots$$

$$y = y_0 + (\eta - \kappa y_0)\left(t - \tfrac{1}{2}\kappa t^2 + \tfrac{1}{6}\kappa^2 t^3 - \cdots\right)$$

$\Rightarrow$ $$= \frac{\eta}{\kappa} + \left(y_0 - \frac{\eta}{\kappa}\right)\left(1 - \kappa t + \frac{1}{2!}\kappa^2 t^2 - \frac{1}{3!}\kappa^3 t^3 + \cdots\right)$$

$$= \frac{\eta}{\kappa} + \left(y_0 - \frac{\eta}{\kappa}\right)e^{-\kappa t}$$

as we know from earlier.

For more complicated equations we may not be able to explicitly sum the series generated, or even be able to give an explicit expression for the coefficients. However, may still be able to evaluate the solution using a computer (but may have convergence issues).

2.2.6 Modelling examples

Here we shall look at some real-world examples

Bank interest

Capital sum $S(t)$ accrues interest at a *rate* r ($r = const$)

i.e. S increases by a proportion $r\,dt$ in time dt

$\Rightarrow S(t + dt) = S(t) + r\,dt\,S(t)$ Banks typically have dt equal to one day.

$$\Rightarrow \frac{S(t+dt) - S(t)}{dt} - rS(t) = 0$$

Banks often say they do things *pro rata*, so we shall approximate the above difference equation with

$$\frac{dS}{dt} - rS = 0 \quad \Rightarrow S = S_0\, e^{rt}$$

S_0 is initial capital.

Of course, the Bank of England changes the base rate from time to time, so $r = r(t)$, leading to $S = S_0 e^{\int r\,dt}$

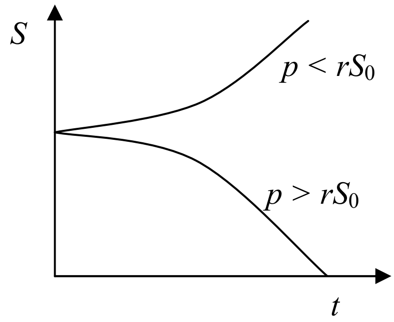
Repaying mortgage

Borrow a sum S_0, subject to interest at rate r but making repayments at rate p. Assuming *pro rata* handling by bank, then the amount owed is

$$\frac{dS}{dt} = rS - p\,; \qquad S(0) = S_0.$$

For the mortgage to decrease, then clearly require $p > rS$.

$$S = \frac{p}{r} + \left(S_0 - \frac{p}{r}\right)e^{rt}$$



Mortgage is repaid when $S = 0 \Rightarrow$ $e^{rt} = \dfrac{p/r}{p/r - S_0} = \dfrac{1}{1 - \dfrac{rS_0}{p}} \Rightarrow t = T \equiv -\dfrac{1}{r}\ln\left(1 - \dfrac{rS_0}{p}\right).$

Total sum paid is of course pT. Note that if p is close to rS_0, there is a large change in pT for a small change in p. Hence, it is generally very beneficial to try to pay your mortgage off faster than you are required to.

To repay your mortgage in T years, you need to make repayments at rate

$$p = \frac{rS_0 e^{rT}}{e^{rt} - 1} = \frac{rS_0}{1 - e^{-rT}}$$

Assume you borrow £100,000 at an annual interest rate of 6% over a 20 year period. Your repayments are ~£8586/year or ~£715/month. In total you repay around £171.7k! Paying things off faster so that you take only 15 years increases your repayments to ~£843/month, but decreases the total cost to ~£151.7k. Enough for a nice car!

Radioactive decay

The probability that a radioactive isotope, such as U, Th, Pu, ^{14}C, will decay in a unit of time is independent of time. Suppose you have a quantity Q, then the decay rate τ^{-1} is independent of time and

$$\frac{dQ}{dt} = -\frac{Q}{\tau} \Rightarrow Q = Q_0\, e^{-t/\tau}.$$

Here τ is the characteristic decay time (time for Q to be reduced by a factor of e), also referred to as the *e-folding time*.

Sometimes we are interested in the *half-life*, $t_{½}$ the time for the radioactivity to reduce by a factor of 2. Noting that $Q = Q_0 e^{-t/\tau} = Q_0 2^{-t/t_{1/2}}$, then $t/\tau = t/t_{½}\ \ln 2 \Rightarrow t_{½} = \tau \ln 2 \approx 0.693\ \tau$.

Newton cooling (or warming)

A block of material with a temperature different to that of the environment in which it sits will lose (or gain) heat from the environment depending on the temperature difference. If the block is made from a good conductor of heat (*e.g.* metal), then it is often a good approximation to assume the temperature of the block is uniform (constant in space).

Suppose H is the heat (thermal energy) of the block, given by

$$H = m\int_{T_r}^{T} C_p \, dT'$$

where T_r is a reference temperature, m is the mass of the block and C_p is the *heat capacity* of the block. For many materials C_p is approximately constant, so $H = m\, C_p(T - T_r)$.

The heat of the block changes due to a *heat flux* Q from the block to the environment as

$$\frac{dH}{dt} = mC_p \frac{dT}{dt} = -Q .$$

If the heat flux depends only on the difference between the temperature of the block and the temperature of the environment, T_0, then

$$Q(T - T_0) = Q_0 + \left.\frac{dQ}{dT}\right|_0 (T - T_0) + \frac{1}{2}\left.\frac{d^2Q}{dT^2}\right|_0 (T - T_0)^2 + \cdots \qquad \text{Taylor series}$$

The zeroth law of thermodynamics requires that there is only a heat flux between two bodies when there is a temperature difference, hence $Q_0 = 0$. If $T-T_0$ is *small*, can ignore quadratic and higher terms, so

$$mC_p \frac{dT}{dt} \approx -\left.\frac{dQ}{dT}\right|_{T=T_0} (T - T_0) .$$

[Approximations like this are the basis of many of the differential equations found in physics.]

Let $\theta = T - T_0$, and $\kappa = \dfrac{1}{mC_p}\left.\dfrac{dQ}{dT}\right|_{T=T_0}$, so

$$\frac{d\theta}{dt} + \kappa\theta = 0 \;\Rightarrow\; \theta = \theta_0\, e^{-\kappa t}.$$

Note that if thermal conductivity of block is poor, then temperature gradients within block make this description inappropriate. The Newton cooling described here ignores heat transfer by natural convection, radiation, *etc.*

Time delay equation

The reaction is out of sync with the state. Lots of systems have feedback of this form.

$$y'(t) = ay(t-\tau); \qquad \tau = const.$$

Integrating:

$$y(t) = y(t_1) + a\int_{t_1}^{t} y(t' - \tau)\, dt'$$

$$= y(t_1) + a\int_{t_1-\tau}^{t-\tau} y(t'')\, dt''$$

Initial conditions are less straight forward than normal: just knowing $y(0)$ is not sufficient!

Suppose $y(t) = Y(t)$ for $0 \le t \le \tau$ "Initial condition"

Then for $\tau \le t \le 2\tau$ $$y(t) = Y(\tau) + a\int_0^{t-\tau} y(t'')\,dt'' = Y(\tau) + a\int_0^{t-\tau} Y(t'')\,dt''$$

and for $2\tau \le t \le 3\tau$ Set $t_1 = 2\tau$

End of Lecture 6

$$y(t) = y(t_1) + a\int_{t_1}^{t-\tau} y(t')\,dt'$$
$$= Y(\tau) + a\int_0^{\tau} y(t')\,dt' + a\int_{\tau}^{t-\tau} y(t')\,dt'$$
$$= Y(\tau) + a\int_0^{\tau} Y(t')\,dt' + a\int_{\tau}^{t-\tau}\left[Y(\tau) + a\int_0^{t'-\tau} y(t'')\,dt''\right]dt'$$
$$= Y(\tau) + a\int_0^{\tau} Y(t')\,dt' + aY(\tau)(t-2\tau) + a^2\int_0^{t-2\tau}\left[\int_0^{t'} Y(t'')\,dt''\right]dt'$$

Integrate the last term by parts, letting $f = \int_0^{t'} Y(t'')\,dt''$ $\Rightarrow f' = Y(t')$ and $g' = 1 \Rightarrow g = t'$

$$y(t) = Y(\tau) + a\int_0^{\tau} Y(t')\,dt' + aY(\tau)(t-2\tau) + a^2\left\{\left[t'\int_0^{t'} Y(t'')\,dt''\right]_0^{t-2\tau} - \int_0^{t-2\tau} t'Y(t')\,dt'\right\}$$

$\Rightarrow$
$$= Y(\tau) + a\int_0^{\tau} Y(t')\,dt' + aY(\tau)(t-2\tau) + a^2\left\{(t-2\tau)\int_0^{t-2\tau} Y(t'')\,dt'' - \int_0^{t-2\tau} t'Y(t')\,dt'\right\}$$
$$= Y(\tau) + a\int_0^{\tau} Y(t')\,dt' + aY(\tau)(t-2\tau) + a^2\int_0^{t-2\tau}(t-2\tau-t')Y(t')\,dt'$$
$$= (1 + a(t-2\tau))Y(t) + a\int_0^{\tau} Y(t')\,dt' + a^2\int_0^{t-2\tau}(t-2\tau-t')Y(t')\,dt'$$

➡ $y'(t) = y(t-1)$ with $Y(t) = 1$ for $0 \le t \le 1$.

$0 \le t \le 1$ $\quad y = 1 \quad$ $dy/dt = 0$

$1 \le t \le 2$ $$y(t) = 1 + \int_0^{t-1} 1\,dt'' = 1 + (t-1) = t$$ $dy/dt = 1$

$2 \le t \le 3$ $$y(t) = y(2) + \int_1^{t-1} t'\,dt' = 2 + \frac{1}{2}\left[(t-1)^2 - 1^2\right] = \frac{1}{2}t^2 - t + 2$$

or
$$y(t) = y(2) + \int_1^{t-1} 1 + (t'-1)\,dt' = y(2) + \int_0^{t-2} 1 + t'\,dt'$$
$$= y(2) + (t-2)^2 + \frac{1}{2}(t-2)^2$$

$$y(t) = Y(\tau) + a\int_0^{\tau} Y(t')\, dt' + aY(\tau)(t-2\tau) + a^2 \int_0^{t-2\tau} (t-2\tau-t')Y(t')\, dt'$$

or

$$= 1 + \int_0^1 1\, dt' + (t-2) + \int_0^{t-2} (t-2-t')\, dt'$$

$$= 1 + 1 + t - 2 + (t-2)^2 - \frac{1}{2}(t-2)^2$$

$$= 2 - t + \frac{1}{2}t^2$$

$$y(t) = y(3) + \int_2^{t-1} y(t')\, dt' = \frac{7}{2} + \int_2^{t-1} \frac{1}{2}t'^2 - t' + 2\, dt'$$

$$= \frac{7}{2} + \left[\frac{1}{6}t'^3 - \frac{1}{2}t'^2 + 2t'\right]_2^{t-1}$$

$3 \le t \le 4$

$$= \frac{7}{2} + \frac{1}{6}\left[t^3 - 3t^2 + 3t - 1 - 8\right] - \frac{1}{2}\left[t^2 - 2t + 1 - 4\right] + 2(t-1-2)$$

$$= \left[\frac{7}{2} - \frac{3}{2} + \frac{3}{2} - 6\right] + \left[\frac{1}{2} + 1 + 2\right]t + \left[-\frac{1}{2} - \frac{1}{2}\right]t^2 + \frac{1}{6}t^3$$

$$= -\frac{5}{2} + \frac{7}{2}t - t^2 + \frac{1}{6}t^3$$

$$y(t) = y(3) + \int_2^{t-1} y(2) + (t'-2) + \frac{1}{2}(t'-2)^2\, dt'$$

or

$$= y(3) + \int_0^{t-3} y(2) + t' + \frac{1}{2}t'^2\, dt'$$

$$= \frac{7}{2} + 2(t-3) + \frac{1}{2}(t-3)^2 + \frac{1}{6}(t-3)^3$$

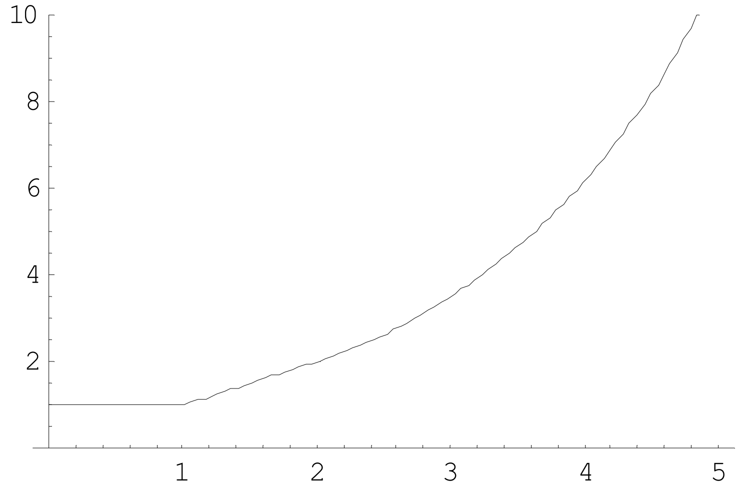
$$y(t) = y(4) + \int_3^{t-1} y\, dt'$$

$4 \le t \le 5$

$$= \frac{37}{6} + \frac{7}{2}(t-4) + (t-4)^2 + \frac{1}{6}(t-4)^3 + \frac{1}{24}(t-4)^4$$

t	*y*(*t*)	left-hand *dy/dt*	right-hand *dy/dt*
1	1	0	1
2	2	1	1
3	7/2	2	2
4	37/6	10/3	?

Function differentiable for $t > 1$



$y'(t) = y(t-\pi)$ with $Y(t) = \sin x$ for $0 \le t \le \pi$.

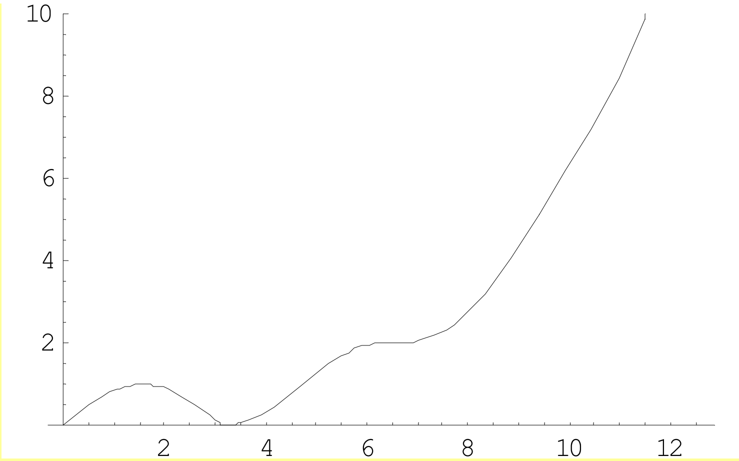
$0 \le t \le \pi$ $$y = \sin x$$

$\pi \le t \le 2\pi$ $$y(t) = \sin\pi + \int_0^{t-\pi} \sin x\, dx = 1 - \cos(t-\pi) = 1 + \cos t$$

$2\pi \le t \le 3\pi$
$$\begin{aligned} y(t) &= y(2\pi) + \int_\pi^{t-\pi} y\, dt'' \\ &= 1 - \cos\pi + \int_\pi^{t-\pi} 1 + \cos t''\, dt'' \\ &= 2 + (t - 2\pi) + \sin(t-\pi) \\ &= 2 - 2\pi + t - \sin t \end{aligned}$$

$3\pi \le t \le 4\pi$
$$\begin{aligned} y(t) &= y(3\pi) + \int_{2\pi}^{t-\pi} y\, dt'' \\ &= 2 - 2\pi + 3\pi - \sin 3\pi + \int_{2\pi}^{t-\pi} 2 - 2\pi + t'' - \sin t''\, dt'' \\ &= 2 - \pi + (2-2\pi)(t-3\pi) + \frac{1}{2}(t-\pi)^2 - 2\pi^2 + \cos(t-\pi) - \cos 2\pi \\ &= 1 - 5\pi + \frac{9}{2}\pi^2 + (2-3\pi)t - \cos t \end{aligned}$$

$4\pi \le t \le 5\pi$
$$\begin{aligned} y(t) &= y(4\pi) + \int_{3\pi}^{t-\pi} y\, dt'' \\ &= \frac{1}{6}\left(-6\pi + 75\pi^2 - 64\pi^3 + (6 - 42\pi + 48\pi^2)t + (6-12\pi)t^2 + t^3 + 6\sin t\right) \end{aligned}$$



2.3 Equations with non-constant coefficients

2.3.1 Basic ideas

Consider

$$P(t)\frac{dy}{dt}+Q(t)y=F(t)$$

Provided $P(t)\neq 0$, then

$$\frac{dy}{dt}+qy=f \qquad q=Q/P;\ f=F/P$$

[Values of t at which $P(t) = 0$ are *singular points* of the differential equation.]

Homogeneous equation

For $f = 0$ can treat as before:

$$\frac{1}{y}\frac{dy}{dt}=-q$$

$\Rightarrow$ $$\ln y=-\int q(t)\,dt$$

$\Rightarrow$ $$y(t)=y_0 e^{-\int_0^t q(t')\,dt'} \qquad y=y_0 \text{ at } t=0.$$

Inhomogeneous equation

From homogeneous solution can see that $ye^{+\int_0^t q(t')\,dt'}=y_0 e^{+\int_0^t q(t')\,dt'}e^{-\int_0^t q(t')\,dt'}=y_0=const$

Try solution of the form $u(t)=ye^{+\int q\,dt'}$

$\Rightarrow$ $$\frac{du}{dt}=\frac{dy}{dt}e^{+\int q\,dt'}+qye^{+\int q\,dt'}=\left(\frac{dy}{dt}+qy\right)e^{+\int q\,dt'}$$

Hence $$e^{-\int q\,dt'}\frac{d}{dt}\left(ye^{+\int q\,dt'}\right)=e^{-\int q\,dt'}\frac{du}{dt}=\frac{dy}{dt}+qy=f$$

so

$$\frac{du}{dt}=e^{+\int q\,dt'}f$$

This transformed equation is said to be **exact** (right-hand side does not depend on u

We can integrate $$u=\int_{t_0}^{t}\left[e^{+\int_{t_0}^{t'} q\,dt''}f(t')\,dt'\right]$$

and so $$y=e^{-\int_{t_0}^{t} q\,dt'}\int_{t_0}^{t}\left[e^{+\int_{t_0}^{t'} q\,dt''}f(t')\,dt'\right]$$

This is the Particular Integral.

Since the ode is linear, the sum of the homogeneous solution and the particular integral is also a solution of the ode. Thus the general solution is

$$y=y_0e^{-\int_{t_0}^{t} q\,dt'}+e^{-\int_{t_0}^{t} q\,dt'}\int_{t_0}^{t}\left[e^{+\int_{t_0}^{t'} q\,dt''}f(t')\,dt'\right]$$

Note that $y(t_0)=y_0$.

2.3.2 INTEGRATING FACTORS

In the previous section we found that by multiplying the equation by $e^{+\int q\,dt'}$ the left-hand side became *integrable*: it was of the form $\frac{d}{dt}(\mathit{something})$. Here, the factor $e^{+\int q\,dt'}$ is called the *Integrating Factor* (IF)

➧ $\frac{dy}{dt}+ty=t$

Integrating factor: $e^{+\int q\,dt'}=e^{\int t'\,dt'}=e^{\frac{1}{2}t^2}$

$$\Rightarrow \qquad e^{\frac{1}{2}t^2}\left(\frac{dy}{dt}+ty\right)=e^{\frac{1}{2}t^2}\frac{dy}{dt}+e^{\frac{1}{2}t^2}ty=\frac{d}{dt}\left(e^{\frac{1}{2}t^2}y\right)=te^{\frac{1}{2}t^2}$$

$$\Rightarrow \qquad e^{\frac{1}{2}t^2}y=\int_{t_0}^{t}t'e^{\frac{1}{2}t'^2}\,dt'=\left[e^{\frac{1}{2}t'^2}\right]_{t_0}^{t}=e^{\frac{1}{2}t^2}-e^{\frac{1}{2}t_0^2}$$

$$\Rightarrow \qquad y=1-e^{\frac{1}{2}\left(t_0^2-t^2\right)}=1-e^{-\frac{1}{2}\left(t^2-t_0^2\right)}$$

As t_0 is arbitrary, can write as $y=1-Ae^{-\frac{1}{2}t^2}$

Using the complete formula,

$$y = y_0 e^{-\int_{t_0}^{t} q\,dt'} + e^{-\int_{t_0}^{t} q\,dt'} \int_{t_0}^{t} \left[e^{+\int_{t_0}^{t'} q\,dt''} f(t')\,dt' \right]$$

$$= y_0 e^{-\int_{t_0}^{t} t\,dt'} + e^{-\int_{t_0}^{t} t\,dt'} \int_{t_0}^{t} \left[e^{+\int_{t_0}^{t'} t\,dt''} t'\,dt' \right]$$

$$= y_0 e^{-\frac{1}{2}\left(t^2 - t_0^2\right)} + e^{-\frac{1}{2}\left(t^2 - t_0^2\right)} \int_{t_0}^{t} \left[e^{+\frac{1}{2}\left(t'^2 - t_0^2\right)} t'\,dt' \right]$$

$$= y_0 e^{-\frac{1}{2}t^2} + e^{\frac{1}{2}t_0^2} e^{-\frac{1}{2}t^2} e^{-\frac{1}{2}t_0^2} \int_{t_0}^{t} \left[e^{\frac{1}{2}t'^2} t'\,dt' \right]$$

$$= y_0 e^{-\frac{1}{2}\left(t^2 - t_0^2\right)} + e^{-\frac{1}{2}t^2} \left[e^{\frac{1}{2}t'^2} \right]_{t_0}^{t}$$

$$= y_0 e^{-\frac{1}{2}\left(t^2 - t_0^2\right)} + e^{-\frac{1}{2}t^2} \left(e^{\frac{1}{2}t^2} - e^{\frac{1}{2}t_0^2} \right)$$

$$= y_0 e^{-\frac{1}{2}\left(t^2 - t_0^2\right)} + 1 - e^{-\frac{1}{2}\left(t^2 - t_0^2\right)}$$

$$= 1 - \left(1 - y_0\right) e^{-\frac{1}{2}\left(t^2 - t_0^2\right)}$$

- $\dfrac{dy}{dx} + y = e^x$

  Don't need to do this, as constant coefficients, but we will to prove it works…

  Integrating factor: $e^{+\int q\,dt'} = e^{\int dx'} = e^x$

  $\Rightarrow \quad e^x \left(\dfrac{dy}{dt} + y \right) = e^x \dfrac{dy}{dt} + e^x y = \dfrac{d}{dt}\left(e^x y\right) = e^x e^x = e^{2x}$

  $\Rightarrow \quad e^x y = \displaystyle\int_{x_0}^{x} e^{2x'}\,dx' = \left[\frac{1}{2} e^{2x'} \right]_{x_0}^{x} = \frac{1}{2}\left(e^{2x} - e^{2x_0} \right)$

  $\Rightarrow \quad y = \dfrac{1}{2} e^{-x} \left(e^{2x} - e^{2x_0} \right) = \dfrac{1}{2}\left(e^x - e^{2x_0} e^{-x} \right) = \dfrac{1}{2} e^x - A e^{-x}$ $\quad x_0$ arbitrary

- $\dfrac{dy}{dx} + y \sin x = e^{\cos x}$

  Integrating factor: $e^{+\int q\,dt'} = e^{\int \sin x'\,dx'} = e^{-\cos x}$

  $\Rightarrow \quad e^{-\cos x} \left(\dfrac{dy}{dt} + y \sin x \right) = e^{-\cos x} \dfrac{dy}{dt} + e^{-\cos x} y \sin x = \dfrac{d}{dt}\left(e^{-\cos x} y \right) = e^{-\cos x} e^{\cos x} = 1$

  $\Rightarrow \quad e^{-\cos x} y = \displaystyle\int_{x_0}^{x} dx' = \left(x - x_0 \right)$

  $\Rightarrow \quad y = \left(x - x_0 \right) e^{\cos x}$ $\quad x_0$ arbitrary

In the above examples:

Equation	Particular Integral (Solution to the full equation)	Complementary Function (Solution to homogeneous equation)
$\frac{dy}{dt}+ty=t$	$y=1$	$y=e^{-\frac{1}{2}t^2}$
$\frac{dy}{dx}+y=e^x$	$y=\frac{1}{2}e^x$	$y=e^{-x}$
$\frac{dy}{dx}+y\sin x=e^{\cos x}$	$y=x\,e^{\cos x}$	$y=e^{\cos x}$

Theorem

If $y=u$ satisfies $\frac{dy}{dt}+q(t)y=f(t)$

and $y=v$ satisfies $\frac{dy}{dt}+q(t)y=0$

then $y=u+A\,v$ satisfies $\frac{dy}{dt}+q(t)y=f(t)$, where A is arbitrary.

Proof: $\frac{d}{dt}(u+Av)+q(u+Av)=\frac{du}{dt}+qu+A\left(\frac{dv}{dt}+qv\right)=f$

v is called a *complementary function* (CF) because it completes the solution.

Note that there is only ever **one** arbitrary constant for a first-order ode (provided y remains bounded). This is true whether the equation is linear or nonlinear.

➡ $\frac{dy}{dx}-xy=x^2$; $y=1$ at $x=0$

Integrating factor: $e^{+\int q\,dt'}=e^{\int -x'dx'}=e^{-\frac{1}{2}x^2}$ [As for first example earlier]

$$\Rightarrow \qquad e^{-\frac{1}{2}x^2}\left(\frac{dy}{dt}-xy\right)=e^{-\frac{1}{2}x^2}\frac{dy}{dt}-xye^{-\frac{1}{2}x^2}=\frac{d}{dt}\left(ye^{-\frac{1}{2}x^2}\right)=e^{-\frac{1}{2}x^2}x^2$$

$$\Rightarrow \qquad e^{-\frac{1}{2}x^2}y=\int e^{-\frac{1}{2}x^2}x^2\,dx$$

Integrate by parts, using $f=x;\ g'=xe^{-\frac{1}{2}x^2}\ \Rightarrow f'=1;\ g=-e^{-\frac{1}{2}x^2}$

$$\Rightarrow \qquad y=e^{\frac{1}{2}x^2}\left\{-xe^{-\frac{1}{2}x^2}+\int e^{-\frac{1}{2}x^2}\,dx\right\}$$

But we cannot do this integral! As it appears often, a function representing the integral has been invented: the *error function*

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

[The factor of $2/\sqrt{\pi}$ is introduced so that $erf(\infty)=1$.]

$$\Rightarrow \qquad y = -x + \frac{\sqrt{\pi}}{2} e^{\frac{1}{2}x^2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + Ae^{\frac{1}{2}x^2}$$

Now since $y(0) = 1$ then $A = 1$

2.3.3 SERIES SOLUTION

We could try a series solution for the previous example: $\frac{dy}{dx} - xy = x^2$

Suppose $y = \sum_{n=0}^{\infty} a_n x^n \;\Rightarrow\; \frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1}$

The boundary condition $y(0) = 1$ gives $a_0 = 1$

Substitute into equation

$$\frac{dy}{dx} - xy = \sum_{n=0}^{\infty} n a_n x^{n-1} - x \sum_{n=0}^{\infty} a_n x^n - x^2$$

$$= \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - a_{n-1} \right] x^n - x^2 = 0 \qquad a_{-1} = 0$$

$$a_{n+1} = \frac{a_{n-1}}{n+1} \qquad n \neq 2$$

$$3a_3 - a_1 - 1 = 0$$

$\Rightarrow$

$$a_0 = 1$$
$$a_1 = a_{-1} = 0$$
$$a_2 = \tfrac{1}{2} a_0 = \tfrac{1}{2}$$
$$a_3 = (1 + a_1)/3 = 1/3$$
$$a_4 = a_2/4 = 1/8$$
$$a_5 = a_3/5 = 1/15$$
$$\cdots$$

$$\Rightarrow \qquad y = 1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{2\times 4}x^4 + \frac{1}{3\times 5}x^5 + \frac{1}{2\times 4\times 6}x^6 + \cdots$$