4 Higher order equations

4.1 Second order linear equations

The general second order linear ordinary differential equation may be written as

$$R(t)\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)y = F(t).$$

This is linear in y (and its derivatives). If $F = 0$ the equation is *homogeneous*; if $F \neq 0$ the equation is inhomogeneous.

Assume that $R(t)$ does not vanish, then can rewrite as

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f$$

where $p(t) = P/R$, $q(t) = Q/R$ and $f(t) = F/R$.

End of Lecture 12

4.1.1 Reduction to first order system

By defining $u_1 = y$ and $u_2 = dy/dt$ we may rewrite the second order equation as a system of first order equations:

$$\frac{du_1}{dt} = u_2 \equiv g_1,$$

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \frac{du_2}{dt} + pu_2 + qu_1 = f$$

$$\Rightarrow \quad \frac{du_2}{dt} = -pu_2 - qu_1 + f \equiv g_2.$$

Define the vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{g} = (g_1, g_2)$, then

$$\frac{d\mathbf{u}}{dt} = \mathbf{g}.$$

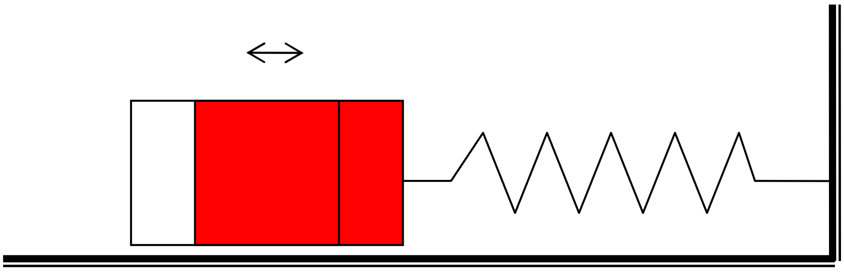
To be able to solve this, we also require initial or boundary conditions. As we have two first order equations, we require two initial/boundary conditions. Note that the coupling allows them to both be on the same variable and/or at different times. For example, we could have $u_1(0)$ and $u_2(0)$, or $u_1(0)$ and $u_1(T)$, or $u_1(0)$ and $u_2(T)$.

As with the first order linear equations, we shall see that the linearity allows us to combine the particular integral solution of the inhomogeneous equations, with the complementary function solution of the homogeneous equation. Here, however, there will be two complementary functions, reflecting the second-order nature of the equation.

4.2 Equations with constant coefficients

Suppose p and q are independent of t.

An example of this is a mass m on a spring (spring constant k), optionally subject to a viscous friction (drag, 2μ):



This gives for unit mass and external forcing f

$$\ddot{y} + 2\mu\dot{y} + ky = f$$

Note: In lectures I used k as the spring constant in this bit (as above), but later in §4.3 I switch to using k^2 for the spring constant. For completeness, I have included in these notes the material both as lectured (normal text), and how it would have been if I had used k^2 throughout for the spring constant (text with violet background).

$$\ddot{y} + 2\mu\dot{y} + k^2 y = f$$

4.2.1 Homogeneous equation

For $f = 0$ (the unforced or freely oscillating case)

$$\ddot{y} + 2\mu\dot{y} + ky = 0 \qquad (*)$$

$$\ddot{y} + 2\mu\dot{y} + k^2 y = 0$$

Recalling that $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$ and $\frac{d^2}{dt^2}e^{\lambda t} = \lambda^2 e^{\lambda t}$ suggests that $y = A\,e^{\lambda t}$ has the correct functional form to satisfy (*).

Substituting $A\,(\lambda^2 + 2\lambda\,\mu + k)e^{\lambda t} = 0,$ $\qquad A\,(\lambda^2 + 2\lambda\,\mu + k)e^{\lambda t} = 0$

which must hold $\forall t$. Hence either $A = 0$ or

$$\lambda^2 + 2\lambda\,\mu + k = 0 \qquad (**)$$

$$\lambda^2 + 2\lambda\,\mu + k^2 = 0$$

with arbitrary A. This is the *characteristic* equation.

Solving $\lambda = -\mu \pm \sqrt{\mu^2 - k}$ $\qquad \lambda = -\mu \pm \sqrt{\mu^2 - k^2}\,.$

There are two possible solutions to the characteristic equation because this is a second order equation. This leads to two solutions to the homogeneous equation – the *complementary functions*. Any linear combination of these will also be a solution, hence

$$y = Ae^{\left(-\mu+\sqrt{\mu^2-k}\right)t} + Be^{\left(-\mu-\sqrt{\mu^2-k}\right)t} = e^{-\mu}\left(Ae^{\sqrt{\mu^2-k}\,t} + Be^{-\sqrt{\mu^2-k}\,t}\right).$$

$$y = Ae^{\left(-\mu+\sqrt{\mu^2-k^2}\right)t} + Be^{\left(-\mu-\sqrt{\mu^2-k^2}\right)t} = e^{-\mu}\left(Ae^{\sqrt{\mu^2-k^2}\,t} + Be^{-\sqrt{\mu^2-k^2}\,t}\right).$$

This linear combination is called the *principle of superposition* and applies to all linear equations.

Clearly the two solutions coincide when $\mu^2 = k$ $\mu^2 = k^2$. We shall return to this situation later.

When $\mu^2 < k$ the square root is imaginary. Since we have not restricted the constants A and B to be real, we may then write

$$y = e^{-\mu t}\left(\hat{A}\cos\left(\sqrt{k-\mu^2}\;t\right) + \hat{B}\sin\left(\sqrt{k-\mu^2}\;t\right)\right).$$

$$y = e^{-\mu t}\left(\hat{A}\cos\left(\sqrt{k^2-\mu^2}\;t\right) + \hat{B}\sin\left(\sqrt{k^2-\mu^2}\;t\right)\right).$$

4.2.2 Principle of superposition

Define the second order linear differential operator L as

$$L \equiv \left[\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\right],$$

then $L[u] \equiv Lu = u'' + p\,u' + q\,u$. This provides us with a compact notation for the differential equation.

Suppose $L[u_1] = 0$ and $L[u_2] = 0$, then the principle of superposition states that $L[Au_1 + Bu_2] = 0$ for arbitrary constants A,B. Proof is by straight forward substitution.

4.2.3 Initial value problems

The two boundary conditions are applied at the same (initial) value of x.

Suppose $u = Au_1 + Bu_2$ where $L[u_1] = 0$ and $L[u_2] = 0$ and u_1 and u_2 are linearly independent (*i.e.* they are the complementary function of $L[u] = 0$).

If we require $u(0) = u_0$ and $u'(0) = v_0$, we therefore have the linear system

$$A\,u_1(0) + B\,u_2(0) = u_0$$
$$A\,u'_1(0) + B\,u'_2(0) = v_0.$$

This has a nontrivial solution provided the determinant

$$\begin{vmatrix} u_1(0) & u_2(0) \\ u'_1(0) & u'_2(0) \end{vmatrix}$$

does not vanish. If the determinant does vanish, then u_1 and u_2 are not (linearly) independent.

4.2.4 Linear independence

Two functions $u_1(x)$ and $u_2(x)$ are *linearly dependent* in an interval I if $\exists$ constants $k_1, k_2 \neq 0$ such that

$$k_1\,u_1(x) + k_2\,u_2(x) = 0$$

$\forall x \in I$. If they are not linearly dependent, then they are *linearly independent*.

Theorem:

Suppose $u_1(x)$ and $u_2(x)$ are differentiable functions in the interval I. Then if the determinant

$$W(u_1,u_2) = \begin{vmatrix} u_1(x) & u_2(x) \\ u'_1(x) & u'_2(x) \end{vmatrix}$$

does not vanish at any $x = x_0$ in I, then u_1 and u_2 are linearly independent in I. Further, if u_1 and u_2 are linearly dependent in I then $W = 0\ \forall x \in I$. This determinant is known as the *Wronskian*.

Proof: Suppose there exits k_1, k_2 such that $k_1 u_1(x) + k_2 u_2(x) = 0$ in I, then we also have $k_1 u_1'(x) + k_2 u_2'(x) = 0$ and substitution into the determinant W proves the result.

Note that we have proven that if the functions are linearly dependent, then $W = 0$. We have not shown that $W = 0$ implies the functions are linearly dependent.

Note there is a similarity here with linear independence in vector algebra.

- $x^\alpha, x^\beta\ (\alpha \neq \beta)$

  $dx^\alpha/dx = \alpha x^{\alpha-1}$; $dx^\beta/dx = \beta x^{\beta-1}$

  $$\Rightarrow \qquad W = \begin{vmatrix} x^\alpha & x^\beta \\ \alpha x^{\alpha-1} & \beta x^{\beta-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix} x^{\alpha+\beta-1} = (\beta-\alpha) x^{\alpha+\beta-1}$$

  which is nonzero (except at $x = 0$)

- $\cos kx, \sin kx\ (k \neq 0)$

  $$\Rightarrow \qquad W = \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix} = k\left(\cos^2 kx + \sin^2 kx\right) = k \neq 0$$

- $e^{\alpha x}, e^{\beta x}\ (\alpha \neq \beta)$

  $$\Rightarrow \qquad W = \begin{vmatrix} e^{\alpha x} & e^{\beta x} \\ \alpha e^{\alpha x} & \beta e^{\beta x} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix} e^{(\alpha+\beta)x} = (\beta-\alpha) e^{(\alpha+\beta)x} \neq 0.$$

Warning: the term "linear" is often omitted from discussions about "linear independence".

4.2.5 ABEL'S THEOREM

If u_1 and u_2 satisfy $L[u] \equiv u'' + p(x)\,u' + q(x)\,u = 0$, where p and q are continuous in interval $x \in I$, then the Wronskian $W(u_1, u_2)$ satisfies

$$W = W_0 e^{-\int p(x)\,dx}$$

where W_0 is a constant.

Consequently, when p remains finite, either $W = 0\ \forall x$ in I, **or** W never vanishes in I.

Proof:

$$u_2\,(u_1'' + p\,u_1' + q\,u_1) = 0$$
$$u_1\,(u_2'' + p\,u_2' + q\,u_2) = 0$$

Subtracting $$u_1 u_2'' - u_2 u_1'' + p(u_1 u_2' - u_2 u_1') = 0$$

$$\Rightarrow \qquad \frac{dW}{dx} + pW = 0 \qquad \text{since } W = u_1 u_2' - u_2 u_1'$$

$$\Rightarrow \qquad W = W_0 e^{-\int p(x)\,dx}$$

Note: this result carries over to higher order equations. For a n^{th} order equation the Wronskian is the determinant of the $n \times n$ matrix

$$W(u_1,u_2,\cdots,u_n) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}.$$

4.2.6 Existence and uniqueness theorems

We saw before that, given a set of initial conditions, we could uniquely determine the solution to the homogeneous equation from two linearly independent solutions. Indeed, the solution is unique regardless of the choice of u_1, u_2 satisfying $L[u] = 0$, and two linearly independent u_1, u_2 always exist. [This statement shall not be proven in this course.]

Consequently, if $W(u_1,u_2) \neq 0$, then

$$u = A\,u_1 + B\,u_2$$

is the most *general solution* of $L[u] = 0$.

End of Lecture 13

4.2.7 Homogeneous examples

- $y'' + 9y = 0$ subject to $y(0) = 0$, $y'(0) = 6$

Characteristic equation $\quad \lambda^2 + 9 = 0$

$\Rightarrow \quad \lambda = \pm 3i \quad$ complex $\Rightarrow$ sinesoidal

General solution $\quad y = A\cos 3x + B\sin 3x \quad$ $A, B = const$

Derivative $\quad y' = -3A\sin 3x + 3B\cos 3x$

At $y(0) = 0 \Rightarrow \quad A = 0$

$y'(0) = 6 \Rightarrow \quad 3B = 6 \Rightarrow B = 2.$

Hence $\quad y = 2\sin 3x.$

- $y'' + 4\,y' + y = 0$ subject to $y(0) = 1$, $y(1) = 0$

Characteristic equation $\quad \lambda^2 + 4\,\lambda + 1 = 0$

$\Rightarrow \quad \lambda = -2 \pm \sqrt{4-1} = -2 \pm \sqrt{3}$

General solution $\quad y = e^{-2t}\left(Ae^{\sqrt{3}\,t} + Be^{-\sqrt{3}\,t}\right)$

$y(0) = 1 \Rightarrow \quad A + B = 1$

$y(1) = 1 \Rightarrow \quad Ae^{-2+\sqrt{3}} + Be^{-2-\sqrt{3}} = 0$

Solving $\quad A = \dfrac{-1}{e^{2\sqrt{3}} - 1}, \quad B = \dfrac{e^{2\sqrt{3}}}{e^{2\sqrt{3}} - 1}$

Hence $$y = e^{-2t}\frac{-e^{\sqrt{3}\,t} + e^{2\sqrt{3}}e^{-\sqrt{3}\,t}}{e^{2\sqrt{3}} - 1}.$$

$y'' + 2\,y' + y = 0$

Characteristic equation $\lambda^2 + 2\,\lambda + 1 = (\lambda + 1)^2 = 0$

$\Rightarrow$ $\lambda = -1$ double root

But we need two complementary functions!

Consider $y = (A + Bt)\,e^{-t}$

$\Rightarrow$ $y' = -(A - B + Bt)\,e^{-t},\ y'' = (A - 2B + Bt)\,e^{-t}$

Substitute $(A - 2B + Bt)\,e^{-t} - 2(A - B + Bt)\,e^{-t} + (A + Bt)\,e^{-t} = 0.$

This is true for any A, B.

4.2.8 Repeated roots

Consider the equation

$$\frac{d^2y}{dt^2} + (2\alpha + \varepsilon)\frac{dy}{dt} + \alpha(\alpha + \varepsilon)\,y = 0$$

The characteristic equation of this

$$\lambda^2 + (2\alpha + \varepsilon)\lambda + \alpha(\alpha + \varepsilon) = (\lambda + \alpha)(\lambda + \alpha + \varepsilon) = 0$$

gives $\lambda = -\alpha$ and $\lambda = -(\alpha + \varepsilon)$, and so the general solution

$$y = A\,e^{-\alpha t} + B\,e^{-(\alpha+\varepsilon)t} = (A + B\,e^{-\varepsilon t})\,e^{-\alpha t}.$$

Now as $\varepsilon \to 0$ the two roots tend towards a repeated root and $e^{-\varepsilon t} \to 1 - \varepsilon t$, thus the general solution tends towards

$$y = [(A + B) - B\,\varepsilon t]\,e^{-\alpha t} = [C + Dt]\,e^{-\alpha t}.$$

Note that as A, B are arbitrary at this point, we can have $\varepsilon \to 0$ but keep $B\varepsilon$ finite. This means that the next term in the Taylor Series expansion of $e^{-\varepsilon t}$ (proportional to ε^2) will vanish.

Another way of looking at the characteristic equation is that in general we may rewrite the differential equation in the form

$$\frac{d^2y}{dt^2} + (\alpha + \beta)\frac{dy}{dt} + \alpha\beta\,y = \left[\frac{d}{dt} + \alpha\right]\left[\frac{d}{dt} + \beta\right]y = 0$$

where $\lambda = -\alpha$ and $\lambda = -\beta$ are the roots of the characteristic equation. The complementary functions $e^{-\alpha t}$ and $e^{-\beta t}$ can then be seen individually to be solutions of the respective first-order equations

$$\left[\frac{d}{dt} + \alpha\right]y = 0 \quad\text{and}\quad \left[\frac{d}{dt} + \beta\right]y = 0.$$

The inverses of these first-order operators are therefore give

$$y = \left[\frac{d}{dt} + \alpha\right]^{-1} 0 = Ae^{-\alpha t} \quad\text{and}\quad y = \left[\frac{d}{dt} + \beta\right]^{-1} 0 = Be^{-\beta t}. \qquad A, B \text{ constants}$$

Hence we may interpret the second order equation as the solution of

$$\left[\frac{d}{dt}+\beta\right]y=\left[\frac{d}{dt}+\alpha\right]^{-1}0=\hat{A}e^{-\alpha t}. \qquad (*)$$

As seen in §2, the solution of this is the complementary function $B\,e^{-\beta t}$ plus the particular integral

$$y=\frac{\hat{A}}{B-\alpha}e^{-\alpha t}=Ae^{-\alpha t},$$

thus the general solution $y = A\,e^{-\alpha t} + B\,e^{-\beta t}$.

When $\alpha = \beta$, (*) becomes

$$\left[\frac{d}{dt}+\alpha\right]y=\left[\frac{d}{dt}+\alpha\right]^{-1}0=\hat{A}e^{-\alpha t}$$

which (as we saw in §2.1.2) yields a particular integral of the form $t\,e^{-\alpha t}$, and thus the general solution $y = (A + Bt)\,e^{-\alpha t}$.

By considering the Wronskian of $e^{-\alpha t}$ and $t\,e^{-\alpha t}$ we can show that the two complementary functions are linearly independent.

4.2.9 PARTICULAR INTEGRAL

As with the first order equation in §2.1.2, the inhomogeneous equation, $L[u] = f$, introduces the need for a *particular integral*.

Suppose for linearly independent u_1, u_2

	$L[u_1] = 0,\ L[u_2] = 0$	complementary functions
and φ is any solution of	$L[\varphi] = f$	principal integral
then	$u = A\,u_1 + B\,u_2 + \varphi$	general solution
and	$L[u] = L[u_1] + L[u_2] + L[\varphi] = 0 + 0 + f = f.$	

Moreover, u is in fact the ***most general solution*** of $L[u] = f$.

This may be demonstrated by supposing that ψ is another solution of $L[u] = f$ (*i.e.* $L[\psi] = f$). Hence

$$L[\psi-\varphi] = L[\psi] - L[\varphi] = f - f = 0,$$

i.e. $\psi - \varphi$ satisfies the homogeneous equation, so we must be able to write

$$\psi - \varphi = C\,u_1 + D\,u_2$$

so

$$\psi = \varphi + C\,u_1 + D\,u_2.$$

Trial solution

As with the first order inhomogeneous equation, we can pose *trial solutions* for the particular integral of the second order inhomogeneous equation.

➡ $y'' + y = 1$

The homogeneous equation gives rise to the characteristic equation $\lambda^2 + 1 = 0$, yielding $\lambda = \pm i$ and complementary functions sin t and cos t.

Consider the trial solution $y = a + bt + ct^2$, and substitute:

$$2c + a + bt + ct^2 = 1$$

As this must hold for all t, we have the particular integral is $y = 1$, and the general solution is

$$y = A \sin t + B \cos t + 1.$$

Clearly, introducing any other powers of t to the trial solution would demonstrate they had zero coefficients.

- $y'' - 3y' - y = t^2 + 2$

The characteristic equation $\lambda^2 - 3\lambda - 1 = 0$ gives $\lambda = (3 \pm \sqrt{13})/2$ and exponential complementary functions.

Try $y = a + bt + ct^2$, and substitute

$$2c - 3(b + 2ct) - (a + bt + ct^2) = (2c - a - 3b) + (-6c - b)t + (-c)t^2 = t^2 + 2$$

Equating coefficients for different powers of t:

$$2c - a - 3b = 2 \qquad t^0$$

$$6c + b = 0 \qquad t^1$$

$$-c = 1 \qquad t^2$$

Solving simultaneously

$$c = -1,$$

$$b = -6c = 6$$

$$a = 2c - 3b - 2 = -2 - 18 - 2 = -22$$

and the general solution

$$y = Ae^{\frac{1}{2}(3+\sqrt{13})t} + Be^{\frac{1}{2}(3-\sqrt{13})t} - t^2 + 6t - 22.$$

- $y'' + 2y' + y = \sin \omega t$

Characteristic equation $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1$ (twice).

Trial solution $y = a \sin \omega t + b \cos \omega t$ and substitute

$$-\omega^2(a \sin \omega t + b \cos \omega t) + 2\omega(a \cos \omega t - b \sin \omega t) + a \sin \omega t + b \cos \omega t = \sin \omega t$$

Equating terms

$$-a\omega^2 - 2b\omega + a = (1-\omega^2)a - 2b\omega = 1$$

$$-b\omega^2 + 2a\omega + b = 2a\omega + (1-\omega^2)b = 0$$

Simultaneous solution gives

$$a = \frac{1-\omega^2}{\left(1+\omega^2\right)^2}, \quad b = \frac{-2\omega}{\left(1+\omega^2\right)^2}$$

and the general solution

$$y = \left(A + Bt\right)e^{-t} + \frac{1-\omega^2}{\left(1+\omega^2\right)^2}\sin \omega t - \frac{2\omega}{\left(1+\omega^2\right)^2}\cos \omega t.$$

- $y'' + y = \cos t$

Characteristic equation $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

Complementary functions $\sin t$ and $\cos t$, so trial function cannot simply be $a \sin t + b \cos t$. Recall a similar problem in §2.1.2, which suggests we try $y = a\,t \sin t + b\,t \cos t$. Substituting

$$(2a \cos t - a\,t \sin t - 2b \sin t - b\,t \cos t) + a\,t \sin t + b\,t \cos t = \cos t$$

Equating terms

$$-at - 2b + at = -2b = 0$$

$$2a - b\,t + bt = 1$$

$$\Rightarrow \qquad a = \tfrac{1}{2}, \quad b = 0$$

$$\Rightarrow \qquad y = A \sin t + B \cos t + \tfrac{1}{2}\, t \sin t.$$

$y'' + 2y' + y = e^{-t}$

The characteristic equation yields a double root $\lambda = -1$, suggesting the solution to the homogeneous problem is $y = (A + Bt)\, e^{-t}$. Hence the trial solution for the particular integral must contain more than just $a\, e^{-t}$ and $b\, t\, e^{-t}$. Try $(a + bt + ct^2)e^{-t}$:

$$(a-2b+2c + (b-4c)t + ct^2)e^{-t} + 2(-a+b + (-b+2c)t - ct^2)e^{-t} + (a + bt + ct^2)e^{-t} = e^{-t}$$

Equating coefficients

$$a-2b+2c - 2a+2b + a \quad = 2c = 1$$

$$b-4c - 2b+4c + b = 0 = 0$$

$$c - 2c + c = 0 = 0$$

Note that the last two do not tell us anything: this is no surprise, since we know that a and b are arbitrary as they represent the homogeneous solution.

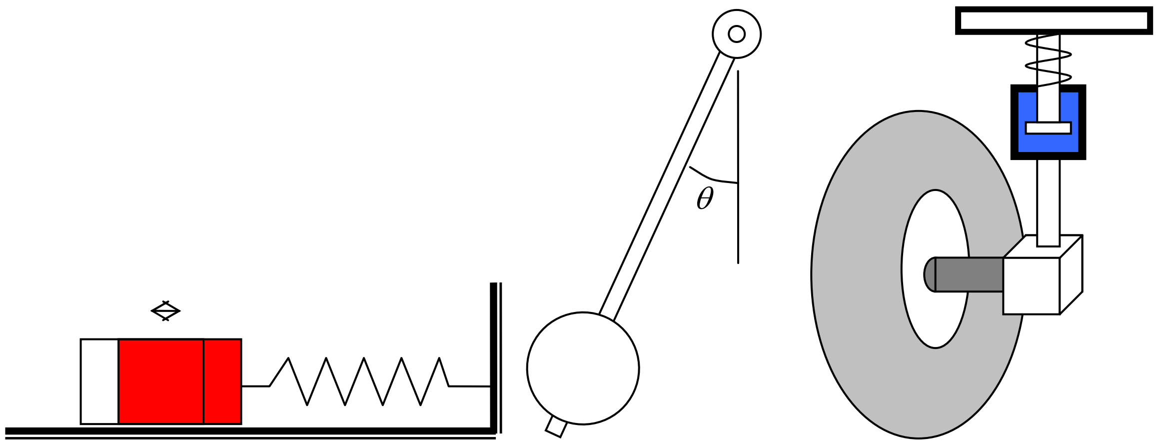
$$\Rightarrow \qquad c = \tfrac{1}{2}$$

Thus the general solution $\qquad y = (A + Bt + \tfrac{1}{2}\, t^2)\, e^{-t}.$

4.3 Applications

4.3.1 Free oscillators

As noted in §4.2, the behaviour of a mass on a spring may be modelled by a second-order linear ordinary differential equation.



If y is the location of the mass, then its acceleration is $a = d^2y/dt^2$, and we assume the force imparted by the spring is $-Ky$, where K is the spring constant (force per unit extension). We shall also assume that the mass is sitting on a lubricating film (*e.g.* oil) or attached to a dashpot so that the friction is proportional to the speed (this will be covered in more detail in the Dynamics course next term), giving drag force $D = 2J\,dy/dt$, say. Following Newton, $F = ma$, where F is the net force, m the mass and a the acceleration, hence

$$-Ky - 2J\frac{dy}{dt} = m\frac{d^2y}{dt^2}.$$

If $k^2 = K/m$ and $\mu = J/m$, then

$$\frac{d^2y}{dt^2} + 2\mu\frac{dy}{dt} + k^2y = 0.$$

WARNING: My notation here is not quite consistent with §4.2 where I had k rather than k^2 in front of the y term.

We get an equation of the same form in a variety of situations, from clock pendulum to car suspension and washing machines.

The characteristic equation

$$\lambda^2 + 2\lambda\,\mu + k^2 = 0 \qquad (**)$$

has solutions

$$\lambda = -\mu \pm \sqrt{\mu^2 - k^2}\,.$$

Undamped oscillation

When $\mu = 0$, then $\lambda = \pm\, ik$, and the homogeneous problem has oscillatory solutions

$$y = A\cos kt + B\sin kt$$

with frequency k rad/s.

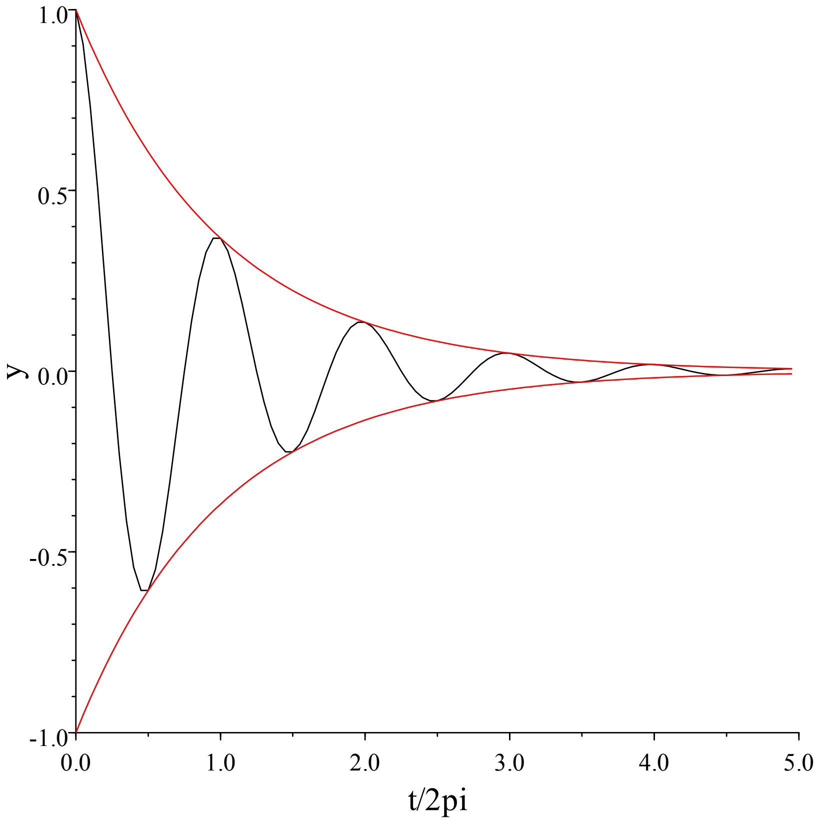
End of Lecture 14

Damped oscillation

When $\mu < k$, the homogeneous problem has oscillatory solutions that decay with time

$$y = e^{-\mu t}(A\cos\Omega t + B\sin\Omega t),$$

where $\Omega^2 = k^2 - \mu^2$. Note that the damping has reduced the frequency.



Critical damping

The decay rate of the oscillation increases as μ increases and the frequency decreases until $\mu = k$, at which point the system is said to be *critically damped* (oscillations have zero frequency).

When $\mu = k$ the characteristic equation gives a double root $\lambda = \mu = k$ and the general solution is

$$y = (A + Bt)\, e^{-\mu t}.$$

Note: Car suspension is roughly critically damped. Too little damping will cause the car to *bounce*, while too much damping will limit the suspension's ability to absorb shocks.

Over damping

When $\mu > k$ the system is *over damped* and there are no oscillations.

4.3.2 Forced oscillator

In the previous section we considered free oscillations, but of course something like a car or a washing machine has a continuous forcing. Suppose this forcing is sinusoidal in nature, then

$$\frac{d^2y}{dt^2} + 2\mu\frac{dy}{dt} + k^2 y = \sin\omega t\ .$$

This has the same set of homogeneous solutions as before, plus a particular integral. Try $y = a \sin \omega t + b \cos \omega t$,

$$\Rightarrow \qquad -\omega^2(a \sin \omega t + b \cos \omega t) + 2\,\omega\mu(a \cos \omega t - b \sin \omega t) + k^2(a \sin \omega t + b \cos \omega t) = \sin \omega t$$

$\Rightarrow$
$$(k^2-\omega^2)a - 2\omega\mu b = 1$$
$$2\omega\mu a + (k^2-\omega^2)b = 0$$

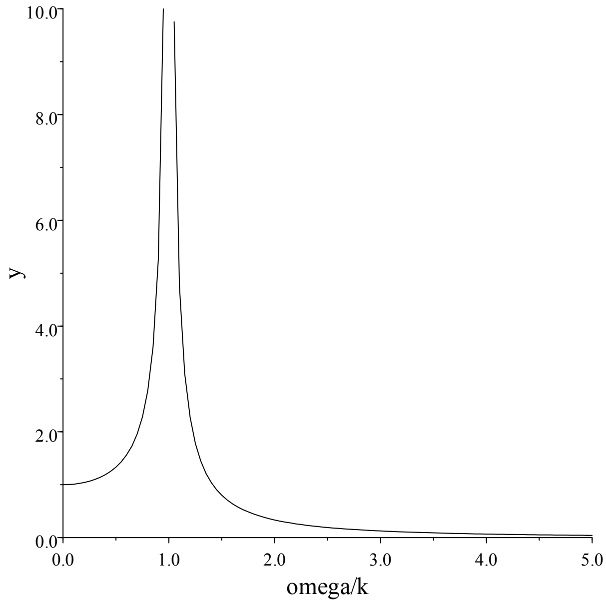
$\Rightarrow$
$$y = \frac{k^2-\omega^2}{\left(k^2-\omega^2\right)^2 + 4\mu^2\omega^2}\sin\omega t - \frac{2\mu\omega}{\left(k^2-\omega^2\right)^2+4\mu^2\omega^2}\cos\omega t$$

Undamped solution

Noting that the particular integral is the *steady* (large time) solution, we can see that for $\mu = 0$ the solution tends towards the steady oscillation

$$y = \frac{1}{k^2-\omega^2}\sin\omega t\,,$$

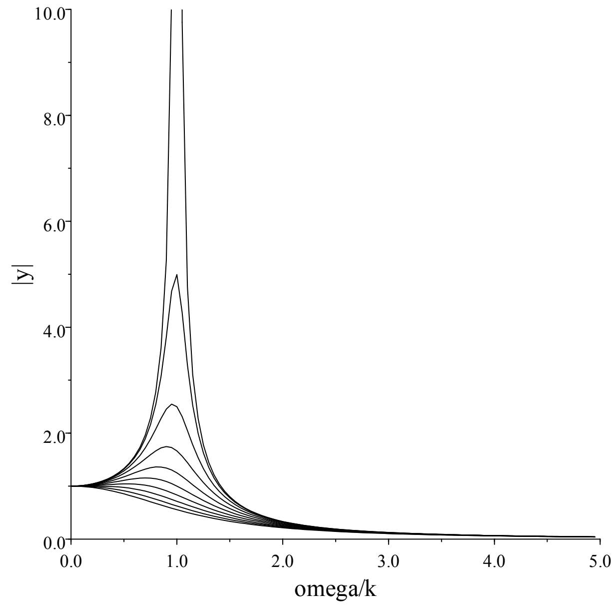
which is singular if the forcing frequency ω matches the frequency of free oscillations, k. This situation is referred to as resonance.



Note that as the forcing frequency comes closer and closer to the resonant frequency, the amplitude of the particular integral increases dramatically.

Damped oscillator

Damping prevents the amplitude from growing without bound as resonance is approached.



4.3.3 RADIOACTIVE SEQUENCES

Suppose A and B are radioactive elements with decay constants λ and μ, respectively, and C is stable. The decay process is

$$A \to B + b,$$

$$B \to C + c.$$

We shall not concern ourselves with the particles b, c emitted. Let $X(t)$, $Y(t)$ and $Z(t)$ be the abundances of A, B and C, with initial values X_0, Y_0 and Z_0, respectively.

$$\frac{dX}{dt} = -\lambda X\,, \tag{1}$$

$$\frac{dY}{dt} = \lambda X - \mu Y\,, \tag{2}$$

$$\frac{dZ}{dt} = \mu Y \tag{3}$$

Note that although Z depends on Y (and thence X), neither X nor Y depend on Z, and so we can consider the first two equations in isolation.

Differentiating (2) $\quad \dfrac{d^2Y}{dt^2} = \lambda \dfrac{dX}{dt} - \mu \dfrac{dY}{dt} = -\lambda^2 X - \mu \dfrac{dY}{dt} \quad$ (using $\dfrac{dX}{dt} = -\lambda X$)

and eliminating X using (2) $\quad \dfrac{d^2Y}{dt^2} = -\lambda\left(\dfrac{dY}{dt} + \mu Y\right) - \mu \dfrac{dY}{dt}$

$$\Rightarrow \quad \frac{d^2Y}{dt^2} + (\lambda + \mu)\frac{dY}{dt} + \lambda\mu Y = 0\,.$$

Characteristic equation $\quad \alpha^2 + (\lambda + \mu)\alpha + \lambda\mu = 0$

$$\Rightarrow \quad Y = H\,e^{-\lambda t} + K\,e^{-\mu t}.$$

At $t = 0$, $Y = Y_0 \Rightarrow$ $H + K = Y_0$.

What is the other condition we need to tie down H and K?

As we know X_0, then we can determine dX/dt at $t = 0$, and so also know that

$$\left.\frac{dY}{dt}\right|_{t=0} = \lambda X_0 - \mu Y_0,$$

Since $\frac{dY}{dt} = -\lambda H e^{-\lambda t} - \mu K e^{-\mu t}$, then $-\lambda H - \mu K = \lambda X_0 - \mu Y_0$

$$\Rightarrow \qquad (\lambda-\mu)K = \lambda X_0 + (\lambda-\mu)Y_0$$

$$K = Y_0 - \frac{\lambda}{\mu-\lambda}X_0, \quad H = \frac{\lambda}{\mu-\lambda}X_0$$

so

$$Y = Y_0 e^{-\mu t} + \frac{\lambda X_0}{\mu-\lambda}\left(e^{-\lambda t} - e^{-\mu t}\right). \qquad (*)$$

Of course, we could have solved this in a different way, since X does not depend on Y:

$$\frac{dX}{dt} = -\lambda X \quad \Rightarrow X = X_0\, e^{-\lambda t}$$

so

$$\frac{dY}{dt} + \mu Y = \lambda X_0 e^{-\lambda t}$$

which we know gives the complementary function $Y = Y_0\, e^{-\mu t}$ and we can seek a particular integral of the form $Y = a\, e^{-\lambda t}$ to recover (*) once the initial condition is imposed.

Note that if $\lambda = \mu$ then, as we have seen before, we have a degenerate case where we need to introduce a solution of the form $a\, t\, e^{-\lambda t}$ to the trial solution (or to the solution of the second order system).

The second approach is easier in this case, but the first approach, using an equation and its derivative to help eliminate a variable, is often useful.

Solving a differential equation is only half the story: we also need to understand the solution. It is instructive to rewrite the solution as

$$Y = Y_0 e^{-\mu t} + \frac{\lambda X_0}{\mu-\lambda}\left(1 - e^{-(\mu-\lambda)t}\right)e^{-\lambda t}.$$

The *first term* is simply the decay of the quantity of B initially in the system.

The *second term* embodies the balance between A being converted to B, and B being converted to C.

Suppose $\mu \gg \lambda$, then B evolves faster than A, and for large time we have

$$Y \approx \frac{\lambda X_0}{\mu-\lambda}e^{-\lambda t} = \frac{\lambda X}{\mu-\lambda}$$

and we have Y move towards a *quasi-equilibrium* with X. [For short time we may consider X as approximately constant.] Overall, there is a rapid initial response towards this quasi-equilibrium, followed by a slow run-down on timescale $1/\lambda$.

Suppose $\mu \ll \lambda$, then initial A will be converted to B faster than B can decay, thus

$$Y \approx (Y_0 + X_0)\, e^{-\mu t}.$$

Suppose $\mu = \lambda$. As we have seen before, this will introduce terms of the form $t\, e^{-\lambda t}$ and we find

$$Y = (Y_0 + \lambda X_0\, t)\, e^{-\lambda t}.$$

To determine Z, the quantity of C, we can solve

$$\frac{dZ}{dt} = \mu Y = \left(Y_0 + \lambda X_0 t\right) e^{-\lambda t} \qquad Z(0) = Z_0$$

to find

$$Z = Z_0 + \frac{Y_0}{\lambda}\left(1 - e^{-\lambda t}\right) - \frac{X_0}{\lambda}\left[\left(1 + \lambda t\right) e^{-\lambda t} - 1\right].$$

4.4 Reduction of order

We have seen that there is a relationship between the differential equation (with constant coefficients) and the polynomial characteristic equation.

For a general polynomial equation of degree n, $P_n(\lambda) = 0$, then if we know one solution $\lambda = \lambda_1$, say, we can divide $P_n(\lambda)$ by $\lambda - \lambda_1$ exactly, and thus form a new polynomial $P_{n-1}(\lambda)$ of degree $n-1$.

For *linear* ordinary differential equations we can undertake a similar process, as was illustrated in §4.2.9. Suppose $L_n[y] = 0$ is a linear ode of order n, then we may (at least in principle) divide the equation by a known solution to obtain a ode of order $n-1$, *i.e.* $L_{n-1}[y'] = 0$. If we can solve this for y', then we can simply integrate the solution to find y. Note that we need not restrict attention to equations with constant coefficients.

Consider

$$L[y] \equiv y'' + p(x)\, y' + q(x)\, y = 0.$$

Suppose u satisfies $L[u] = 0$ (hence u is a solution to the homogeneous equation). We then define a new function

$$v = \psi u$$

$$\Rightarrow \qquad v' = \psi' u + \psi u'$$

$$v'' = \psi'' u + 2\, \psi' u' + \psi u''$$

$$\Rightarrow \qquad L[v] = \psi\,(u'' + pu' + qu) + \psi'' u + 2\psi' u' + p\psi' u = 0.$$

Now the first term in brackets vanishes, so we may rewrite this as

$$\frac{\psi''}{\psi'} = -2\frac{u'}{u} - p\,.$$

Integrating $\ln|\psi'| = -2\ln|u| - \int p\,dt \;\Rightarrow\; \psi' = u^{-2}e^{-\int p\,dx}$

hence $$\psi = \int u(x')^{-2} e^{-\int^{x'} p\,dx''}\,dx'$$

so that $$v = u(x)\,\psi(x) = u(x)\int u(x')^{-2} e^{-\int^{x'} p\,dx''}\,dx' \quad \text{is a solution.}$$

➡ $y'' - 5y' + 6y = 0.$

Characteristic equation $\lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0 \quad \Rightarrow \lambda = 2, 3$

Taking the e^{2x} solution, $\psi' = u^{-2}e^{-\int p\,dx} = e^{-4x}e^{-\int^x -5\,dx} = e^{-4x}e^{5x} = e^x$,

and integrating $\psi = e^x$

so $v = \psi u = e^x e^{2x} = e^{3x}$

is another solution.

End of Lecture 15

➡ $y'' + 2k\,y' + k^2 y = 0$

Characteristic equation gives double root $\lambda = -k$.

Take $u = e^{-kx}$ as one solution, then $\psi' = u^{-2}e^{-\int p\,dx} = e^{2kx}e^{-\int^x 2k\,dx} = e^{2kx}e^{-2kx} = e^0 = 1$.

Integrating $\psi = x$

so the other complementary function is $u\,\psi = x\,e^{-kx}$, and the general solution is

$$y = (A + B\,t)\,e^{-kx}$$

4.5 Generalisation to higher orders

The techniques we have developed for second order odes may be applied to higher order equations. Suppose we have the n^{th} order equation

$$L[y] \equiv a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \qquad a_i = const$$

This gives the polynomial characteristic equation

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda' + a_0 y = 0$$

with roots (potentially complex) $\lambda = \lambda_1, \lambda_2, \ldots\lambda_n$ and complementary functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x}$ being individual solutions of first order equations of the form $y' - \lambda_1 y = 0$.

Suppose one of the roots, λ_k is repeated m times. From the discussion on repeated roots in §4.2.8, we might predict that this will produce a series of complementary functions of the form

$$\left(A_0 + A_1 x + A_2 x^2 + \cdots + A_{m-1}x^{m-1}\right)e^{\lambda_k x}.$$

Recall integrating factors from §2.3.2 that for $y' + qy = 0$ then

$$I\left(\frac{dy}{dx}+qy\right)=\frac{d}{dx}(Iy) \quad \text{or} \quad \left(\frac{d}{dx}+q\right)y=\frac{1}{I}\frac{d}{dx}(Iy),$$

where I is the integrating factor

$$I=e^{+\int q\,dx}.$$

Now for our repeated root λ_k we have

$$\left(\frac{d}{dx}-\lambda_k\right)\left(\frac{d}{dx}-\lambda_k\right)\cdots\left(\frac{d}{dx}-\lambda_k\right)y=\left(\frac{d}{dx}-\lambda_k\right)^m y=0.$$

Noting that $\left(\frac{d}{dx}-\lambda_k\right)y=\frac{1}{I}\frac{d}{dx}(Iy)$, then

$$\left(\frac{d}{dx}-\lambda_k\right)^m y=\left(\frac{d}{dx}-\lambda_k\right)^{m-1}\left(\frac{1}{I}\frac{d}{dx}(Iy)\right)$$

$$=\left(\frac{d}{dx}-\lambda_k\right)^{m-2}\left(\frac{1}{I}\frac{d}{dx}\left(I\frac{1}{I}\frac{d}{dx}(Iy)\right)\right)=\left(\frac{d}{dx}-\lambda_k\right)^{m-2}\left(\frac{1}{I}\frac{d^2}{dx^2}(Iy)\right)$$

$$=\left(\frac{d}{dx}-\lambda_k\right)^{m-3}\left(\frac{1}{I}\frac{d^3}{dx^3}(Iy)\right)$$

$$=\frac{1}{I}\frac{d^m}{dx^m}(Iy)=0$$

Hence, integrating this m^{th} order equation,

$$Iy = A_0 + A_1x + A_2x^2 + \ldots + A_{m-1}x^{m-1},$$

and for $y' - \lambda_k y = 0$, the integrating factor is $I=e^{-\lambda_k x}$, so

$$y=\left(A_0+A_1x+A_2x^2+\cdots+A_{m-1}x^{m-1}\right)e^{\lambda_k x},$$

as expected.

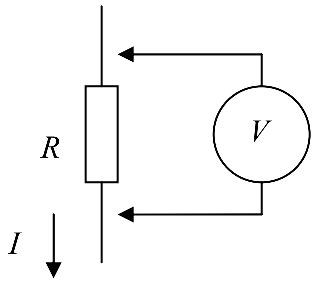
4.6 More on forcing

4.6.1 ELECTRICAL CIRCUITS

Here we consider electrical circuits that contain resistors, capacitors and inductors, connected by perfect conductors (wires; of course, real copper wire is not a perfect conductor).

Ohms law

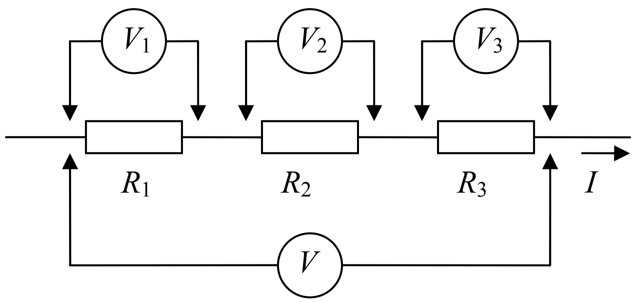
The resistor is the most basic component of a circuit (other than a wire). The *resistance* R relates the voltage V across the resistor to the current I that passes through it.



$$V = I\,R.$$

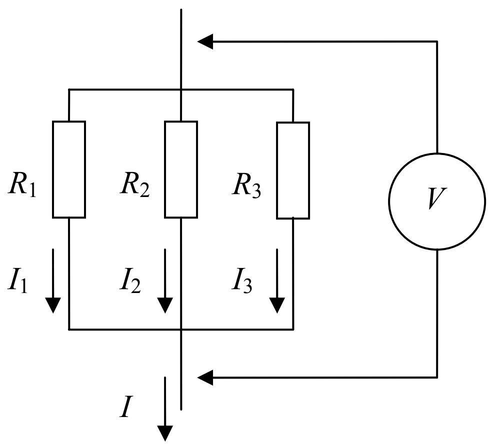
Current is measured in amps (A; coulombs per second; one coulomb contains 6.25×10^{18} electrons) and, by convention, flows from positive to negative. [In fact, the electrons flow from negative to positive.] Voltage, sometimes referred to as *potential*, is measured in volts (V), and resistance in ohms (Ω).

Resistors in series, with no intermediate connections, must each have the same current through them, so the effective resistance is additive.



$$I = \frac{V}{R_1 + R_2 + R_3} = \frac{V_1}{R_1} = \frac{V_2}{R_2} = \frac{V_3}{R_3}$$

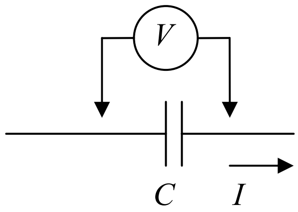
Resistors in parallel share the same voltage, with each contributing to the current.



$$I = I_1 + I_2 + I_3 = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}.$$

Capacitors

Capacitors (sometimes referred to as *condensers*) consist of two conductors separated by an insulator. The conductors accumulate a charge (electrons) in response to the application of a potential (voltage) across them. The charge is created or dissipated by a current flowing into or out of the capacitor, but no current passes from one conductor to the other.



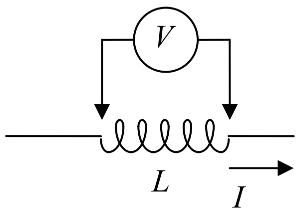
The voltage across the capacitor lags behind the current:

$$\frac{dV}{dt} = \frac{I}{C}.$$

The capacitance, C is measured in Farads, representing charge Q (coulombs) per volt.

Inductors

Inductors typically consist of a coiled conductor that creates a magnetic field. The magnetic field is generated by the current passing through the coil, but a changing magnetic field in turn induces a voltage in the coil.



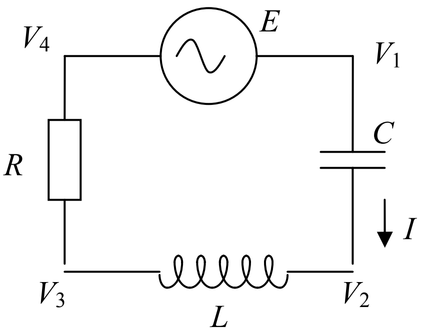
The current lags behind the voltage:

$$V = L\frac{dI}{dt},$$

and the inductance L is measured in Henry.

Simple circuit

Consider a simple circuit excited by an oscillating voltage:



Suppose the circuit is excited by the voltage $E = E_0\, e^{i\omega t}$ [E_0 is complex and we assume the actual voltage is the real part of $E_0\, e^{i\omega t}$, thus the complex amplitude E_0 also contains the phase information.]

Conservation of charge implies here that each of the components must also see the same current *I*.

$$\frac{d}{dt}\left(V_2 - V_1\right) = \frac{I}{C}$$

$$V_3 - V_2 = L\frac{dI}{dt} \qquad \frac{d}{dt}\left(V_3 - V_2\right) = L\frac{d^2 I}{dt^2}$$

$$V_4 - V_3 = RI \qquad \frac{d}{dt}\left(V_4 - V_3\right) = R\frac{dI}{dt}$$

Adding the contributions for the individual components, and noting that $V_4 - V_1$ must be equal to *E*:

$$\frac{d}{dt}\left(V_4 - V_1\right) = L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = i\omega E_0 e^{i\omega t} ,$$

a second order linear ordinary differential equation with constant coefficients. Rearranging

$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{i\omega E_0}{L}e^{i\omega t} .$$

We can see that the resistance *R* leads to damping, and that the frequency of the undamped free oscillation is $\frac{1}{\sqrt{LC}}$. More generally, the characteristic equation has solutions

$$\lambda = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} ,$$

For the particular integral, try the form $I = a\, e^{i\omega t}$ (*a* complex):

$$-a\omega^2 + ia\omega\frac{R}{L} + \frac{a}{LC} = \frac{i\omega E_0}{L}$$

$$\Rightarrow \quad a = \frac{iE_0}{L}\frac{\omega}{\frac{1}{LC}-\omega^2+i\omega\frac{R}{L}} = \frac{i\omega CE_0}{1-LC\omega^2+i\omega RC}$$

$$= \frac{E_0C\omega\left(\omega RC+i\left(1-LC\omega^2\right)\right)}{\left(1-LC\omega^2\right)^2+\left(\omega RC\right)^2}$$

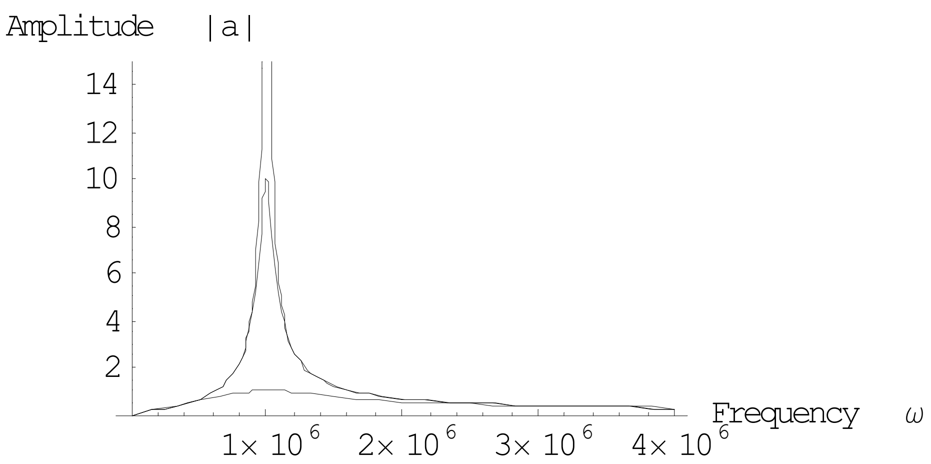
The particular integral $a\,e^{i\omega t}$ has the same form as we saw previously for the resonance of a mass on a spring,

$$y = \frac{k^2-\omega^2}{\left(k^2-\omega^2\right)^2+4\mu^2\omega^2}\sin\omega t - \frac{2\mu\omega}{\left(k^2-\omega^2\right)^2+4\mu^2\omega^2}\cos\omega t$$

except that the forcing has an additional factor of ω in it since it is the differential of E rather than E itself that is important in driving the circuit.

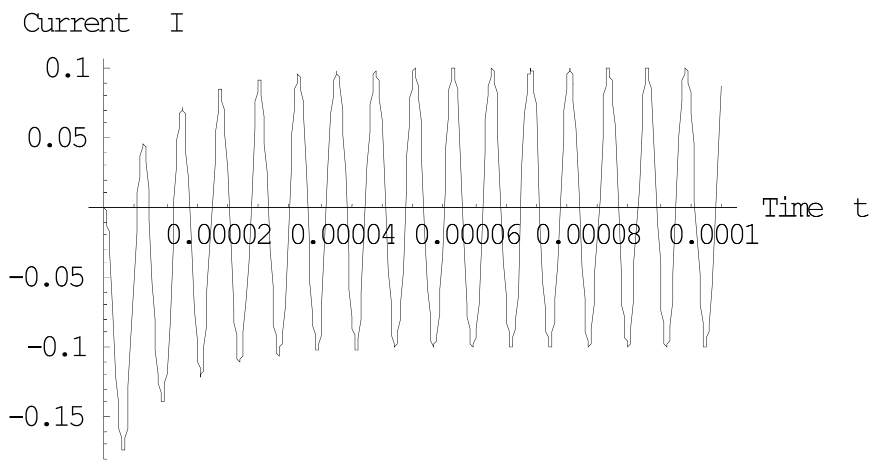
The modulus of the complex amplitude

$$|a| = \frac{E_0C\omega\left(\left(1-LC\omega^2\right)^2-\left(\omega RC\right)^2\right)}{\left(1-LC\omega^2\right)^2+\left(\omega RC\right)^2}$$



Amplitude with frequency for L = 1μH, C = 1μF and R = 0.01, 0.1 and 1Ω.

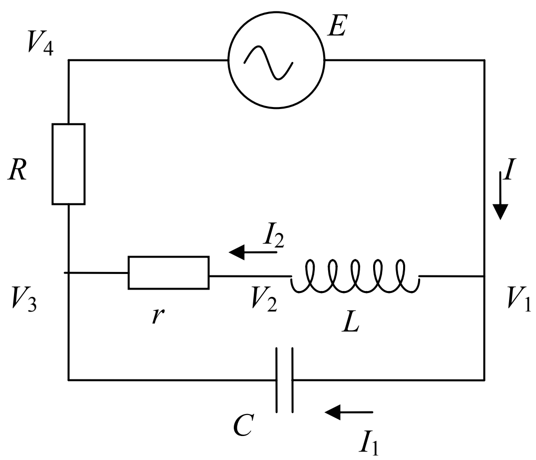
The resonance peak is damped by the presence of the resistive element in the circuit.



$L = 10\ \mu$H, $C = 1\ \mu$F, $R = 10\Omega$, $\omega = 1$ MHz.

Typical tuning circuit

This is the basic idea of the tuning circuit in a radio or television, however, the amplitude of the resonant signal is strongly affected by the resistance, *R*. A typical tuning circuit is a little more complex than the previous example. This may be illustrated by



Analysis is similar, but with

$I\,R = V_4 - V_3$ $$\frac{d}{dt}\left(V_4 - V_3\right) = R\frac{dI}{dt}$$

$I = I_1 + I_2$

$$\frac{d}{dt}\left(V_3 - V_1\right) = \frac{I_1}{C}$$

$V_2 - V_1 = L\dfrac{dI_2}{dt}$ $$\frac{d}{dt}\left(V_2 - V_1\right) = L\frac{d^2 I_2}{dt^2}$$

$I_2\,r = V_3 - V_2$ $$\frac{d}{dt}\left(V_3 - V_2\right) = r\frac{dI_2}{dt}$$

Eliminating V_2 $$V_3 - V_1 = L\frac{dI_2}{dt} + rI_2$$

Noting that inductive and capacitative branches see the same voltage difference

$$\frac{I_1}{C} = L\frac{d^2I_2}{dt^2} + r\frac{dI_2}{dt}$$

Equating $V_4 - V_1$ with E $$V_4 - V_1 = L\frac{dI_2}{dt} + rI_2 + RI = E_0e^{i\omega t}$$

Noting $I = I_1 + I_2$

$$L\frac{dI_2}{dt} + rI_2 + R\left(I_1 + I_2\right) = L\frac{dI_2}{dt} + \left(R + r\right)I_2 + RC\left(L\frac{d^2I_2}{dt^2} + r\frac{dI_2}{dt}\right)$$

$$= LRC\frac{d^2I_2}{dt^2} + \left(L + rRC\right)\frac{dI_2}{dt} + \left(R + r\right)I_2 = E_0e^{i\omega t}$$

$$\Rightarrow \qquad L\frac{d^2I_2}{dt^2} + \left(\frac{L}{RC} + r\right)\frac{dI_2}{dt} + \frac{1}{C}\left(1 + \frac{r}{R}\right)I_2 = \frac{E_0}{RC}e^{i\omega t}$$

Similar to before, but with

$$R \to r + \frac{L}{RC}$$

$$\frac{1}{C} \to \frac{1}{C}\left(1 + \frac{r}{R}\right)$$

$$E_0 \to \frac{-iE_0}{\omega RC}.$$

The characteristic equation

$$L\lambda^2 + \left(\frac{L}{RC} + r\right)\lambda + \frac{1}{C}\left(1 + \frac{r}{R}\right) = 0$$

has solutions

$$\lambda = -\frac{1}{2L}\left(\frac{L}{RC} + r\right) \pm \frac{1}{2L}\sqrt{\left(\frac{L}{RC} + r\right)^2 - \frac{4L}{C}\left(1 + \frac{r}{R}\right)}.$$

Suppose $R \to \infty$ and $r \to 0$, then

$$\lambda \to -\frac{1}{2L}\left(\frac{L}{RC} + r\right) \pm \frac{1}{2L}\sqrt{-\frac{4L}{C}}$$

$$= -\frac{1}{2L}\left(\frac{L}{RC} + r\right) \pm i\sqrt{\frac{1}{LC}}$$

$$== -\frac{1}{2L}\left(\frac{L}{RC} + r\right) \pm i\omega_0$$

where $\omega_0 = (LC)^{-1/2}$ is the oscillation in the $L - C$ part of the circuit, with little leakage into the reset of the circuit as R is large. The first term is small and represents the slow exponential decay, both due to the small resistance in series with the inductor (causing dissipation in the $L - C$ network) and due to the small leakage through R.

For particular integral, again try a solution of the form $I_2 = a\,e^{i\omega t}$:

$$-a\omega^2 L + ia\omega\left(\frac{L}{RC} + r\right) + a\frac{1}{C}\left(1 + \frac{r}{R}\right) = \frac{E_0}{RC}$$

$$\Rightarrow \quad a = -a\omega^2 L + ia\omega\left(\frac{L}{RC} + r\right) + a\frac{1}{C}\left(1 + \frac{r}{R}\right) = \frac{-E_0/RCL}{\omega^2 - \omega_0^2\left(1 + \frac{r}{R}\right) - i\omega\left(\frac{1}{RC} + \frac{r}{L}\right)}$$

$$= \frac{\omega^2 - \omega_1^2 + i\omega\kappa}{\left(\omega^2 - \omega_1^2\right)^2 + \omega^2\kappa^2}\frac{E_0}{RCL}$$

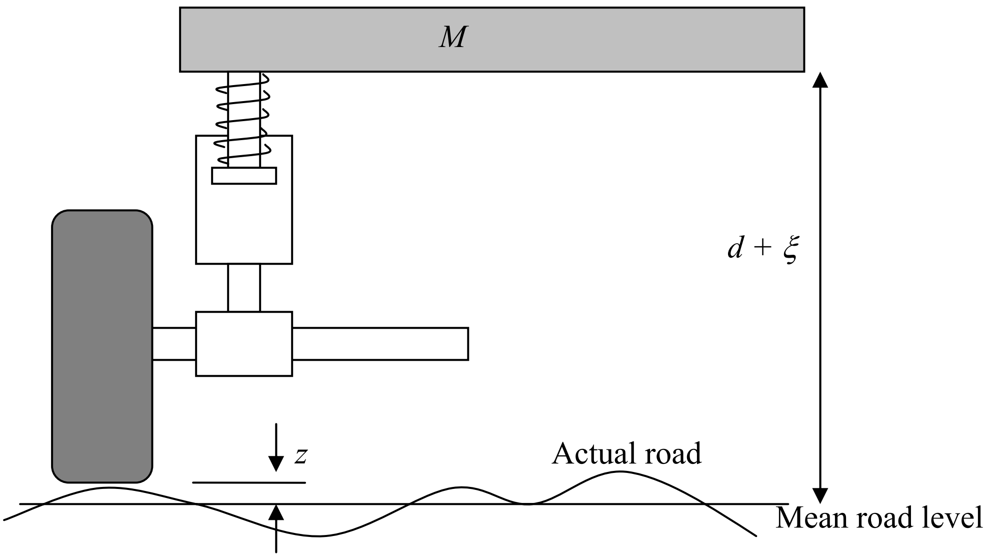
where $\omega_1^{\,2} = \omega_0^{\,2}(1 + r/R)$ and $\kappa = r/L + 1/RC$. The functional form of the dependence on ω is, of course, the same as before.

When $r \ll \sqrt{(L/C)}$ and $R \gg \sqrt{(L/C)}$ then $\kappa \ll \omega_0 = 1/\sqrt{(LC)}$, and $r/R \ll 1$, so $\omega_1 \approx \omega$ and the circuit will oscillate at a resonant frequency close to $1/\sqrt{(LC)}$.

End of Lecture 16

4.6.2 CAR SUSPENSION

Consider again our idealised car suspension



Let d be the equilibrium height of the car above the road surface, and $d + \xi$ be the instantaneous height above the mean road level, so ξ is the departure form the equilibrium location.

Excess upward force on car due to spring $S = -\gamma(\xi - z)$.

Upward force on car due to dashpot $D = -2J\left(\dot{\xi} - \dot{z}\right)$.

Assuming wheel has no mass and the tyre is rigid,

$$M\ddot{\xi} + 2J\left(\dot{\xi} - \dot{z}\right) + \gamma\left(\xi - z\right) = 0\,.$$

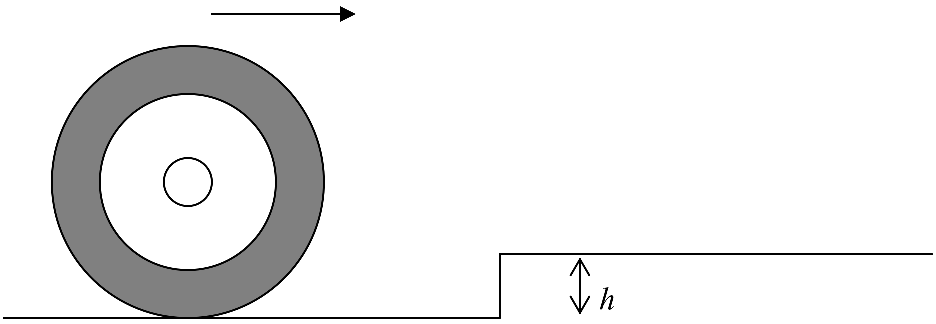
[We also assume the wheel remains in contact with the road.] Let $\mu = J/M$ and $\sigma^2 = \gamma/M$, then

$$\ddot{\xi} + 2\mu\dot{\xi} + \sigma^2\xi = 2\mu\dot{z} + \sigma^2 z\,.$$

As noted earlier in §4.3.2, an ideal car is critically damped, so $\mu = \sigma$. [A real car typically has $\mu > \sigma$ to allow for future wear of the dashpot.]

Mounting a curb

Suppose at time $t = 0$ the car goes over a step in the road from $z = 0$ for $t < 0$ to $z = h$ for $t > 0$.



If the car is travelling at a constant speed U, there is an equivalence between position and time through $x = U\,t$. Road surface may be described as

$$z = h\,H(t)$$

where $H(t)$ is the Heaviside step function (see §1.11). The vertical *speed* of this surface is therefore

$$\dot{z} = h\,\delta\left(t\right)$$

where $\delta(t)$ is the Dirac delta function. Note that we are assuming here that car wheel moves vertically upward as it changes level, whereas the trajectory will be more gradual for a typical wheel with a diameter much larger than the step height.

The equation therefore becomes

$$\ddot{\xi} + 2\mu\dot{\xi} + \sigma^2\xi = 2\mu h\delta\left(t\right) + \sigma^2 hH\left(t\right).$$

Clearly, for $t < 0$ nothing happens and $\xi = 0$. As car goes over curb, ξ adjusts continuously: the spring supplies a finite force, although the dashpot provides an infinite one because $\dot{z}$ is infinite but $\dot{\xi}$ is finite.

For $t > 0$, we therefore have

$$\ddot{\xi} + 2\mu\dot{\xi} + \sigma^2\xi = \sigma^2 h\,,$$

with the initial condition $\xi(0) = 0$. But need a second one, on $\dot{\xi}$, say.

To get $\dot{\xi}$, integrate equations over instant of crossing the step,

$$\int_{0_-}^{0_+} \xi\,dt = 0\,, \qquad \int_{0_-}^{0_+} \dot{\xi}\,dt = 0 \text{ because } \dot{\xi} \text{ is finite}$$

$$\int_{0_-}^{0_+} \ddot{\xi}\, dt = \left[\dot{\xi}\right]_{0_-}^{0_+} = \dot{\xi}(0_+), \text{ since } \dot{\xi}(0_-) = 0$$

$$\int_{0_-}^{0_+} z\, dt = 0, \qquad \int_{0_-}^{0_+} \dot{z}\, dt = [z]_{0_-}^{0_+} = h$$

so
$$\int_{0_-}^{0_+} \ddot{\xi} + 2\mu\dot{\xi} + \sigma^2\xi\, dt = \dot{\xi}(0_+) = \int_{0_-}^{0_+} 2\mu\dot{z} + \sigma^2 z\, dt = 2\mu h .$$

Here we have the *impulse* that accelerates the car upwards due to the dash pot rather than simply the spring.

The particular integral is clearly $\xi = h$, while in the critical damping case, $\mu = \sigma$, and the complementary functions are

$$\xi = (A + B\, t)\, e^{-\sigma t}$$

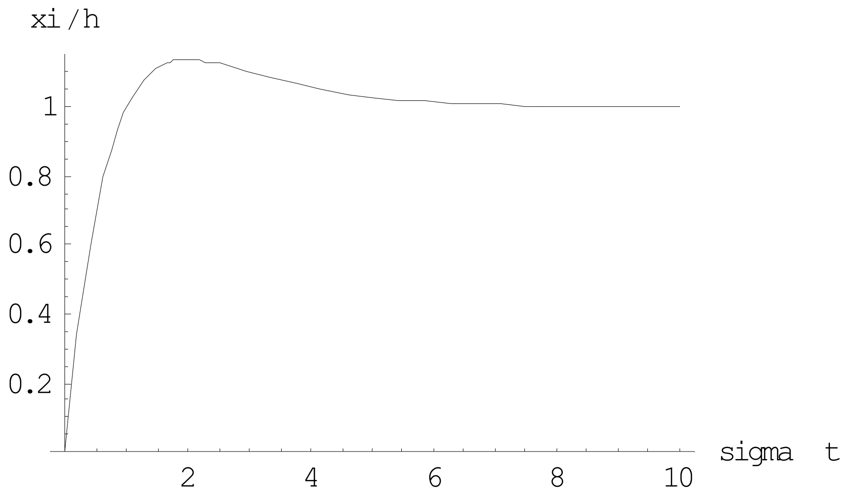
Thus the general solution

$$\xi = (A + B\, t)\, e^{-\sigma t} + h$$

$$\Rightarrow \qquad \dot{\xi} = \sigma(h - Bt)e^{-\sigma t} + Be^{-\sigma t}$$

satisfies the boundary condition $\xi(0) = 0 \Rightarrow A = -h$, and $\dot{\xi}(0) = 2\mu h = 2\sigma h \Rightarrow B = \sigma h$.

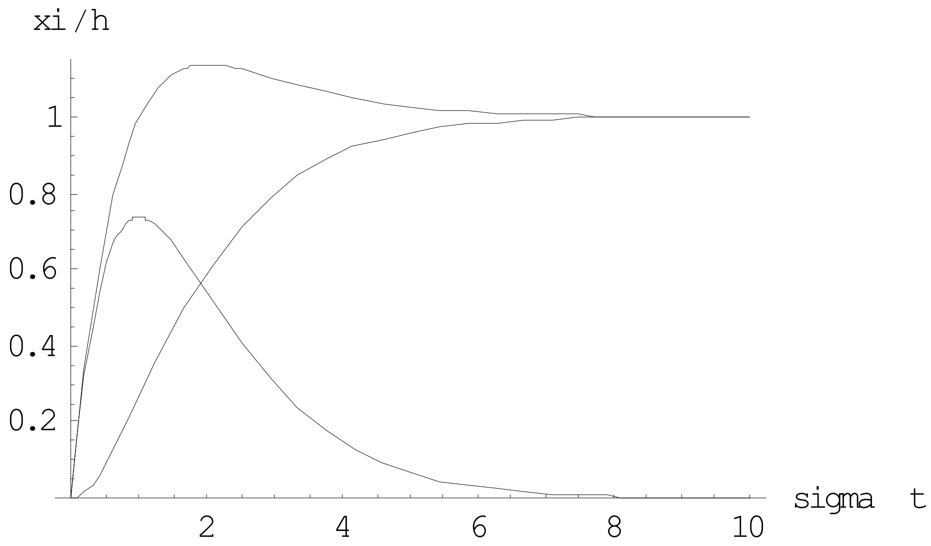
Hence
$$\xi = [1 - (1 - \sigma t)\, e^{-\sigma t}]\, h.$$



Maximum height car reaches is attained when $\dot{\xi} = \sigma h(2 - \sigma t)e^{-\sigma t} = 0 \Rightarrow \sigma t = 2 \Rightarrow$ $\xi/h = 1 + e^{-2} \approx 1.14$.

If there were no impulse at $t = 0$, then the velocity at $t = 0$ would vanish giving $B = -\sigma h$ and $\xi/h = 1 - (1 + \sigma t)\, e^{-\sigma t}$.

Conversely, if there were just the impulse – representing a very short isolated bump such as a speed ramp at speed – then we would have $\xi = (A + B\, t)\, e^{-\sigma t}$ with $\xi(0) = 0$ and $\dot{\xi}(0) = 2\sigma h$ giving $A = 0$, $B = 2\sigma h$.



If the car had been underdamped, $\mu < \sigma$, then going over the curb or over a speed ramp will set up oscillations.

Suppose $\mu = ½\,\sigma$, then for a curb taken at speed we have for, $t > 0$,

$$\ddot{\xi} + \sigma\dot{\xi} + \sigma^2\xi = \sigma^2 h\,,$$

giving the solution to the characteristic equation

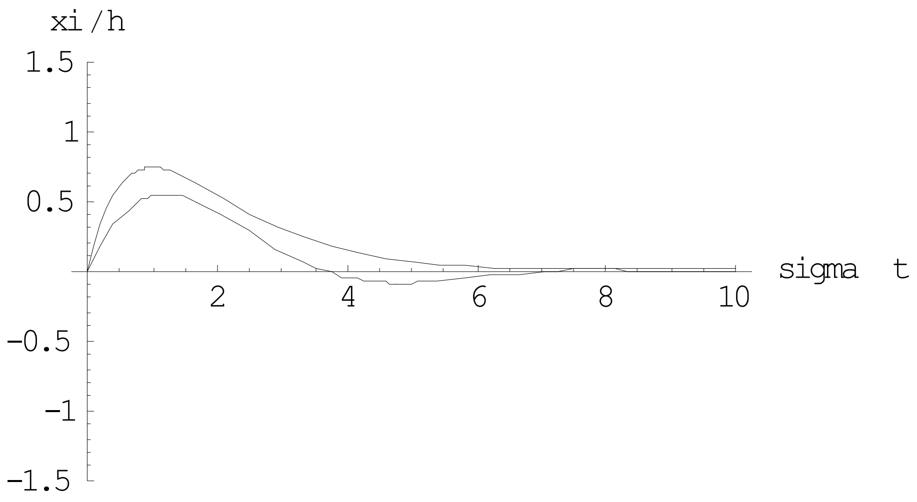
$$\lambda = -\frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4} - \sigma^2} = \frac{\sigma}{2}\left(-1 \pm i\sqrt{3}\right)$$

and the general solution

$$\xi = \left(A\cos\left(\frac{\sqrt{3}}{2}\sigma t\right) + B\sin\left(\frac{\sqrt{3}}{2}\sigma t\right)\right)e^{-\frac{1}{2}\sigma t}.$$

The initial condition $\xi(0) = 0$ gives $A = 0$, while $\dot{\xi} = \frac{1}{2}B\left(\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sigma t\right) - \sin\left(\frac{\sqrt{3}}{2}\sigma t\right)\right)e^{-\frac{1}{2}\sigma t}$ must give $\dot{\xi}(0) = 2kh = \sigma h$, so $B = \frac{2}{\sqrt{3}}h$ and the solution is

$$\frac{\xi}{h} = \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sigma t\right)e^{-\frac{1}{2}\sigma t}$$



Response to an impulse with critical damping (upper) and half critical damping (lower).

Normal road surface

A normal road surface will have undulations over a variety of wave lengths. However, we can understand much by considering a road surface with simple sinusoidal undulations of constant wave length and frequency. [The idea of superposition allows us to simply add the response for different surfaces. This links in with the idea of Fourier decomposition you will come across later in the tripos.]

Suppose $$z = h\cos\beta x = h\cos\beta Ut$$

where U is the speed of the car and $2\pi/\beta$ and h are the wavelength and amplitude of the undulations of the road surface. For a car with critically damped suspension we therefore have

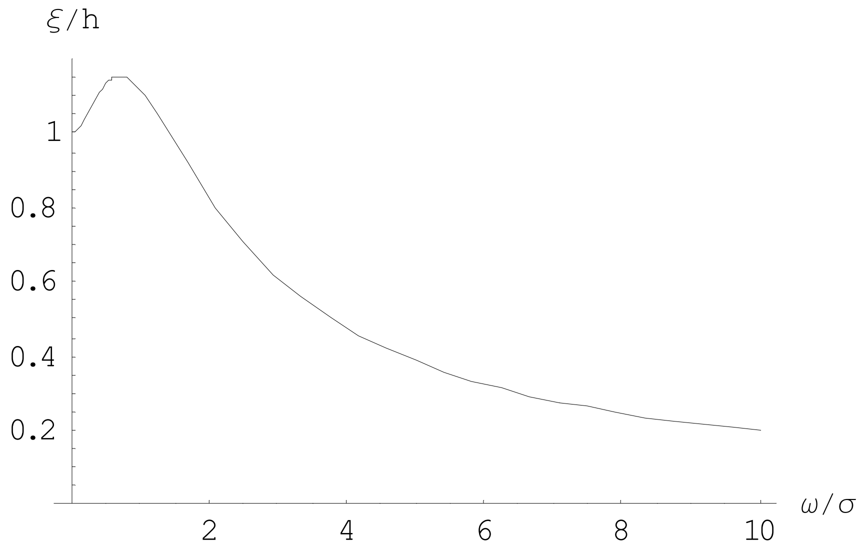
$$\ddot{\xi} + 2\sigma\dot{\xi} + \sigma^2\xi = 2\sigma\dot{z} + \sigma^2 z = h\sigma\,\mathrm{Re}\left(\left(2i\omega + \sigma\right)e^{i\omega t}\right)$$

where $\omega = \beta U$. For a car moving steadily, the initial transients will dye out and only the particular integral will remain. Trying a solution of the form $a\,e^{i\omega t}$ leads to

$$-a\omega^2 + 2ia\omega\sigma + a\sigma^2 = h\sigma\left(2i\omega + \sigma\right)$$

$$\Rightarrow \qquad a = \frac{h\sigma\left(2i\omega + \sigma\right)}{\left(\sigma^2 - \omega^2\right) + 2i\omega\sigma} = \frac{h\sigma\left(2i\omega + \sigma\right)}{\left(\sigma + i\omega\right)^2}$$

$$\Rightarrow \qquad |a|^2 = \frac{h^2\sigma^2\left(\sigma^2 + 4\omega^2\right)}{\left(\sigma^2 - \omega^2\right)^2 + 4\omega^2\sigma^2}$$



As $\omega \to \infty$, $a \to 2\sigma h/\omega$.

Driving very slowly means you go up and down over every bump. Driving fast minimises the vertical motion of the car (so long as your suspension does not *bottom out*). The performance is generally worse at lower speeds.

Of course we need to worry about the tyres on a real car: both add to the suspension and damping (just as well for those of us who ride bikes without suspension as the flexing of the frame provides very little damping).

4.7 Non-constant coefficients

Consider $$x^2 u'' + (1-\lambda-\mu)x u' + \lambda\mu u = 0$$ for λ, μ constant.

Try a solution of the form $u = x^\alpha$. Substitution leads to

$$[\alpha(\alpha-1) + (1-\lambda-\mu)\alpha + \lambda\mu]\, x^\alpha = 0.$$

For a non-trivial solution, the expression in square brackets must vanish. This is the *indicial equation*. Rearranging this for α

$$\alpha^2 - (\lambda+\mu)\,\alpha + \lambda\mu = (\alpha - \lambda)(\alpha - \mu) = 0$$

shows that $u = x^\lambda$ and $u = x^\mu$ are independent solutions to the differential equation (at least provided $\lambda \neq \mu$).

More generally, for any equation of the form

$$a_0 y + a_1 x y' + a_2 x^2 y'' + a_3 x^3 y''' + \ldots = 0 \qquad (*)$$

is said to be *homogeneous in x* and possesses solutions of the form $y = x^\alpha$. Moreover, the values of α are solutions to a polynomial equation, the *indicial equation*, analogous to the characteristic equation for equations with constant coefficients.

We may consider this differential equation in a different way by introducing a new independent variable ξ, say, such that $x\dfrac{d}{dx} = \dfrac{d}{d\xi}$. Thus if we rewrite (*) in the form

$$A_0 y + A_1\left(x\frac{d}{dx}\right)y + A_2\left(x\frac{d}{dx}\right)^2 y + A_3\left(x\frac{d}{dx}\right)^3 y + \cdots = 0$$

then $$A_0 y + A_1\frac{dy}{d\xi} + A_2\frac{d^2y}{d\xi^2} + A_3\frac{d^3y}{d\xi^3} + \cdots = 0$$

the complementary functions for which are of the form $y = e^{\alpha\xi}$.

Noting that $x\frac{d}{dx} = \frac{d}{d\xi} \Rightarrow \frac{dx}{d\xi} = x \Rightarrow \xi = \ln x$, then $y = e^{\alpha\,\xi} = e^{\alpha\ln x} = x^{\alpha}$ as before.

From this relationship between equations with constant coefficients and equations that are homogeneous in x, we know how to handle cases such as repeated roots. For equations with constant coefficients we found the solution to the homogeneous equation to be of the form

$$y = (A + Bt + Ct^2 + \ldots)\, e^{\alpha t}.$$

Correspondingly, for equations that are homogeneous in x with repeated roots of the indicial equation, we will have solutions of the form

$$y = (A + B\ln x + C\,(\ln x)^2 + \ldots)\, x^{\alpha}.$$

We shall return to this later.

End of Lecture 17

4.8 Singular points

For an equation of the form

$$R(t)\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)\,y = F(t).$$

singular points of the equation occur when $R(t)$ vanishes. More generally, a singular point occurs at points where the coefficient in front of the highest derivative vanishes.

Consider again $$x^2 u'' + (1-\lambda-\mu)x\,u' + \lambda\mu\,u = 0$$ for λ, μ constant.

which we have seen gives the indicial equation

$$\alpha^2 - (\lambda+\mu)\,\alpha + \lambda\mu = (\alpha-\lambda)(\alpha-\mu) = 0$$

and complementary functions $u = x^{\lambda}$ and $u = x^{\mu}$ that are independent solutions to the differential equation (at least provided $\lambda \neq \mu$).

Recall the Wronskian

$$W = \begin{vmatrix} x^{\lambda} & x^{\mu} \\ \lambda x^{\lambda-1} & \mu x^{\mu-1} \end{vmatrix} = (\lambda+\mu)\,x^{\lambda+\mu-1},$$

which vanishes at $x = 0$ if $\lambda + \mu > 1$. Writing the equation as

$$u'' + \frac{1-\lambda-\mu}{x}u' + \frac{\lambda\mu}{x^2}u = 0,$$

then we can see that this is consistent with Abel's theorem (§4.2.5):

$$W = W_0 e^{-\int p\,dx} = W_0 e^{-(1-\lambda-\mu)\int\frac{1}{x}dx} = W_0 e^{-(1-\lambda-\mu)\ln|x|}$$
$$= W_0\,|x|^{\lambda+\mu-1}$$

where $p = (1 - \lambda - \mu)/x$. Hence W vanishes at $x = 0$ if $\lambda + \mu > 1$ and p diverges at $x = 0$.

If λ and μ are not both positive integers, then at least one of x^λ, x^μ is not well behaved at $x = 0$ in the sense that some of its derivatives are unbounded and a Taylor Series does not exist for it. For example, if $\lambda < 0$, then x^λ diverges, whereas if $\lambda = 3/2$, then the second derivative diverges at $x = 0$.

Something special must happen at the singular point. In the neighbourhood of this the coefficient of the highest derivative is very small, so there is a possibility of either at least one of the solutions behaving 'badly' (such as $u'' \to \infty$ to keep $x^2 u''$ finite to balance u if $u \neq 0$ at $x = 0$), or if $R(x)\, y''$ does become small, the equation behaves as though it is an equation of lower order. In the latter case, the linearly independent solutions that existed away from the singular point may no longer be linearly independent near the singular point, as is shown by the Wronskian vanishing at the singular point.

4.8.1 Series solution near singular point

To establish what happens near a singular point we shall use a series solution. However, unlike the previous examples, we shall pose

$$y = x^\alpha \sum_{n=0}^{\infty} a_n x^n \qquad a_n \neq 0$$

for an equation of the form

$$R(x)\, y'' + P(x)\, y' + Q(x)\, y = 0$$

in which P, Q and R are continuous and differentiable. Our choice of the form of the series solution is motivated by the discussion in §4.4 where, given a solution u to $L[y] = 0$, we sought a second solution of the form $v = \psi u$. Here we know that there is a solution of the form x^α and we will now seek a second solution.

As we will be equating coefficients for different powers of x, we will also need Taylor Series expansions of $P(x)$, $Q(x)$ and $R(x)$ about the singular point. Our requirement that P, Q and R are continuous and differentiable means that this is possible (or at least can be made possible, *e.g.* if $Q = x^2/(x-a)$ then multiply through by $x-a$ if analysing the solution about $x = a$). Hence, assuming the singular point is $x = 0$,

$$P = P_0 + P_1 x + P_2 x^2 + \ldots,$$
$$Q = Q_0 + Q_1 x + Q_2 x^2 + \ldots,$$
$$R = R_0 + R_1 x + R_2 x^2 + \ldots.$$

➡ $L[y] \equiv x^2 y'' + 3x\, y' + (1+x)y = 0$

Note that P, Q and R are already power series in x, but the equation is not homogeneous in x.

Pose $y = x^\alpha \sum_{n=0}^{\infty} a_n x^n \;\Rightarrow\; \dfrac{dy}{dx} = x^\alpha \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n-1}$ and $\dfrac{d^2y}{dx^2} = x^\alpha \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n-2}$

Substituting

$$x^{\alpha}\sum_{n=0}^{\infty} x^{2}\left(n+\alpha\right)\left(n+\alpha-1\right)a_{n}x^{n-2}+3x\left(n+\alpha\right)a_{n}x^{n-1}+\left(1+x\right)a_{n}x^{n}$$

$$=x^{\alpha}\sum_{n=0}^{\infty} x^{n}\left[\left(n+\alpha\right)\left(n+\alpha-1\right)a_{n}+3\left(n+\alpha\right)a_{n}+\left(1+x\right)a_{n}\right]=0$$

Rearranging

$$x^{\alpha}\sum_{n=0}^{\infty} x^{n}\left[\left(\left(n+\alpha\right)\left(n+\alpha+2\right)+1\right)a_{n}+a_{n-1}\right]=0 \qquad a_{-1}\equiv 0$$

Since $x^{n+\alpha}$ are linearly independent $\forall n$, then square brackets must vanish $\forall n$.

For $n = 0$ $\quad [\alpha(\alpha+2)+1]a_n = (\alpha+1)^2\, a_n = 0 \quad$ since $a_{-1} \equiv 0$

$\Rightarrow \quad \alpha = -1$

and for $n > 0$

$$a_{n}=\frac{-a_{n-1}}{\left(n+\alpha\right)\left(n+\alpha+2\right)+1}$$

Noting that this is only valid when $\alpha = -1$, we have

$$a_{n}=\frac{-a_{n-1}}{\left(n-1\right)\left(n+1\right)+1}=\frac{-a_{n-1}}{n^{2}}$$

[Note that the n^2 in the demoninator ensures rapid convergence when $x = O(1)$].

a_0 arbitrary (from boundary conditions)

$$a_1 = -a_0$$

$$a_2 = a_0/2^2 = \tfrac{1}{4}\, a_0$$

$$a_{3}=\frac{-a_{0}}{2^{2}3^{2}}=\frac{-a_{0}}{\left(3!\right)^{2}}$$

…

$$\Rightarrow \qquad y=\frac{a_{0}}{x}\sum_{n=0}^{\infty}\left(-1\right)^{n}\frac{x^{n}}{\left(n!\right)^{2}}$$

We now have one solution (complementary function) for the equation, but since the equation is second order there must be a second solution.

There is more than one approach, but we shall use what we know about reduction of order in §4.4. We proceed by rewriting the equation in the form

$$y'' + 3\, x^{-1}\, y' + (1+x)x^{-2} y = 0.$$

Let $u = x^{-1}\,(1 - x + x^2/4 - \dots)$ be the known solution, then $y = v \equiv u\ \psi$ is another solution, where

$$\psi' = u^{-2}e^{-\int p\,dx}$$

and $p = 3/x$. Hence

$$\psi' = u^{-2}e^{-3\ln x} = u^{-2}x^{-3} = \frac{x^2}{\left(1 - x + \frac{1}{4}x^2 - \cdots\right)^2}x^{-3}$$ using binomial expansion

$$= x^{-1}\left(1 + 2x + \cdots\right) = \frac{1}{x} + 2 + \cdots$$

Integrating $\psi = \ln x + 2x + \ldots$

Thus $y = u \ln x + x^{\alpha} \times \textit{power series}$

[Could equivalently write $y = u\ (\ln x + \textit{power series})$.]

$L[y] \equiv x^2 y'' + x y' + (x - \eta^2)y = 0$

Pose $y = x^{\alpha}\sum_{n=0}^{\infty} a_n x^n \Rightarrow \frac{dy}{dx} = x^{\alpha}\sum_{n=0}^{\infty}(n+\alpha)a_n x^{n-1}$ and $\frac{d^2y}{dx^2} = x^{\alpha}\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n-2}$

and substitute

$$x^{\alpha}\sum_{n=0}^{\infty} x^2(n+\alpha)(n+\alpha-1)a_n x^{n-2} + x(n+\alpha)a_n x^{n-1} - \eta^2 a_n x^n + x a_n x^n$$

$$= x^{\alpha}\sum_{n=0}^{\infty}\left[(n+\alpha)(n+\alpha-1) + (n+\alpha) - \eta^2\right]a_n x^n + a_n x^{n+1}$$ $a_{-1} = 0$

$$= x^{\alpha}\sum_{n=0}^{\infty}\left[\left[(n+\alpha)^2 - \eta^2\right]a_n + a_{n-1}\right]x^n$$

For $n = 0$ get indicial equation $\alpha^2 = \eta^2 \Rightarrow \alpha = \pm\eta$.

$n > 0$ $$a_n = \frac{a_{n-1}}{(n+\alpha)^2 - \eta^2}$$

Have two sets of solutions. For $\alpha = +\eta$,

$$a_n = \frac{a_{n-1}}{n(n+2\eta)}$$

If η is integer, then $a_0 = \textit{arbitrary}$ from boundary conditions

$$a_1 = \frac{a_0}{1+2\eta},$$

$$a_2 = \frac{a_0}{(1+2\eta)}\frac{1}{2(2+2\eta)} = \frac{(2\eta)!}{2!(2+2\eta)!}a_0,$$

$$\ldots$$

$$a_n = \frac{(2\eta)!}{n!(n+2\eta)!}a_0$$

and $$y = a_0\sum_{n=0}^{\infty}\frac{(2\eta)!}{n!(n+2\eta)!}x^{\eta+n}$$

[Note: Factorials can be generalised for non-integer η.]

For the second solution to the indicial equation we had $\alpha = -\eta$. Hence

$$a_n = \frac{a_{n-1}}{(n+\alpha)^2 - \eta^2} = \frac{a_{n-1}}{n(n-2\eta)}.$$

But if 2η is integer, then the denominator vanishes at $n = 2\eta$ and $a_2\eta = \infty$… Thus we can see a simple series expansion does not work!

4.8.2 FROBENIUS

Not examinable!

There is a way around this: the *method of Frobenius*. However, this is not an examinable part of this course. The idea is that if $\alpha = \alpha_1$ is a repeated root of the indicial equation giving a solution $y = u = u(x;\alpha)$ at $\alpha = \alpha_1$ that satisfies $L[u] = 0$, then $L[u]$ must also be stationary with respect to α at $\alpha = \alpha_1$, so we also have $L\left[\left.\frac{\partial u}{\partial \alpha}\right|_{\alpha=\alpha_1}\right] = 0$.

Recall earlier that we found solutions of the form x^α and $x^\alpha \ln x$ when we had a repeated root. We could have predicted this from the above discussion as $\frac{\partial}{\partial \alpha} x^\alpha = \frac{\partial}{\partial \alpha} e^{\alpha \ln x} = \ln x\, e^{\alpha \ln x} = x^\alpha \ln x$.

4.8.3 LEGENDRE'S EQUATION

Consider

$$(1 - x^2)\, y'' - 2\, x\, y' + \lambda(\lambda + 1)\, y = 0$$

for $-1 \le x \le 1$. This represents axisymmetric oscillations of a sphere when $x = \cos\theta$ ($\Rightarrow 0 \le \theta \le 1$).

Clearly there are singular points at $x = \pm 1$ (corresponding to $\theta = 0, \pi$, the poles). We are interested in the solution between the singularities.

To study the equation near the singular points it is convenient to introduce $\xi = x + 1$, which transforms the equation to

$$\xi(2-\xi)y_{\xi\xi} + 2(1-\xi)y_\xi + \lambda(\lambda+1)y = 0$$

and pose

$$y = \xi^\alpha \sum_{n=0}^{\infty} a_n \xi^n \qquad a_0 \neq 0.$$

End of Lecture 18

Substitute into the equation

$$\xi^\alpha \sum_{n=0}^{\infty} \xi(2-\xi)(n+\alpha)(n+\alpha-1)a_n\xi^{n-2} + 2(1-\xi)(n+\alpha)a_n\xi^{n-1} + \lambda(\lambda+1)a_n\xi^n$$

$$= \sum_{n=0}^{\infty}\left[2(n+\alpha+1)(n+\alpha)a_{n+1} - (n+\alpha)(n+\alpha-1)a_n + 2(n+\alpha+1)a_{n+1} - 2(n+\alpha)a_n + \lambda(\lambda+1)a_n\right]\xi^{n+\alpha}$$

$$= \sum_{n=0}^{\infty}\left[2(n+\alpha+1)^2 a_{n+1} - \left[(n+\alpha)(n+\alpha+1) + \lambda(\lambda+1)\right]a_n\right]\xi^{n+\alpha}$$

with $a_{-1} = 0$. Setting $n = -1$ gives the indicial equation

$$2\alpha^2 a_0 = 0,$$

giving $\alpha = 0$ (double root).

The recurrence relation

$$a_{n+1} = \frac{n(n+1) - \lambda(\lambda+1)}{2(n+1)^2} a_n .$$

However, as $n \to \infty$, we have $a_{n+1}/a_n \to ½$ and so the series posed for y will diverge (at least for sufficiently large x) unless it terminates. We thus need λ to be a non-negative integer so that series gives $a_{\lambda+1} = 0$. The solution y is therefore a polynomial of degree λ.

4.8.4 IRREGULAR SINGULAR POINTS

Not examinable!

Not all singular points are this simple. Here we are looking at a *regular singular point*; solutions in the neighbourhood of a regular singular point never diverge faster than a power of x. Singularities that diverge faster than a power of x are *essential singularities* and occur in the neighbourhood of an *irregular singular point*.

$x^3 y'' + \lambda x y' + \mu y = 0$

Pose $$y = x^\alpha \sum_{n=0}^{\infty} a_n x^n \qquad a_n \neq 0$$

$$\Rightarrow \quad x^\alpha \sum_{n=0}^{\infty} x^3 (n+\alpha)(n+\alpha-1) a_n x^{n-2} + \lambda x (n+\alpha) a_n x^{n-1} + \mu a_n x^n$$
$$= \sum_{n=0}^{\infty} \left[(n+\alpha-1)(n+\alpha-2) a_{n-1} + \left(\lambda(n+\alpha)+\mu\right) a_n \right] x^{n+\alpha} = 0 \qquad a_{-1} = 0$$

$n = 0 \Rightarrow$ $$\lambda\alpha + \mu = 0$$

so $\alpha = -\mu/\lambda$. Although this is a second order equation, we appear to have only one root and the indicial equation is linear rather than quadratic…

$$a_n = \frac{(n+\alpha-1)(n+\alpha-2)}{\lambda(n+\alpha)+\mu} a_{n-1} = \frac{\left(n-\frac{\mu}{\lambda}-1\right)\left(n-\frac{\mu}{\lambda}-2\right)}{\lambda\left(n-\frac{\mu}{\lambda}\right)+\mu} a_{n-1}$$

Assuming μ/λ is not a positive integer, then $a_n a_{n-1} \sim n \to \infty$ as $n \to \infty$, so series diverges rapidly if it does not terminate. Hence, whenever $x \neq 0$ the solution diverges; it has a zero radius of convergence and the series solution is not valid anywhere. (This does not necessarily mean there is no valid solution to the equation.)

Could try solutions of a different form in $x^3 y'' + \lambda x y' + \mu y = 0$. Here try

$$y = e^\varphi \Rightarrow y' = \varphi' e^\varphi \text{ and } y'' = (\varphi'' + \varphi^2)\, e^\varphi$$

$$\Rightarrow \quad x^2 (\varphi'' + \varphi^2)\, e^\varphi + \lambda x\, \varphi' e^\varphi + \mu\, e^\varphi = 0$$

hence $$\varphi'' + \varphi^2 + \lambda x^{-2}\varphi' + \mu x^{-3} = 0.$$

The second and third terms suggest $\varphi' = -\lambda/x^2$ as an appropriate form. Substitute

$$2\lambda/x^3 + (\lambda-\lambda)/x^4 + \mu/x^3 = 0$$

which is dominated by the second term as $x \to 0$, hence $\varphi \sim \lambda/x$, and $y = e^{\lambda/x}$ as $x \to 0$.

4.8.5 CONDITIONS FOR SINGULAR POINT TO BE REGULAR

For the equation

$$R(x)\, y'' + P(x)\, y' + Q(x)\, y = 0$$

singular points occur at $x = x_0$ where $R(x) = 0$. For this singular point to be regular, we require both

$$\lim_{x\to x_0}\left[(x-x_0)\frac{P(x)}{R(x)}\right] \text{ and } \lim_{x\to x_0}\left[(x-x_0)^2\frac{Q(x)}{R(x)}\right]$$

to be finite. This means that R/P and R/Q must not tend to zero faster than they do in an equation that is homogeneous in x at that singular point.

Thus in $$x^r\, y'' + A\, x^p\, y' + B\, x^q\, y = 0$$

we must have $$r - p \le 1 \text{ and } r - q \le 2$$

for the singular point to be regular at $x = 0$.

4.9 Particular integrals

As we have seen, the general solution to an inhomogeneous equation is the sum of the complementary function(s) (solutions to the homogeneous problem) and a particular integral that satisfies the inhomogeneous problem. Although the particular integral is not unique, such solutions differ only by some linear combination of the complementary functions.

4.9.1 EQUATIONS THAT ARE HOMOGENEOUS IN x

Consider $$x^2\, u'' + (1-\lambda-\mu)x\, u' + \lambda\mu\, u = 1$$ for λ, μ constant.

As we saw in §4.7, solutions have the form x^α, leading to the indicial equation

$$\alpha^2 - (\lambda+\mu)\,\alpha + \lambda\mu = (\alpha-\lambda)(\alpha-\mu) = 0$$

which reveals that $u = x^\lambda$ and $u = x^\mu$ are independent solutions to the differential equation (at least provided $\lambda \ne \mu$).

For the particular integral, as with the equations with constant coefficients, we try a solution of the form $u = a + bx + cx^2$. Substituting

$$x^2(2c) + (1-\lambda-\mu)x(b+2cx) + \lambda\mu\,(a+bx+cx^2) = 1,$$

hence $c = 0$, $b = 0$ and $a = 1/\lambda\mu$. Hence the general solution is

$$u = Ax^\lambda + Bx^\mu + \frac{1}{\lambda\mu}.$$

In general, a polynomial right-hand side will require a polynomial form for the particular integral. The technique we have used here and previously is to pose a form with unknown parameters, substitute and then solve to find the unknown parameters.

Consider $$x^2 u'' - 2x u' + 2u = x + x^2,$$

the indicial equation suggests the complementary functions x and x^2. As with equations with constant coefficients, we would then expect to have other terms in the principal integral. We might think of trying $a + bx + cx^2 + dx^3 + ex^4$. Substituting

$$x^2(2c+6dx+12ex^2) - 2x(b+2cx+3dx^2+4ex^3) + 2(a+bx+cx^2+dx^3+ex^4) = x + x^2.$$

Inspection shows that this requires $a = 0$, $d = 0$, $e = 0$, but cannot be satisfied for any values of b or c.

Recall that for $y'' + 2y' + y = e^{-t}$ in §4.2.9 we had complementary functions of the form $(A+Bt)e^{-t}$ and required a trial function of the form $at^2 e^{-t}$. We can get some guidance here by recalling that for a double root in the indicial equation we needed to introduce a solution of the form $(A + B \ln x)x^{\alpha}$, so we could try $u = ax \ln x + bx^2 \ln x$.

Now $$u' = a \ln x + a + 2bx \ln x + bx,$$

$$u'' = a/x + 2b \ln x + 2b + b$$

Substituting

$$(ax + 2bx^2 \ln x + 3bx^2) - 2(ax \ln x + 2bx^2 \ln x + ax + bx^2) + 2(ax \ln x + bx^2 \ln x) = x + x^2$$

Equating different terms

x:	$a - 2a = 1$	$\Rightarrow a = -1$
x^2:	$2b - 2b + 2b = 1$	$\Rightarrow b = ½$
$x \ln x$	$-2a + 2a = 0$	OK
$x^2 \ln x$	$2b - 4b + 2b = 0$	OK

hence the particular integral is

$$u = -x \ln x + ½\, x^2 \ln x$$

and the general solution

$$u = Ax + Bx^2 - x \ln x + ½\, x^2 \ln x.$$

If the indicial equation had a double route, and the right-hand side had contained one of the complementary equations, then we would have had to introduce terms of the form $(\ln x)^2 x^{\alpha}$, *etc.* This is no surprise given the equivalence between equations with constant coefficients and those that are homogeneous in the independent variable.

4.9.2 Method of variation of parameters

What do we do if our guess at a suitable form for the particular integral fails?

Consider the general second order linear ordinary differential equation

$$L[y] \equiv y'' + py' + qy = f,$$

where p, q and f are all functions of the independent variable x.

Two known complementary functions

Let $y = u_1$ and $y = u_2$ are complementary functions solving the homogeneous problem ($L[u_1] = 0$, $L[u_2] = 0$). We could then try solutions of the form

$$y = \alpha(t)\, u_1 + \beta(t)\, u_2.$$

How do we choose $\alpha(t)$ and $\beta(t)$? Noting that

$$y' = \alpha\, u_1' + \beta\, u_2' + \alpha' u_1 + \beta' u_2$$

we make α and β (which are, at present, arbitrary) *look like constants* by requiring

$$\alpha' u_1 + \beta' u_2 = 0$$

The second derivative is then

$$y'' = \alpha\, u_1'' + \beta\, u_2'' + \alpha' u_1' + \beta' u_2'.$$

Substituting into the ode,

$$y'' + py' + qy = (\alpha\, u_1'' + \beta\, u_2'' + \alpha' u_1' + \beta' u_2') + p(\alpha\, u_1' + \beta\, u_2') + q(\alpha\, u_1 + \beta\, u_2) = f$$

noting that $\alpha(u_1'' + p\, u_1' + q\, u_1) = 0$ and $\alpha(u_2'' + p\, u_2' + q\, u_2) = 0$, then we require

$$\alpha' u_1' + \beta' u_2' = f.$$

Hence we have two simultaneous equations for α' and β':

$$\alpha' u_1 + \beta' u_2 = 0$$

$$\alpha' u_1' + \beta' u_2' = f.$$

$\Rightarrow$
$$\alpha'(u_1' u_2 - u_1\, u_2') = u_2 f$$

$\Rightarrow$
$$\alpha = \int \frac{-u_2 f}{u_1 u_2' - u_1' u_2}\, dx$$

and
$$\beta = \int \frac{u_1 f}{u_1 u_2' - u_1' u_2}\, dx\,.$$

Of course the denominator here is just the Wronskian – for this to make sense we must have u_1 and u_2 linearly independent.

➧ Car driving up curb (§4.6.2): $\ddot{\xi} + 2\sigma\dot{\xi} + \sigma^2\xi = 2\sigma h\delta(t) + \sigma^2 hH(t)$

Suspension critically damped, characteristic equation has double root and the complementary functions are $u_1 = e^{-\sigma t}$, $u_2 = t\, e^{-\sigma t}$. The Wronskian is

$$W = \begin{vmatrix} 1 & t \\ -\sigma & 1-\sigma t \end{vmatrix} e^{-2\sigma t} = e^{-2\sigma t},$$

and noting $\int_{-\infty}^{t} f(s)\,\delta(s)\,ds = f(0)H(t)$ and $\int_{-\infty}^{t} f(s)\,H(s)\,ds = H(t)\int_{0}^{t} f(s)\,ds$

$$\alpha = \int \frac{-u_2 f}{W}\,dt = \int_{-\infty}^{t} \frac{-se^{-\sigma s}}{e^{-2\sigma s}} \sigma h\left(2\delta(s) + \sigma H(s)\right) ds$$

then
$$= -\sigma^2 hH(t)\int_0^t se^{\sigma s}\,ds = -\sigma hH(t)\left\{\left[se^{\sigma s}\right]_0^t - \int_0^t e^{\sigma s}\,ds\right\}$$
$$= -h\left((\sigma t - 1)e^{\sigma t} + 1\right)H(t)$$

$$\beta = \int \frac{u_1 f}{W}\,dt = \int_{-\infty}^{t} \frac{e^{-\sigma s}}{e^{-2\sigma s}} \sigma h\left(2\delta(s) + \sigma H(s)\right) ds$$

and
$$= \sigma hH(t)\left(2 + \sigma\int_0^t e^{\sigma s}\,ds\right) = \sigma hH(t)\left(2 + e^{\sigma t} - 1\right)$$
$$= \sigma hH(t)\left(1 + e^{\sigma t}\right)$$

Hence, the particular integral is

$$\begin{aligned}\xi = \alpha u_1 + \beta u_2 &= -hH(t)\left((\sigma t - 1)e^{\sigma t} + 1\right)e^{-\sigma t} + \sigma hH(t)\left(1 + e^{\sigma t}\right)te^{-\sigma t} \\ &= hH(t)\left(1 - \sigma t - e^{-\sigma t} + \sigma te^{-\sigma t} + \sigma t\right) \\ &= hH(t)\left(1 - (1 - \sigma t)e^{-\sigma t}\right)\end{aligned}.$$

The general solution

$$\xi = h\,H(t)\,(1 - (1 - \sigma t)e^{-\sigma t}) + (A + Bt)e^{-\sigma t},$$

needs to be matched by initial conditions that $\xi = 0$ for $t = -\infty$, hence $A = B = 0$ and the general solution is simply

$$\xi = h\,H(t)\,(1 - (1 - \sigma t)e^{-\sigma t})$$

which, apart from the presence of the step function, is the same as we found earlier for $t > 0$ in §4.6.2.

End of Lecture 19

4.9.3 Only one complementary function known

Consider the inhomogeneous equation

$$L[y] = f.$$

If you only know one solution, $y = u$, say, to the homogeneous equation $L[y] = 0$, could search out a second solution using the approach we used earlier (§4.4) to find a second solution of the form $y = \psi u$. However, if we also need to find the particular integral, it might be better to combine the two steps.

Set
$$y = \psi u,$$

and seek ψ such that $L[y] \equiv y'' + py' + qy = f$. Substituting,

$$\psi u'' + 2\,\psi' u' + \psi'' u + p(\psi u' + \psi' u) + q\,\psi u = f.$$

Noting that $\psi(u'' + pu' + qu) = 0$, then

$$2\,\psi' u' + \psi'' u + p\psi' u = f,$$

$$\Rightarrow \qquad \psi'' + \left(2\frac{u'}{u} + p\right)\psi' = \frac{f}{u},$$

which is a first order linear equation for ψ' for which we have already discussed solution techniques. Ultimately, ψ is a double integral; the constants of integration lead to the general solution.

- Car driving up curb, again: $\ddot{\xi} + 2\sigma\dot{\xi} + \sigma^2\xi = 2\sigma h\delta(t) + \sigma^2 hH(t)$

  Suppose we know one complementary function, $u = e^{-\sigma t}$,

$$\xi = \psi u = \psi e^{-\sigma t},$$

require

$$\psi'' + \left(2\frac{u'}{u} + p\right)\psi' = \psi'' + (-2\sigma + 2\sigma)\psi' = \psi''$$

$$= \frac{f}{u} = \sigma h e^{\sigma t}\left(2\delta(t) + \sigma H(t)\right)$$

$$\Rightarrow \quad \psi' = \sigma h \int e^{\sigma t}\left(2\delta(t) + \sigma H(t)\right) dt = \sigma h\left[2H(t) + \sigma H(t)\int_0^t e^{\sigma t}\, dt\right] + b$$

$$= \sigma h H(t)\left(2 + e^{\sigma t} - 1\right) + b = \sigma h H(t)\left(1 + e^{\sigma t}\right) + b$$

$$\Rightarrow \quad \psi = \int \sigma h H(t)\left(1 + e^{\sigma t}\right) + b\, dt = \sigma h H(t)\int_0^t \left(1 + e^{\sigma s}\right) ds + bt + c$$

$$= \sigma h H(t)\left(t + \frac{1}{\sigma}\left(e^{\sigma t} - 1\right)\right) + bt + c$$

so again $\xi = u\psi = e^{-\sigma t}\left[hH(t)\left(\sigma t + e^{\sigma t} - 1\right) + bt + c\right] = hH(t)\left(1 - (1 - \sigma t)e^{-\sigma t}\right) + (bt + c)e^{-\sigma t}$,

with the arbitrary constants of integration b and c disappearing as $\xi = 0$ for $t < 0$.

4.10 Coupled systems

4.10.1 Converting to a coupled system

As we saw in §4.1.1, we may reduce a second order linear differential equation into a pair of coupled first order equations. For

$$y'' + py' + qy = f,$$

define $z = y'$, then

$$y' = z,$$

$$z' = f - pz - qy.$$

We can do a similar thing for any order of linear ode: an n^{th} order equation may be converted into a system of n first order equations.

4.10.2 Converting to a single equation

Conversely, any pair of coupled equations, such as

$$y' = ay + bz + f,$$

$$z' = cy + dz + g,$$

can be converted into a second order equation for y, say, by rewriting the first equation as

$$z = \frac{y' - ay - f}{b}$$

and substituting into the second

$$\frac{(y'' - ay' - a'y - f')b - (y' - ay - f)b'}{b^2} = cy + \frac{d}{b}(y' - ay - f) + g$$

$\Rightarrow$
$$y'' - \left(a + \frac{b'}{b} + d\right)y' + \left(ad - bc - a' + \frac{ab'}{b}\right)y = f' - \frac{b'f}{b} - df + g\ .$$

Obviously, we could do the same thing for z to give an equivalent second order equation for that instead.

4.10.3 Solving coupled systems

Consider

$$y' + 4y - 24z = 4\,e^x$$

$$z' - y + 2z = e^x.$$

Method 1

Use one of the equations to eliminate one variable, ending up with a second order system.

$$y = z' + 2z - e^x$$

$\Rightarrow$ $$y' = z'' + 2z' - e^x$$

Substitute $$z'' + 2z' - e^x + 4(z' + 2z - e^x) - 24z = 4e^x$$

$\Rightarrow$ $$z'' + 6z' - 16z = 9e^x$$

Characteristic equation gives $\lambda = -8, 2$, so complementary functions e^{-8x} and e^{2x}. For particular integral try $z = a\,e^x$

$\Rightarrow$ $$a + 6a - 16a = 9 \qquad \Rightarrow a = -1$$

so $$z = A\,e^{-8x} + B\,e^{2x} - e^x.$$

and $$z' = -8A\,e^{-8x} + 2B\,e^{2x} - e^x$$

so $$y = z' + 2z - e^x$$

$$= -8A\,e^{-8x} + 2B\,e^{2x} - e^x + 2(A\,e^{8x} + B\,e^{-2x} - e^x) - e^x$$

$$= -6A\,e^{-8x} + 4B\,e^{2x} - 4\,e^x.$$

This is a simple, but cumbersome approach, particularly for equations where the coefficients are functions of x. It can, however, make finding the particular integrals more complex than necessary.

Method 2

Add a linear combination of the two equations. Here, add μ times the second equation to the first:

$1\times$ $$y' + 4y - 24z = 4\,e^x$$

$\mu\times$ $$\mu(z' - y + 2z) = \mu e^x$$

add $$(y + \mu z)' + (4 - \mu)y + (-24 + 2\mu)z = (4+\mu)e^x$$

Rearrange

$$(y+\mu z)' + (4-\mu)\left[y + \frac{-24+2\mu}{4-\mu}z\right] = (4+\mu)e^x.$$

So far μ has been arbitrary. We now choose μ such that the term in square brackets is equal to $y + \mu z$. Thus

$$\mu = \frac{-24+2\mu}{4-\mu}$$

$\Rightarrow$ $$\mu^2 - 2\mu - 24 = 0$$

which has solutions $\mu = -4, 6$.

For $\mu = -4$:

$$u = y - 4z,$$

and $$u' + 8u = 0$$

$\Rightarrow$ $$u = A\,e^{-8x}.$$

For $\mu = 6$:

$$v = y + 6z,$$

and $$v' - 2v = 10e^x$$

$\Rightarrow$ $$v = B\,e^{2x} - 10e^x.$$

We now have a pair of solutions for u and v. These are known as the *normal modes* of the system, each having just one term in the complementary function.

u: $$y - 4z = A\,e^{-8x}$$

v: $$y + 6z = B\,e^{2x} - 10e^x$$

Solve simultaneously for y and z:

$$10z = -Ae^{-8x} + Be^{2x} - 10e^x$$

$\Rightarrow$ $$z = -(A/10)e^{-8x} + (B/10)e^{2x} - e^x$$

and $$y = (3A/5)e^{-8x} + (2A/5)\,e^{2x} - 4x.$$

Note: The constants A and B are not the same as with the previous method.

Method 3: Matrix method

This is really a generalisation of the second method.

Write the system of first order linear equations in matrix form:

$$\mathbf{w}' - \mathbf{M}\mathbf{w} = \mathbf{b}.$$

For our present system

$$\mathbf{w} = \begin{pmatrix} y \\ z \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix}, \text{ and } \ \mathbf{b} = \begin{pmatrix} 4e^x \\ e^x \end{pmatrix}.$$

For the complementary functions, try $\mathbf{w} = \mathbf{w}_0\, e^{\lambda x}$,

$\Rightarrow$ $$\lambda \mathbf{w}_0\, e^{\lambda x} - \mathbf{M}\mathbf{w}_0\, e^{\lambda x} = 0,$$

$\Rightarrow$ $$\mathbf{M}\mathbf{w}_0 = \lambda \mathbf{w}_0$$

and λ are the eigenvalues of $\mathbf{M}$, with $\mathbf{w}_0$ the corresponding eigenvectors. Of course, the eigenvalues must satisfy

$$|\mathbf{M} - \lambda \mathbf{I}| = 0.$$

For the present example,

$$\begin{vmatrix} -4-\lambda & 24 \\ 1 & -2-\lambda \end{vmatrix} = (-4-\lambda)(-2-\lambda) - 24 = \lambda^2 + 6\lambda - 16 = (\lambda+8)(\lambda-2),$$

hence $\lambda = -8, 2$, as before. The corresponding eigenvectors are

$$\mathbf{w}_{01} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}, \quad \mathbf{w}_{02} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

and the solution to homogeneous equation

$$\begin{pmatrix} y \\ z \end{pmatrix} = A\mathbf{w}_{01} e^{-8x} + B\mathbf{w}_{02} e^{2x} = A\begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8x} + B\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2x}.$$

For the particular integral, try $\mathbf{w} = \mathbf{a}\, e^x$,

$\Rightarrow$ $$\mathbf{a}e^x - \mathbf{M}\mathbf{a}e^x = \mathbf{b}$$

$\Rightarrow$ $$\left[\mathbf{I} - \begin{bmatrix} -4 & 24 \\ 1 & -2 \end{bmatrix}\right]\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{bmatrix} 5 & -24 \\ -1 & 3 \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$\Rightarrow$ $$a_1 = -4,\ a_2 = -1,$$

giving the general solution

$$\begin{pmatrix} y \\ z \end{pmatrix} = A\begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8x} + B\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2x} + \begin{pmatrix} -4 \\ -1 \end{pmatrix} e^x.$$

This method is readily generalised to systems comprising a larger number of coupled equations. It can also be generalised to handle more complex right-hand sides.

4.10.4 Coupled Higher Order Systems

For a system of higher order equations, can either reduce to a larger system of first order equations, or extend the previous ideas.

➧ $y'' - y - 10z = K \cos 3x$,

$z'' + 5y + 14z = -\cos 3x$,

subject to $y, y', z, z' = 0$ at $x = 0$.

Using the second method, multiplying the second equation by μ and adding to the first

$$\left(y+\mu z\right)''+\left(-1+5\mu\right)\left[y+\frac{-10+14\mu}{-1+5\mu}z\right]=\left(K-\mu\right)\cos 3x.$$

Selecting μ so that term in square brackets has same form as first term

$$\mu=\frac{-10+14\mu}{-1+5\mu}$$

$\Rightarrow$ $$5\mu^2 - 15\mu + 10 = 0$$

$\Rightarrow$ $$\mu = 1, 2.$$

For $\mu = 1$, have

$$u'' + 4u = (K - 1)\cos 3x$$

$\Rightarrow$ $$u = A\cos 2x + B\sin 2x - ((K-1)/5)\cos 3x$$

and for $\mu = 2$,

$$v'' + 9v = (K - 2)\cos 3x$$

$\Rightarrow$ $$v = C\cos 3x + D\sin 3x + ((K-2)/6)\,x\sin 3x$$

Initial conditions give u, u', v, $v' = 0$ at $x = 0$, so

$$u = ((K-1)/5)(\cos 2x - \cos 3x),$$

$$v = ((K-2)/6)\,x\sin 3x$$

which may be solved simultaneously for y and z:

$$y=\frac{2}{5}\left(K-1\right)\left(\cos 2x-\cos 3x\right)-\frac{1}{6}\left(K-2\right)x\sin 3x,$$

$$z=-\frac{1}{5}\left(K-1\right)\left(\cos 2x-\cos 3x\right)+\frac{1}{6}\left(K-2\right)x\sin 3x.$$

4.11 Second order difference equations

Homogeneous equation

Consider the difference equation

$$y_{n+2} + \lambda y_{n+1} + \mu y_n = 0.$$

Noting the similarity to the second order ode $y'' + qy' + ry = 0$ which has solutions of the form $y = e^{\alpha x}$, try a solution of the form $y_n = a^n = e^{\alpha n}$ ($\alpha = \ln a$):

$$(a^2 + \lambda a + \mu)a^n = 0.$$

For a non-trivial solution,

$$a=\frac{1}{2}\left(-\lambda\pm\sqrt{\lambda^2-4\mu}\right).$$

Like the second order ode, the second order linear homogeneous difference equation has two solutions.

As with odes, if there are repeated roots, (*e.g.* if $\lambda^2 = 4\mu$ in the above), then try $y = n\,a^n$ for the second solution:

$$(n+2)a^{n+2} + \lambda(n+1)a^{n+1} + (n/4)\lambda^2 a^n = 0$$

$$\Rightarrow \quad (n+2)a^2 + \lambda(n+1)a + (n/4)\lambda^2 = 0$$

$$a = -\frac{1}{2}\frac{n+1}{n+2}\lambda \pm \sqrt{\frac{1}{4}\left(\frac{n+1}{n+2}\right)^2 \lambda^2 - \frac{1}{4}\frac{n}{n+2}\lambda^2}$$

$$\Rightarrow \quad = -\frac{1}{2}\frac{n+1}{n+2}\lambda \pm \frac{1}{2}\frac{\lambda}{n+2}\sqrt{n^2+2n+1-n^2-2n}$$

$$= -\frac{1}{2}\frac{n+1\pm 1}{n+2}\lambda$$

and so both a^n and $n\,a^n$ are solutions when $a = -\lambda/2$.

Particular integral

Consider $$y_{n+2} + \lambda\, y_{n+1} + \mu\, y_n = c.$$

Again there is an analogy with the ode $y'' + qy' + ry = f$.

Try $y_n = A \Rightarrow$ $$(1 + \lambda + \mu)\, A = c$$

$$\Rightarrow \quad A = c/(1 + \lambda + \mu).$$

If $1 + \lambda + \mu = 0$, then inspection will show that $y_n = const$ is a solution to the homogeneous equation. We can try instead $y_n = A\,n$. If we have a repeated root, then would need to use $y_n = A\,n^2$, *etc.*

Reduction of order

In the same way as we can convert a second order linear ode into a pair of coupled first order linear odes, we can convert a second order difference equation into a system of coupled first order difference equations.

Consider $$y_{n+2} + \lambda\, y_{n+1} + \mu\, y_n = f_n.$$

Let $\alpha + \beta = \lambda$ and $\alpha\beta = \mu$, then

$$y_{n+2} + \alpha\, y_{n+1} + \beta(y_{n+1} + \alpha\, y_n) = f_n.$$

Defining $z_n = y_{n+1} + \alpha\, y_n$, then we have the system

$$y_{n+1} + \alpha\, y_n = z_n,$$

$$z_{n+1} + \beta\, z_n = f_n.$$

End of Lecture 20