5 Nonlinear equations

5.1 Approach

It is often difficult or impossible to obtain explicit solutions to higher order or coupled systems of nonlinear ordinary differential equations. The techniques we shall discuss here are aimed primarily at discovering something of the character of the solutions. For simplicity, we shall concentrate on second order systems; the techniques may readily be generalised to higher order systems. Further, as all the techniques we will employ here can be applied to linear equations, we will often use linear equations to help our understanding.

For much of the discussion it is convenient to cast our nonlinear differential equation as a system of first order, nonlinear ordinary differential equations. Any n^{th} order ordinary differential equation

$$g(y,y',y'',\ldots,y^{(n)};t) = 0$$

may be written in the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x},t\right) \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

It is often convenient to think of $\mathbf{x}(t)$ as the *position* of a point in n–dimensional space which *moves* as time t advances. [Note that $\mathbf{f}(\mathbf{x},t)$ is a function and so single-valued.] One reason why we shall concentrate on second order nonlinear ordinary differential equations, which give rise to a system of two first order equations, is that plotting $\mathbf{x}$ in two dimensions is relatively easy. However, everything we shall do generalises to more dimensions.

We shall also concentrate on autonomous systems, *i.e.* systems that do not have any explicit time dependence and $\mathbf{f} = \mathbf{f}(\mathbf{x})$.

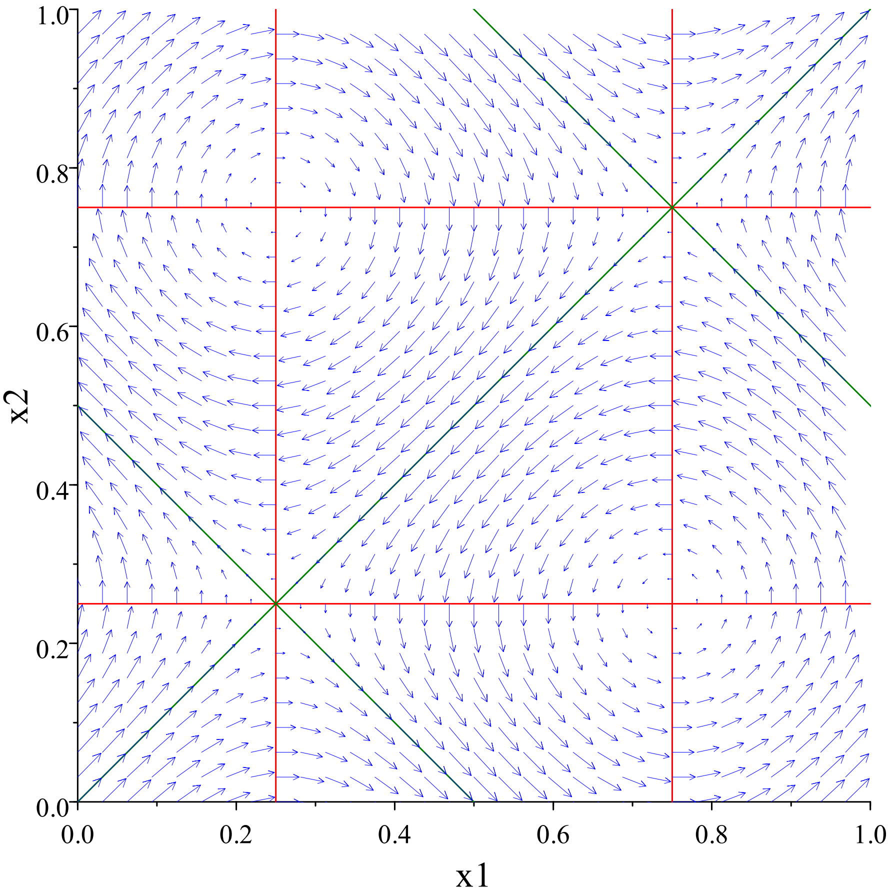
5.1.1 TRAJECTORIES IN THE PHASE PLANE

The n–dimensional space containing all possible solution vectors $\mathbf{x}(t)$ is known as ***phase space***. For a second order system, the phase space is two-dimensional and often referred to as the ***phase plane***. A solution $\mathbf{x}(t)$ to the system of differential equations satisfying a particular set of initial conditions forms a *trajectory* in the phase plane. The phase plane is closely related to the phase portrait we explored when looking at first order systems.

The phase plane is also closely related to the flow maps we considered for first order systems. Here we can interpret $d\mathbf{x}/dt$ as velocity vectors. For example, if

$$\mathbf{f}\left(\mathbf{x}\right) \equiv \begin{pmatrix} \cos 2\pi x_2 \\ \cos 2\pi x_1 \end{pmatrix},$$

we can begin to construct the solution in the phase plane by noting that the vectors will be horizontal along lines where $\cos 2\pi x_1$ vanishes (*i.e.* $x_1 = 1/2,\ 3/2$) and vertical where $\cos 2\pi x_2$ vanishes (*i.e.* $x_2 = 1/2,\ 3/2$). We can also see that the vectors have unit slope when $|\cos 2\pi x_1| = |\cos 2\pi x_2| \Rightarrow x_1 - x_2 = n,\ x_1 + x_2 = (n+1)/2$.



Note that unlike the earlier flow maps, here the vectors can point to both the left and right.

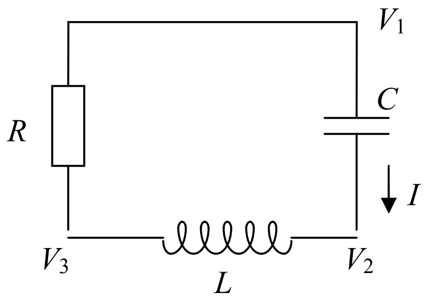
As with the earlier flow maps, we can gain a *feeling* for the solution by following the arrows. Here the trajectories we plot out the direction in phase space, but not directly the time.

Drawing phase plane trajectories for a non-autonomous system is clearly more difficult as the trajectories will change as a function of time.

5.1.2 LINEAR EXAMPLES

Electrical circuit

Consider a simple electrical circuit, similar to that we looked at in §4.6.1:



Capacitor $$\frac{d}{dt}\left(V_2 - V_1\right) = \frac{I}{C}$$

Inductor $$L\frac{dI}{dt} = V_3 - V_2$$

Resistor $$V_1 - V_3 = R\,I.$$

Eliminating V_3 and defining $V = V_2 - V_1$ gives

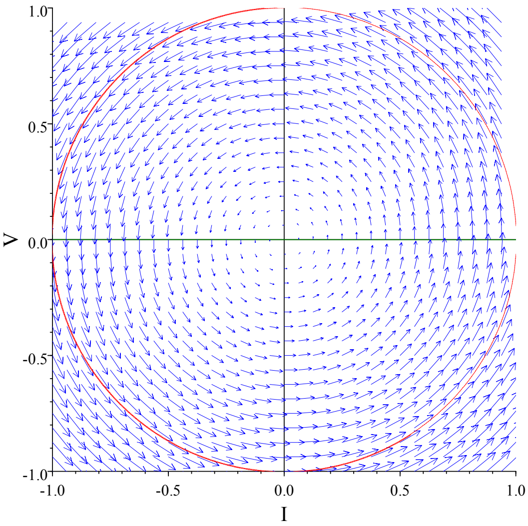
$$\frac{dV}{dt} = \frac{I}{C},$$

$$\frac{dI}{dt} = -\frac{V}{L} - \frac{RI}{L}.$$

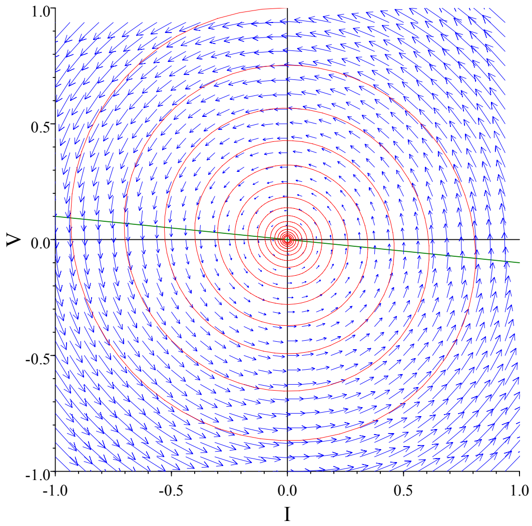
Define $x_1 = I$ and $x_2 = V$, then

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\mathbf{x}.$$

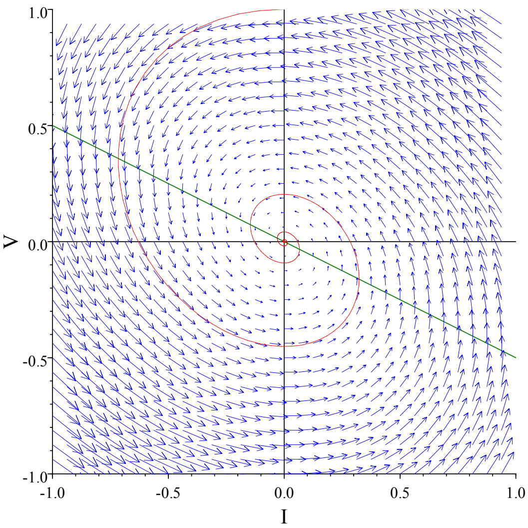
Clearly our vectors will be horizontal when $x_1 = 0$. If $R = 0$, then the vectors will be vertical when $x_2 = 0$, but if $R \neq 0$, then they are vertical when $R\,x_1 + x_2 = 0 \Rightarrow x_2 = -R\,x_1$.



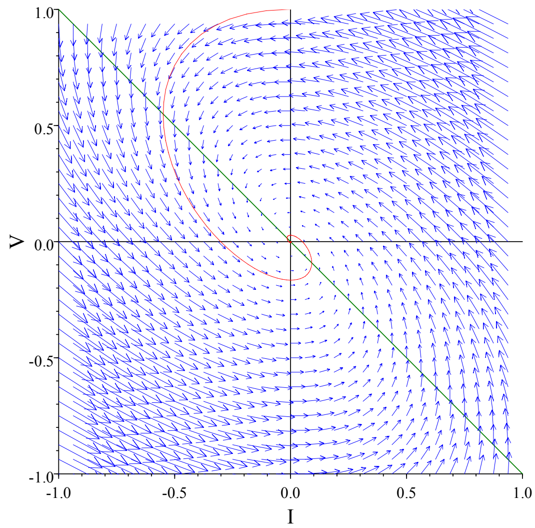
$R = 0$



$R = 0.1$



$R = 0.5$



$R = 1$

Solution curves above for For $V(0) = 1$, $I(0) = 0$ with $L = 1$, $C = 1$.

Car suspension

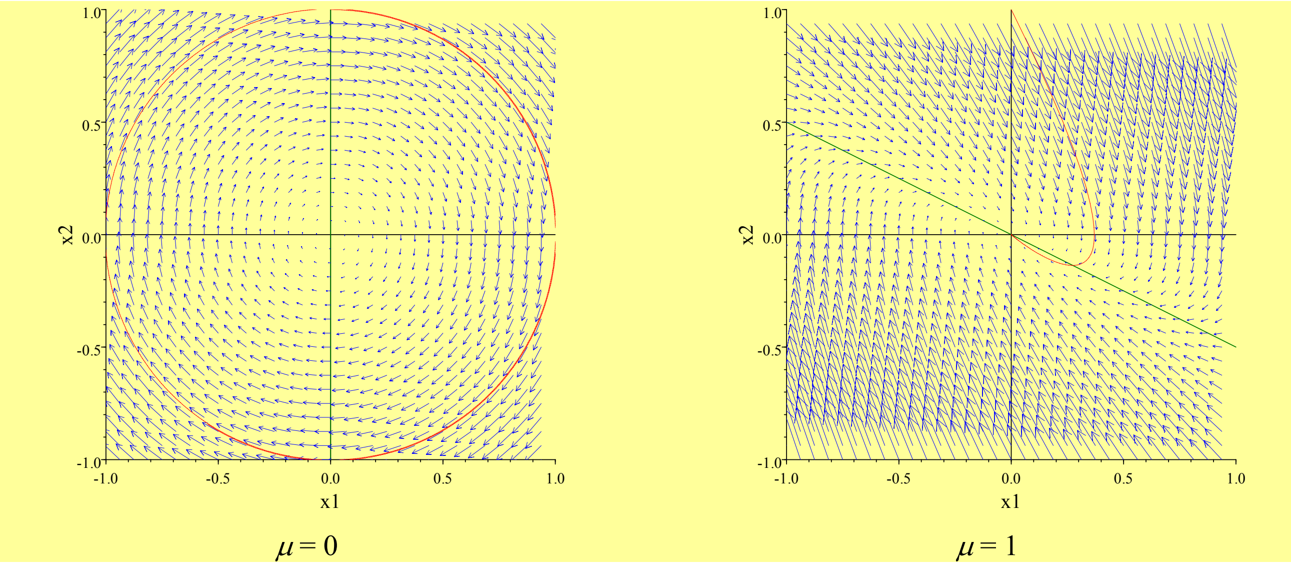
Recall in §4.3.1 for car suspension we found

$$\ddot{\xi} + 2\mu\dot{\xi} + \sigma^2\xi = 2\mu\dot{z} + \sigma^2 z\ .$$

There is more than one choice of how to write this as a system of equations, but here we select $x_1 = \xi$ and $x_2 = d\xi/dt$. This gives for a smooth road ($z = 0$)

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\sigma^2 & -2\mu \end{bmatrix}\mathbf{x}\ .$$

Clearly the vectors will be vertical when $x_2 = 0$. The vectors will be horizontal when $-\sigma^2 x_1 - 2\mu\, x_2 = 0 \Rightarrow x_2 = -(\sigma^2/2\mu)x_1$.



Phase plane not unique

There is no unique phase plane for a given differential equation. For example, as we saw in §4.10.3, there can be some advantage in taking linear combinations of the natural variables. For example, in

$$\dot{\mathbf{u}} = \mathbf{M}\mathbf{u}\,,$$

where

$$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix},$$

we may to choose our new variables as

$$\mathbf{v} = \begin{pmatrix} -x + 2y \\ -x + y \end{pmatrix}.$$

Substituting in for x and y leads to the new linear system

$$\dot{\mathbf{v}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{v}\,.$$

This is effectively a rotation of the axes of the original problem, and in this case we have been able to decouple the two equations.

[The method of selecting new variables to decouple the equations is similar to the matrix method outlined in §4.10.3, but details are beyond the scope of this course.]

Note that we need not choose our new variables as a linear combination of the original variables, although for a system that is already linear this would be the normal approach.

5.2 Elementary phase plane analysis

The basic recipe is:

1. Find fixed points, *i.e.* equilibrium solutions

2. Construct solutions in neighbourhood of fixed points by linearization of the governing equations
3. Join up the fixed points.

As $\mathbf{f}(\mathbf{x},t)$ is single valued, then trajectories cannot cross. Trajectories also cannot end, although they can *stop* at a fixed point.

5.2.1 SADDLE POINTS – EIGENVALUES OF OPPOSITE SIGN

Consider the equation

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}.$$

Clearly this has a fixed point ($d\mathbf{x}/dt = 0$) when $\mathbf{x} = 0$.

The eigenvalues of the system are given by $(1-\lambda)^2-4 = 0 \Rightarrow \lambda = -1, 3$, and the eigenvectors

$$\mathbf{q}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

As $$\frac{dx_1}{dt} = x_1 + x_2, \quad \frac{dx_2}{dt} = 4x_1 + x_2,$$

then $$\frac{dx_2}{dx_1} = \frac{4x_1 + x_2}{x_1 + x_2}.$$

Hence

(Green)	$dx_2/dx_1 = 0$	on $x_2 = -4x_1$,
(Green)	$dx_2/dx_1 = \infty$	on $x_2 = -x_1$
	$dx_2/dx_1 = 4$	on $x_2 = 0$
	$dx_2/dx_1 = 1$	on $x_1 = 0$.

Consider the solution along the eigenvectors, *i.e.* if $\mathbf{x} = r\mathbf{q}_i$, where $\mathbf{q}_i$ is the eigenvector corresponding to eigenvalue λ_i. In this case the right-hand side becomes

$$\mathbf{Mx} = r\mathbf{M}\mathbf{q}_i = r\lambda_i\mathbf{q}_i$$

so we have $d\mathbf{x}/dt$ is parallel to $\mathbf{q}_i$ and

$$\frac{d\mathbf{x}}{dt} = \mathbf{q}_i\frac{dr}{dt} = r\lambda_i\mathbf{q}_i$$

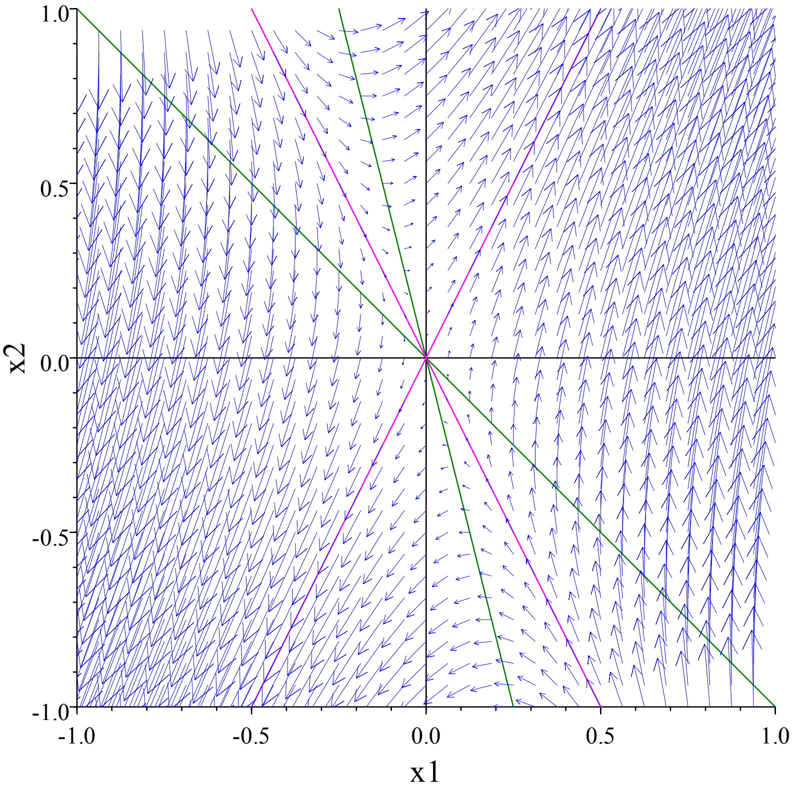
so a solution on an eigenvector remains on the eigenvector and moves as $r = Ae^{\lambda_i t}$ (with A arbitrary constant of integration). Note that this applies even if the eigenvalues are not of opposite signs.

Here we can see that

(Magenta) on $\mathbf{x} = \mathbf{q}_1$, $d\mathbf{x}/dt = r\lambda_1\mathbf{q}_1$

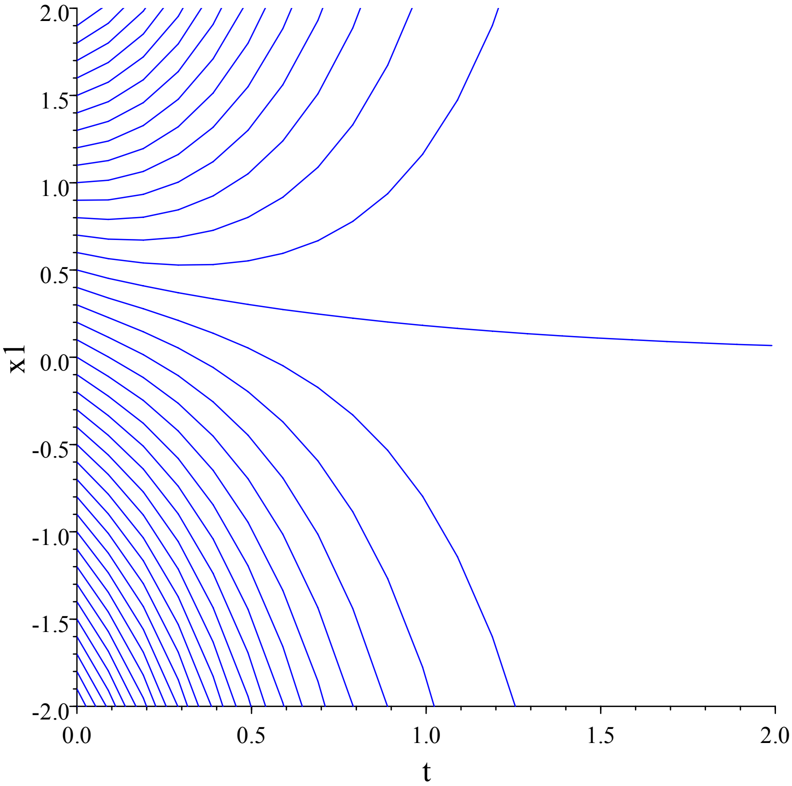
(solutions parallel to $\mathbf{q}_1$; converging on origin since $\lambda_1 = -1 < 0$ so $dr/dt = -r$)

(Magenta) on $\mathbf{x} = \mathbf{q}_2$, $d\mathbf{x}/dt = r\lambda_2\mathbf{q}_2$
(solutions parallel to $\mathbf{q}_2$; diverging from origin since $\lambda_2 = 3 > 0$ so $dr/dt = 3r$)

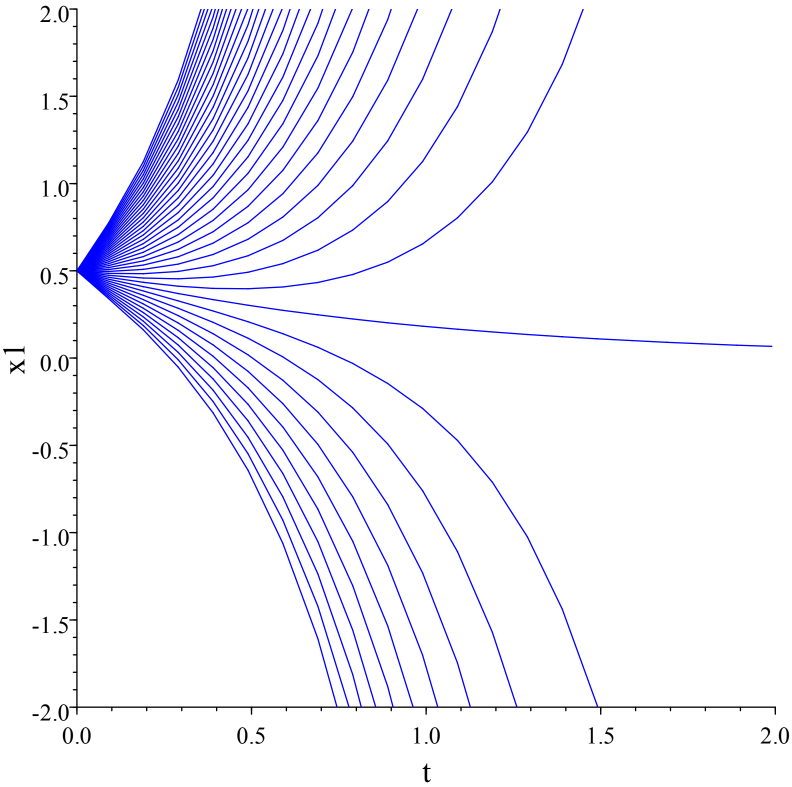


Note the domain is divided by the eigenvectors into four regions. The two eigenvectors intersect at the fixed point. Along the $\mathbf{q}_1$ vector, with $\lambda = -1$, the solution *converges* (relatively slowly) on the fixed point. Along the $\mathbf{q}_2$ vector with $\lambda = 3$, the solution *diverges* (somewhat more rapidly) from the fixed point. This type of fixed point is referred to as a *saddle point*. Note that a solution close to the $\mathbf{q}_1$ vector will diverge from that vector as the fixed point is approached. Conversely, a solution close to the $\mathbf{q}_2$ vector will converge on that vector as the solution moves away from the fixed point.

We may gain further insight by looking at the time behaviour of one of the variables, x_1, say.



Solutions with $x_2(0) = -1$ for a range of $x_1(0)$.



Solutions with $x_1(0) = ½$ and x_2 ranging between -2 and 2.

What happens at the fixed point? Obviously $d\mathbf{x}/dt = 0$ and the solution does not change, but it takes infinitely long to get there along $\mathbf{q}_1$.

We have seen that slope increases as x_1 increases at fixed x_2

slope increases as x_2 increases at fixed x_1

⇒ trajectories get closer to the eigenvectors as distance form the saddle point increases.

Here, arrows show that the solution trajectories go from $\mathbf{q}_1$ towards $\mathbf{q}_2$. [Note that along the eigenvectors, the time dependence is $e^{\lambda t}$, so the solution *slows down* along $\mathbf{q}_1$ as e^{-t}, and *speeds up* along $\mathbf{q}_2$ as e^{3t}.]

The *speed* at which the point moves along the trajectory increases as distance from the saddle point increases (*i.e.* the arrows are longer), at least when it is not near to the closest approach:

$$speed^2 = \dot{x}_1^2 + \dot{x}_2^2 = (4x_1 + x_2)^2 + (x_1 + x_2)^2 .$$

5.2.2 Nodes – two real negative eigenvalues

Consider the linear system governed by

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -4 & \sqrt{3} \\ \sqrt{3} & -2 \end{bmatrix} \mathbf{x} .$$

The matrix here is real and symmetric (Hermitian), and so the eigenvalues are real and the eigenvectors are orthogonal:

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$$\begin{vmatrix} -4-\lambda & \sqrt{3} \\ \sqrt{3} & -2-\lambda \end{vmatrix} = (4+\lambda)(2+\lambda) - 3 = \lambda^2 + 6\lambda + 5 = (\lambda+5)(\lambda+1) = 0$$

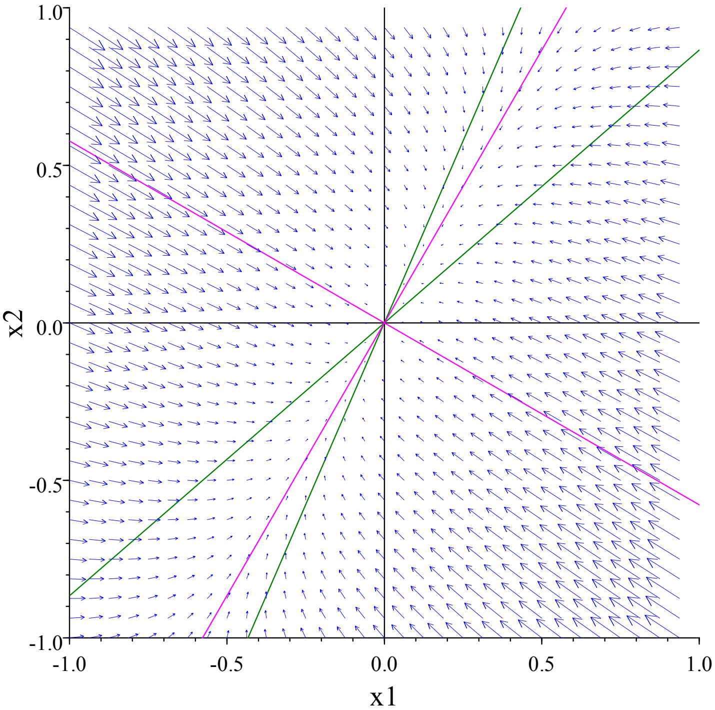
$\Rightarrow \lambda = -5, -1$ $\qquad \mathbf{q}_1 = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}, \quad \mathbf{q}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}.$

Now $\dfrac{dx_2}{dx_1} = \dfrac{\sqrt{3}x_1 - 2x_2}{-4x_1 + \sqrt{3}x_2}$, so $\qquad dx_2/dx_1 = 0$ when $x_2 = \dfrac{\sqrt{3}}{2}x_1$,

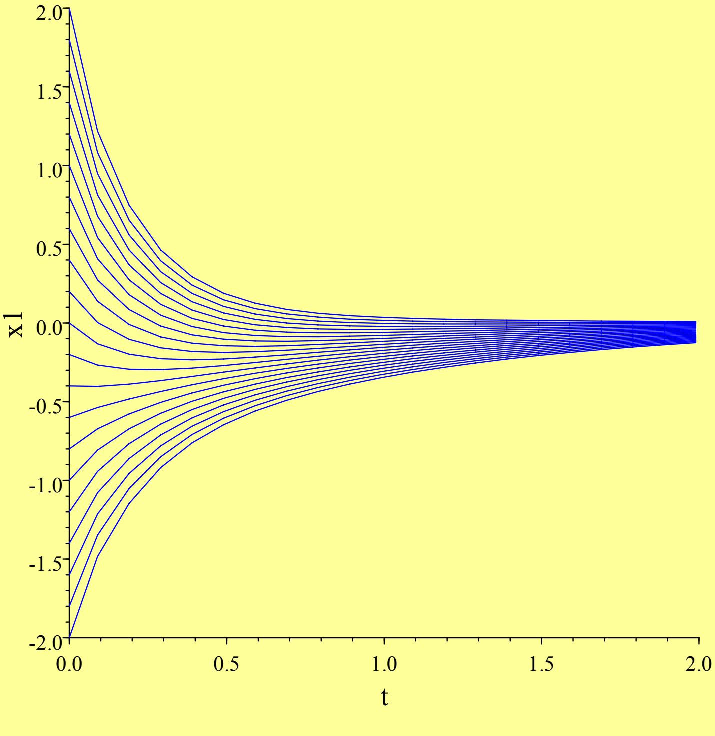
$dx_2/dx_1 = \infty$ when $x_2 = \dfrac{4}{\sqrt{3}}x_1$,

and the solution is parallel to the eigenvectors along the eigenvectors.

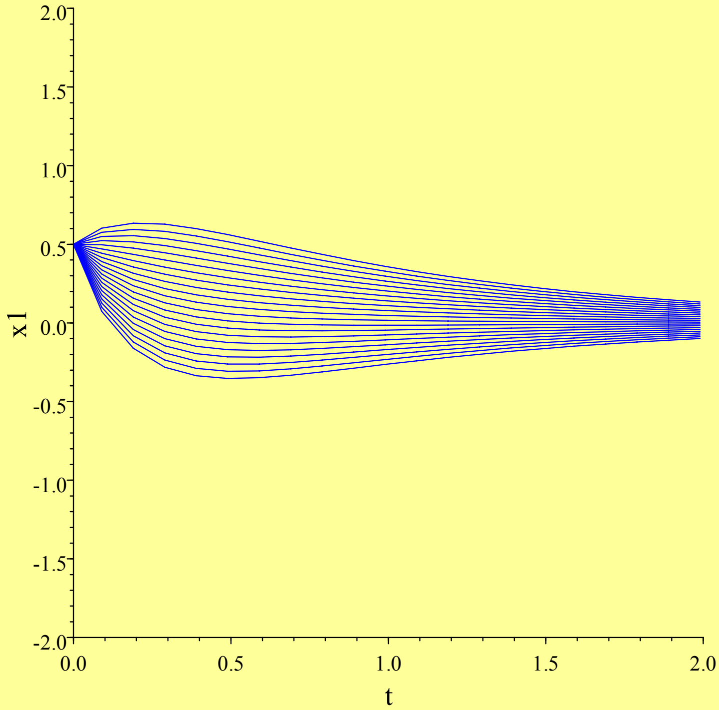
Obviously the fixed point is $\mathbf{x} = 0$.



Green lines indicate slope zero and infinite, magenta lines are eigenvectors.



Solutions with $x_2(0) = -1$ for a range of $x_1(0)$.



Solutions with $x_1(0) = ½$ and x_2 ranging between -2 and 2.

Obviously $d\mathbf{x}/dt = 0$ at the fixed point and the solution does not change, but it takes infinitely long to get there. The solution approaches the solution converging towards the $\mathbf{q}_2$ vector (along which progress goes like e^{-t}) as this decays more slowly than the e^{-5t} along the $\mathbf{q}_1$ vector.

Negative eigenvectors mean all solutions converge and this is a *node*.

We can work out the shape of the trajectories in the phase plane by defining

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \end{bmatrix}.$$

Now, since the matrix is symmetric (and we have normalised the eigenvectors), then $\mathbf{q}_1$ and $\mathbf{q}_2$ are orthogonal, so $\mathbf{Q}^T\mathbf{Q}$ is the identity, and we can rewrite our system

$$\frac{d\mathbf{x}}{dt} = \mathbf{Mx}$$

as

$$\mathbf{Q}^T \frac{d\mathbf{x}}{dt} = \mathbf{Q}^T\mathbf{MQQ}^T\mathbf{x}\ .$$

Defining $\xi = \mathbf{Q}^T\mathbf{x}$, then

$$\frac{d\xi}{dt} = \mathbf{Q}^T\mathbf{MQ}\xi = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}\xi$$

($\mathbf{Q}^T\mathbf{MQ}$ is diagonal with the eigenvalues along the trace). From this we can see that

$$\frac{d\xi_2}{d\xi_1} = \frac{\xi_2}{5\xi_1} \quad \Rightarrow \xi_1 = c\ \xi_2^{\,5} \qquad c = const.$$

5.2.3 NODES – TWO REAL POSITIVE EIGENVALUES

Consider the system
$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{bmatrix}\mathbf{x}.$$

This is identical to the previous example except that time is reversed and the solutions progress in the opposite direction. Rather than converging on the fixed point $\mathbf{x} = 0$, the solutions diverge from this point. The fixed point is still referred to as a *node*, but is *repulsive* or *unstable*, in contrast to the *attractive* or *stable* node in the previous section.

5.2.4 SPIRAL POINTS – COMPLEX EIGENVALUES

Consider the system
$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}\mathbf{x},$$

with eigenvalues given by

$$\begin{vmatrix} -\frac{1}{2}-\lambda & 1 \\ -1 & -\frac{1}{2}-\lambda \end{vmatrix} = \left(\lambda + \frac{1}{2}\right)^2 + 1 = \lambda^2 + \lambda + \frac{5}{4} = 0$$

$\Rightarrow$
$$\lambda = -\tfrac{1}{2} \pm i$$

and
$$\mathbf{q}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Along first eigenvector $\mathbf{x} = r\,\mathbf{q}_1$, we have $\mathbf{x} = r\mathbf{q}_1 = A\mathbf{q}_1 e^{\lambda_1 t}$ (A arbitrary). Now since λ_1 is complex, we have

$$\begin{aligned}\mathbf{x} = A\mathbf{q}_1 e^{\lambda_1 t} &= A\begin{pmatrix} 1 \\ i \end{pmatrix} e^{\left(-\frac{1}{2}+i\right)t} = A\begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{1}{2}t}\left(\cos t + i\sin t\right) \\ &= A\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{1}{2}t} + iA\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-\frac{1}{2}t}\end{aligned}.$$

Similarly, for λ_2,
$$\begin{aligned}\mathbf{x} = B\mathbf{q}_2 e^{\lambda_2 t} &= B\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-\frac{1}{2}t}\left(\cos t - i\sin t\right) \\ &= B\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{1}{2}t} - iB\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-\frac{1}{2}t}\end{aligned},$$
B arbitrary.

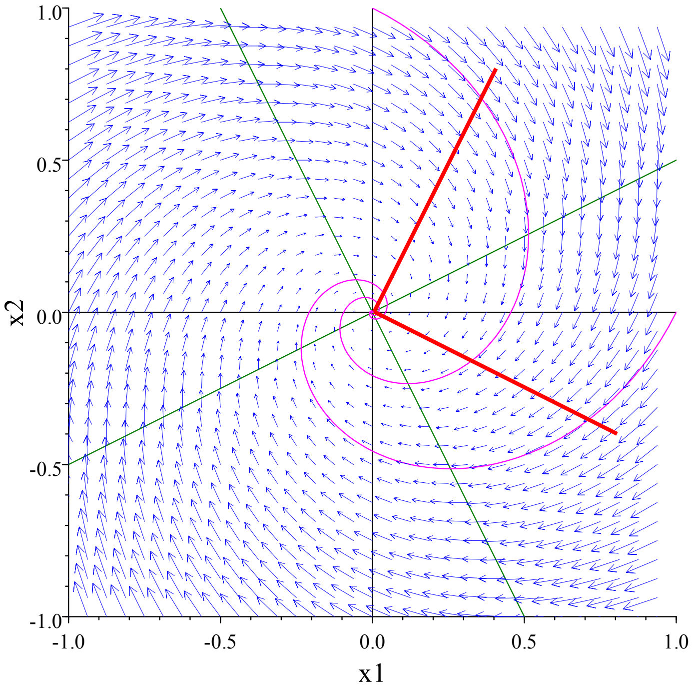
These solutions are (multiples of the) complex conjugates. We can thus find two real vectors from the linear combinations:

$$\boldsymbol{\xi}_1 = r_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{1}{2}t}, \quad \boldsymbol{\xi}_2 = r_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-\frac{1}{2}t}$$
r_1, r_2 arbtirary.

which rotate and are orthogonal at all times (doing a similar analysis with real eigenvalues has them remaining in the same direction for all times).

We also have
$$\frac{dx_2}{dx_1} = \frac{-x_1 - \frac{1}{2}x_2}{-\frac{1}{2}x_1 + x_2}$$

which vanishes when $x_2 = -2x_1$, and is infinite when $x_2 = ½\, x_1$.



In this example, the solutions spiral into the *spiral point* at $\mathbf{x} = 0$ in a negative rotational sense (due to the negative sign in front of the sin t term for the first eigenvector).

A spiral point is stable (spirals towards the fixed point) if the real part of the eigenvalue is negative (i.e. $Re(\lambda) < 0$). If $Re(\lambda) > 0$ then the spiral point is unstable and the solution spirals away from the fixed point.

5.2.5 STABLE CENTRE – IMAGINARY EIGENVALUES

Simple harmonic motion

For
$$\frac{d^2y}{dt^2} + \omega^2 y = 0,$$

set $x_1 = y$, $x_2 = dy/dt$, then
$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\mathbf{x}$$

with imaginary eigenvalues $\lambda = \pm i\omega$ and the eigenvectors

$$\mathbf{x} = r\mathbf{q}_1 = A\begin{pmatrix} -i \\ \omega \end{pmatrix}e^{i\omega t} = A\begin{pmatrix} -i \\ \omega \end{pmatrix}(\cos\omega t + i\sin\omega t) = A\begin{pmatrix} \sin\omega t \\ \omega\cos\omega t \end{pmatrix} + iA\begin{pmatrix} -\cos\omega t \\ \omega\sin\omega t \end{pmatrix},$$

$$\mathbf{x} = r\mathbf{q}_2 = B\begin{pmatrix} i \\ \omega \end{pmatrix}e^{-i\omega t} = B\begin{pmatrix} i \\ \omega \end{pmatrix}(\cos\omega t - i\sin\omega t) = B\begin{pmatrix} \sin\omega t \\ \omega\cos\omega t \end{pmatrix} - iB\begin{pmatrix} -\cos\omega t \\ \omega\sin\omega t \end{pmatrix},$$

(A, B arbitrary), so

$$\xi_1 = r_1 \frac{\mathbf{q}_1 + \mathbf{q}_2}{2} = r_1 \begin{pmatrix} \sin \omega t \\ \omega \cos \omega t \end{pmatrix},$$

$$\xi_2 = r_2 \frac{\mathbf{q}_1 - \mathbf{q}_2}{2} = r_2 \begin{pmatrix} -\cos \omega t \\ \omega \sin \omega t \end{pmatrix},$$

(r_1, r_2 arbitrary), and ξ_2 is $\pi/2$ behind ξ_1. The eigenvectors here are circular and do not converge on spiral point.

Solution in polar coordinates

For spiral points, it is often easier to visualise the structure using polar coordinates. Let

$$x_1 = r \cos \theta, x_2 = r \sin \theta,$$

then for $\omega = 1$

$$\frac{\dot{r}}{r}\cos\theta - \dot{\theta}\sin\theta = -\frac{1}{2}\cos\theta + \sin\theta,$$

$$\frac{\dot{r}}{r}\sin\theta + \dot{\theta}\cos\theta = -\cos\theta - \frac{1}{2}\sin\theta.$$

Multiplying first equation by $\cos \theta$ and the second by $\sin \theta$, and adding gives

$$\frac{\dot{r}}{r} = -\frac{1}{2} \quad \Rightarrow r = r_0\, e^{-\frac{1}{2}t},$$

and

$$\dot{\theta} = -1 \quad \Rightarrow \theta = t_0 - t,$$

hence

$$x_1 = r_0\, e^{-\frac{1}{2}t} \cos(t - t_0),$$

$$x_2 = -r_0\, e^{-\frac{1}{2}t} \sin(t - t_0).$$

5.2.6 Improper node – repeated eigenvalues

Consider

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & 2 \end{bmatrix} \mathbf{x},$$

with eigenvalues given by

$$\begin{vmatrix} 1-\lambda & -\frac{1}{2} \\ \frac{1}{2} & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) + \frac{1}{4},$$

$$= \lambda^2 - 3\lambda + \frac{9}{4} = \left(\lambda - \frac{3}{2}\right)^2 = 0$$

so $\lambda = 3/2$ (twice). The first eigenvector

$$\mathbf{q}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

gives rise to the complementary function $\xi_1 = A\,\mathbf{q}_1\, e^{3t/2}$, but clearly we need a second complementary function. Recalling that for second order equations we introduced a function of the form $t\,e^{\lambda t}$ under such circumstances, we try the form

$$\xi_2 = B(\mathbf{q}_1\, t + \mathbf{q}_2)\, e^{\lambda t}.$$

Substituting $$B\mathbf{q}_1\, e^{\lambda t} + \lambda\, B\,(\mathbf{q}_1\, t + \mathbf{q}_2)\, e^{\lambda t} = B\mathbf{M}\,(\mathbf{q}_1\, t + \mathbf{q}_2)\, e^{\lambda t},$$

which simplifies to $$\mathbf{q}_1 + \lambda\,\mathbf{q}_2 = \mathbf{M}\,\mathbf{q}_2$$

$\Rightarrow$ $$[\mathbf{M} - \lambda\mathbf{I}]\,\mathbf{q}_2 = \mathbf{q}_1.$$

Here, $$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} q_{21} \\ q_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow q_{21} + q_{22} = -2.$$

We have an underdetermined system. Arbitrarily choose $q_{21} = c \Rightarrow q_{22} = -2 - c$, so our second complementary function is

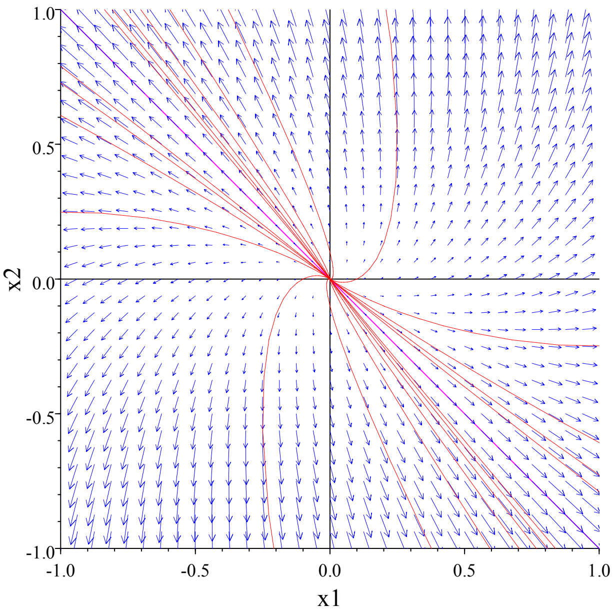
$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{3t/2} + c \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t/2} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^{3t/2}.$$

Since the second term is in the first complementary function we can select $c = 0$ without loss of generality, and the general solution is

$$\mathbf{x} = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} (A + Bt) + B \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right) e^{3t/2}.$$

Plotting in the phase plane

As $t \to \infty$, the $Bt\, e^{3t/2}$ term will dominate, and trajectories become parallel with the eigenvector. As $t \to -\infty$, they approach the fixed point in the opposite direction. For $B > 0$, solutions head towards positive x_1 *beneath* (as …+ $B(0, -2)$) the eigenvector. Similarly, for $B < 0$, solutions head towards negative x_1 *above* the eigenvector.



The node is an *improper node*: for each segment of the eigenvector the solutions emerge from only one side.

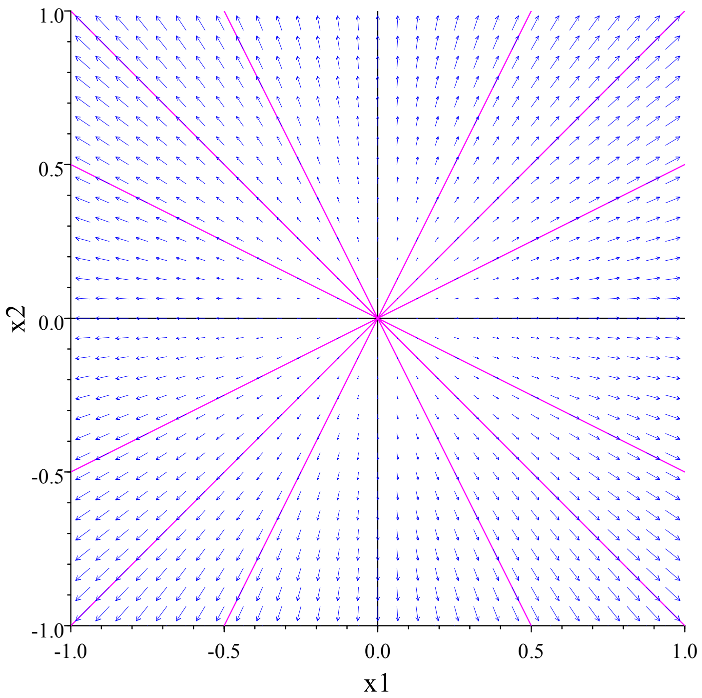
Here the improper node is unstable/repulsive because $\lambda > 0$. If $\lambda < 0$ then the node would be stable/attractive.

5.2.7 PROPER NODE – REPEATED EIGENVALUES

For a real symmetric matrix, can always identify two distinct eigenvectors, even if we have repeated eigenvalues. Consider

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x},$$

which has the repeated eigenvalue $\lambda = 1$. Now as any two orthogonal vectors are eigenvectors then all directions are the same and the solution must symmetrically diverge ($\lambda > 0$) or converge ($\lambda < 0$) on the fixed point.



5.3 Equilibrium and stability

5.3.1 Autonomous systems

Consider the generic autonomous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}).$$

The phase portrait has a unique trajectory through any point with slope

$$\frac{dx_2}{dx_1} = \frac{f_2}{f_1}.$$

Critical (fixed) points are where $\mathbf{f}(\mathbf{x}) = 0$.

We can analyse the stability near these fixed points in a manner similar to that we have used previously for first order equations.

Suppose $\mathbf{x}_0$ is one of the fixed points, then let

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{y},$$

where $\mathbf{y}$ is a small perturbation about the equilibrium. Now

End of Lecture 22

$$\frac{dx_i}{dt} = \frac{dx_{0i}}{dt} + \frac{dy_i}{dt} = f_i(\mathbf{x}) = f_i(x_{0j}) + \left.\frac{\partial f_i}{\partial x_j}\right|_{x_j = x_{0j}} y_j + \frac{1}{2}\left.\frac{\partial^2 f_i}{\partial x_j \partial x_k}\right|_{x_j = x_{0j}} y_j y_k + \cdots = 0.$$

Linearise
$$\frac{dy_i}{dt} \approx \left.\frac{\partial f_i}{\partial x_j}\right|_{x_j = x_{0j}} y_j$$

which may be written in the form

$$\frac{d\mathbf{y}}{dt} = \mathbf{My}\,, \text{ where } \; M_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_j = x_{0j}} ,$$

which is, of course, a linear system of the type we have already studied.

Hence we can see that the full equation is approximated by the linear system in the neighbourhood of the fixed point, *provided* at least one component of A_{ij} is non zero.

5.3.2 CLASSIFICATION OF FIXED POINTS

Suppose near a fixed point

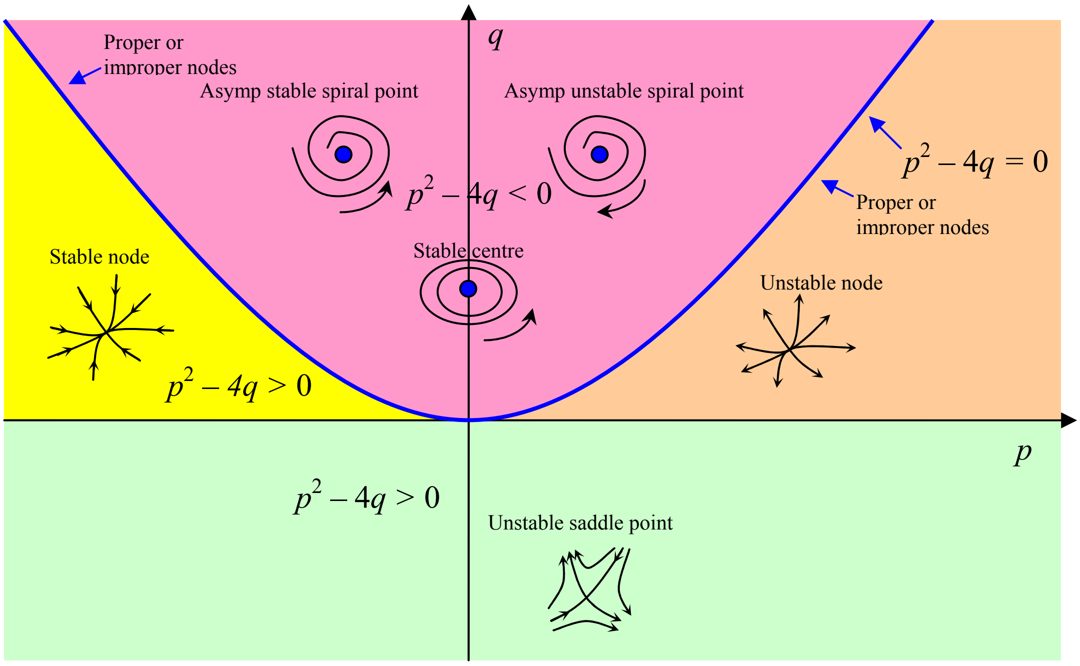
$$\frac{d\mathbf{y}}{dt} = \mathbf{My}\,,$$

then we obtain the eigenvalues of **M** as the roots of

$$|\mathbf{M} - \lambda\mathbf{I}| = \lambda^2 + p\lambda + q = 0\,.$$

Clearly $$\lambda = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q}\,,$$

and the sign of $p^2 - 4q$ determines whether the eigenvalues are real or complex, and hence, as we have seen, the nature of the fixed point.



Note that for the spiral and centre circles, p and q alone are insufficient to determine the sense of the rotation. Similarly, p and q alone cannot distinguish proper from improper nodes.

5.3.3 MATRICES IN CANONICAL FORM

Matrices in canonical form are the simplest representations of different kinds of behaviour.

Real eigenvalues

For real eigenvalues η, μ, then $$\mathbf{M} = \begin{bmatrix} \eta & 0 \\ 0 & \mu \end{bmatrix},$$

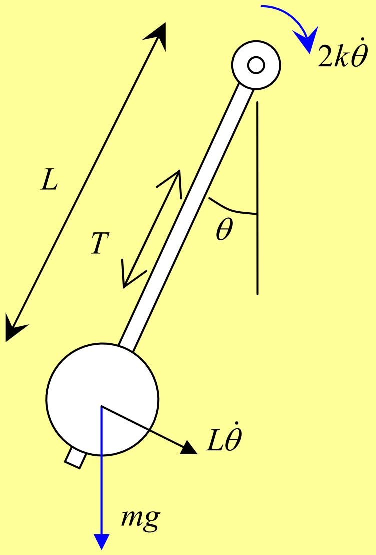
which can describe a node or a saddle (depending on whether λ and μ are the same or different signs).

Complex eigen values

For eigen values $\eta \pm i\omega$ $$\mathbf{M} = \begin{bmatrix} \eta & -\omega \\ \omega & \eta \end{bmatrix},$$

which can describe spirals and circle centres.

5.4 Simple pendulum



5.4.1 Small oscillations

For small amplitude oscillations of a simple pendulum we may derive the linear approximation in which $\sin\theta \approx \theta$ so that $T = mg$, the restoring force is $mg\,\theta$, the acceleration is $L\ddot{\theta}$ and the friction is $2k\dot{\theta}$. This gives

$$\ddot{\theta} + 2k\dot{\theta} + \frac{g}{L}\theta = 0,$$

which we could write $v = L\, d\theta/dt$ and rewrite the second order equation as the first order system

$$\frac{d}{dt}\begin{pmatrix} \theta \\ v \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -g & -2k \end{bmatrix}\begin{pmatrix} \theta \\ v \end{pmatrix}.$$

This approximation, however, ceases to be valid when θ stops being small.

5.4.2 Finite amplitude oscillations

The system of equations retains a similar form but with $g\sin\theta$ replacing $g\,\theta$, thus making the system nonlinear:

$$\dot{\mathbf{x}} = \frac{d}{dt}\begin{pmatrix}\theta\\ v\end{pmatrix} = \begin{pmatrix}\dfrac{v}{L}\\ -g\sin\theta - 2kv\end{pmatrix} \equiv \mathbf{f}(\mathbf{x}).$$

The fixed points are clearly $(\theta,v) = (n\pi,0)$.

$$\frac{\partial f_i}{\partial x_j} = \begin{bmatrix}0 & \dfrac{1}{L}\\ -g\cos\theta & -2k\end{bmatrix}$$

Vertically downwards

For n even, $\cos\theta = 1$, the fixed point is with the pendulum hanging vertically downwards. In the neighbourhood of this fixed point write the linear expansion

$$\dot{\mathbf{y}} = \frac{\partial f_i}{\partial x_j}\mathbf{y} = \begin{bmatrix}0 & \dfrac{1}{L}\\ -g & -2k\end{bmatrix}\mathbf{y},$$

which has the same form as the equation for a pendulum with a small amplitude oscillation. The eigenvalues

$$\begin{vmatrix}-\lambda & \dfrac{1}{L}\\ -g & -2k-\lambda\end{vmatrix} = \lambda^2 + 2k\lambda + \frac{g}{L} = 0 \;\Rightarrow\; \lambda = -k \pm i\sqrt{\frac{g}{L} - k^2}.$$

If $g/L > k$ then

Complex eigenvalues $\Rightarrow$ spiral

Negative real part $\Rightarrow$ spirals inwards

Eigenvectors of matrix

$$\mathbf{q}_1 = \begin{pmatrix}-k - i\sqrt{\dfrac{L}{g} - k^2}\\ 1\end{pmatrix},\quad \mathbf{q}_2 = \begin{pmatrix}-k + i\sqrt{\dfrac{L}{g} - k^2}\\ 1\end{pmatrix}.$$

Let $\omega^2 = L/g - k^2$, then complementary functions

$$\mathbf{q}_1 e^{(-k+i\omega)t} = \begin{pmatrix}-k - i\omega\\ 1\end{pmatrix}(\cos\omega t + i\sin\omega t)e^{-kt}$$

$$= \begin{pmatrix}-k\cos\omega t + \omega\sin\omega t\\ \cos\omega t\end{pmatrix}e^{-kt} - i\begin{pmatrix}\omega\cos\omega t + k\sin\omega t\\ -\sin\omega t\end{pmatrix}e^{-kt},$$

$$\mathbf{q}_2 e^{(-k-i\omega)t} = \begin{pmatrix} -k+i\omega \\ 1 \end{pmatrix} (\cos\omega t - i\sin\omega t) e^{-kt}$$

$$= \begin{pmatrix} -k\cos\omega t + \omega\sin\omega t \\ \cos\omega t \end{pmatrix} e^{-kt} + i \begin{pmatrix} \omega\cos\omega t + k\sin\omega t \\ -\sin\omega t \end{pmatrix} e^{-kt},$$

are complex conjugates. Can look at sum and difference,

$$\xi_1 = A \begin{pmatrix} -k\cos\omega t + \omega\sin\omega t \\ \cos\omega t \end{pmatrix} e^{-kt}, \quad \xi_2 = B \begin{pmatrix} \omega\cos\omega t + k\sin\omega t \\ -\sin\omega t \end{pmatrix} e^{-kt}$$

and see that will spiral in clockwise direction (*e.g.* inspect behaviour near $t = 0$).

Vertically upwards

If n is odd, then $\cos\theta = -1$ and fixed point has critical point vertically above the pivot and the linearised system gives

$$\dot{\mathbf{y}} = \frac{\partial f_i}{\partial x_j}\mathbf{y} = \begin{bmatrix} 0 & \frac{1}{L} \\ g & -2k \end{bmatrix} \mathbf{y},$$

with the corresponding eigenvalues

$$\begin{vmatrix} -\lambda & \frac{1}{L} \\ g & -2k-\lambda \end{vmatrix} = \lambda^2 + 2k\lambda - \frac{g}{L} = 0 \;\Rightarrow\; \lambda = -k \pm \sqrt{\frac{g}{L} + k^2}.$$

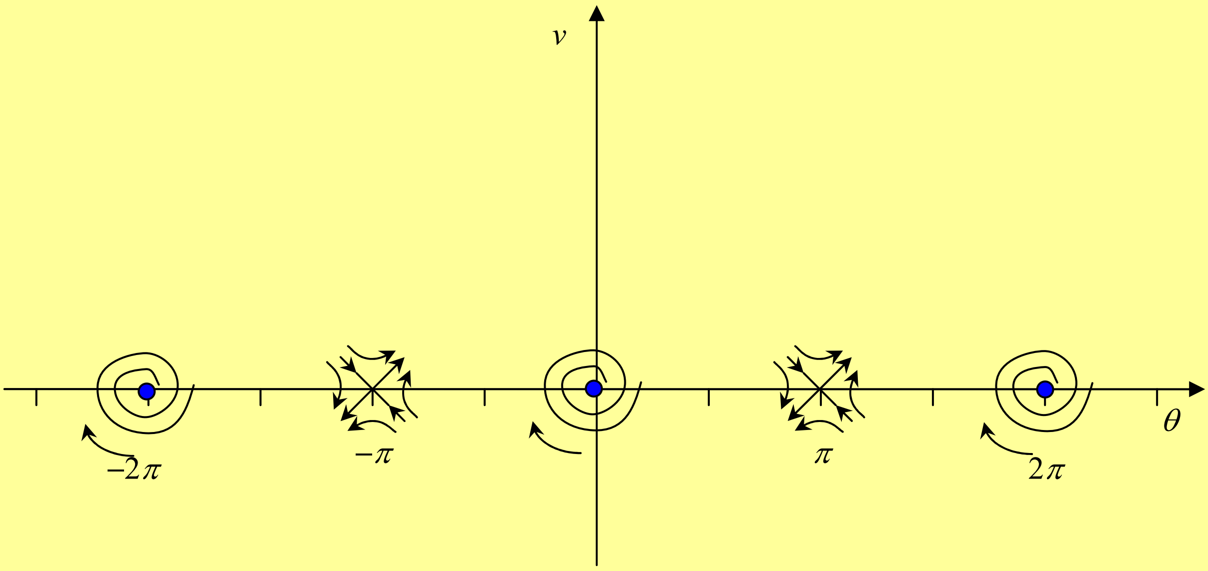
are real and of opposite sign. Hence they represent a saddle point. The eigenvectors

$$\mathbf{q}_1 = \begin{pmatrix} \frac{1}{L\lambda} \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{k + \sqrt{\frac{g}{L} + k^2}}{g} \\ 1 \end{pmatrix} \quad \text{for} \quad \lambda = -k + \sqrt{\frac{g}{L} + k^2} > 0$$

and

$$\mathbf{q}_2 = \begin{pmatrix} \frac{1}{L\lambda} \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{k - \sqrt{\frac{g}{L} + k^2}}{g} \\ 1 \end{pmatrix} \quad \text{for} \quad \lambda = -k - \sqrt{\frac{g}{L} + k^2} < 0,$$

so converges from second and fourth quadrants, and diverges in first and third quadrants.



5.4.3 PENDULUM PHASE PLANE

From the full equation

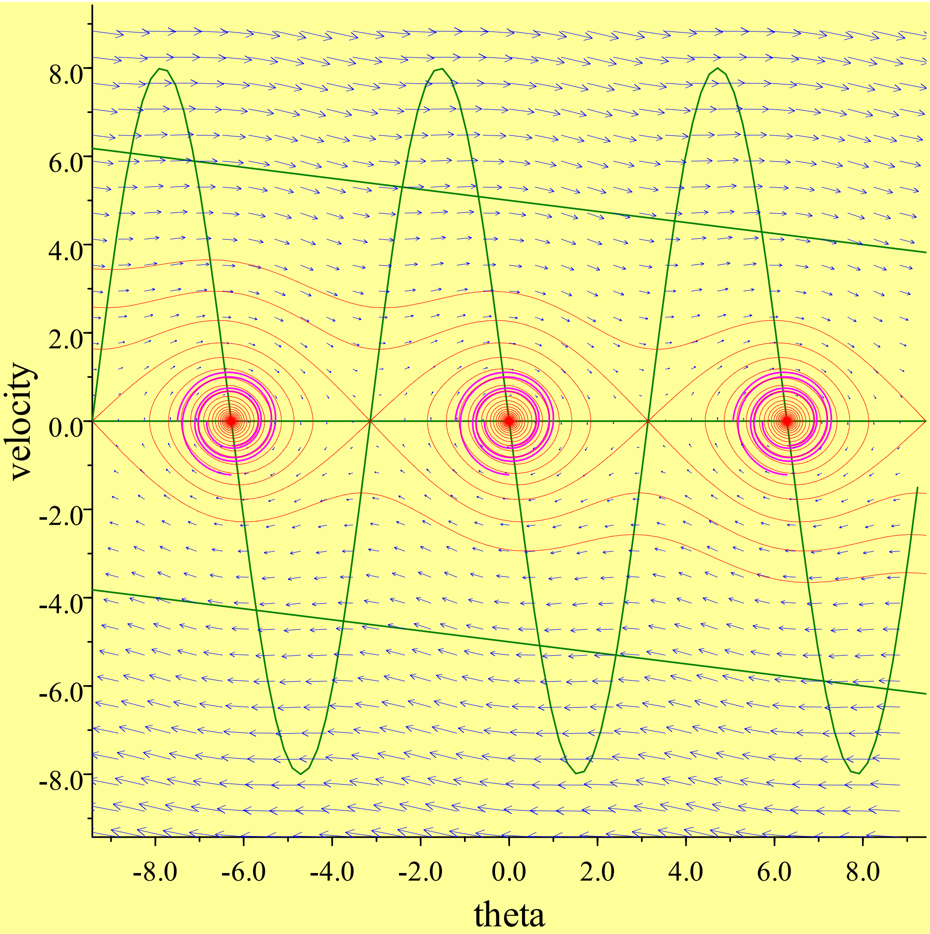
$$\dot{\mathbf{x}} = \frac{d}{dt}\begin{pmatrix}\theta \\ v\end{pmatrix} = \begin{pmatrix}\dfrac{v}{L} \\ -g\sin\theta - 2kv\end{pmatrix} \equiv \mathbf{f}(\mathbf{x})$$

we can see that $d\theta/dt$ vanishes at $v = 0$, so solutions are vertical through $v = 0$. Similarly, dv/dt vanishes (the solution is horizontal) when $v = -\frac{1}{2}(g/k)\sin\theta$.

More generally,
$$\frac{dv}{d\theta} = -\frac{g\sin\theta + 2kv}{v/L} = -gL\frac{\sin\theta}{v} - 2kL,$$

so mean slope at large $|v|$ for solution is $-2kL$. Oscillation about this mean slope decreases as v increases.

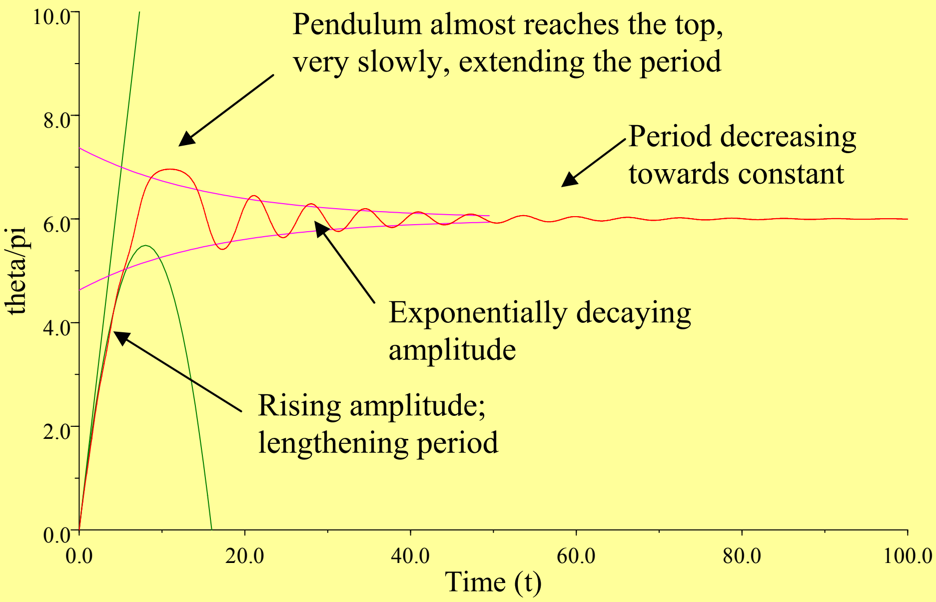
For large $|v|$ we therefore have $\theta = \int v\, dt = v_0 t(1 - kt)$.



The *basin of attraction* is the region that contains all solutions ending up in a given stable spiral point (or node).

A *separatrix* bounds each side of the basin of attraction. Sparatrices divide the phase plane into regions where the solution is attracted to or repulsed from a given fixed/critical point.

5.4.4 PENDULUM BEHAVIOUR



Initial condition: $\theta(0) = 0$, $v(0) = v_0$.

For $L = 1$:

Very early time: $v = v_0 \Rightarrow \theta = v_0\, t$.

Early time: $v = v_0 - 2kt \Rightarrow \theta = v_0 t\,(1 - kt)$.

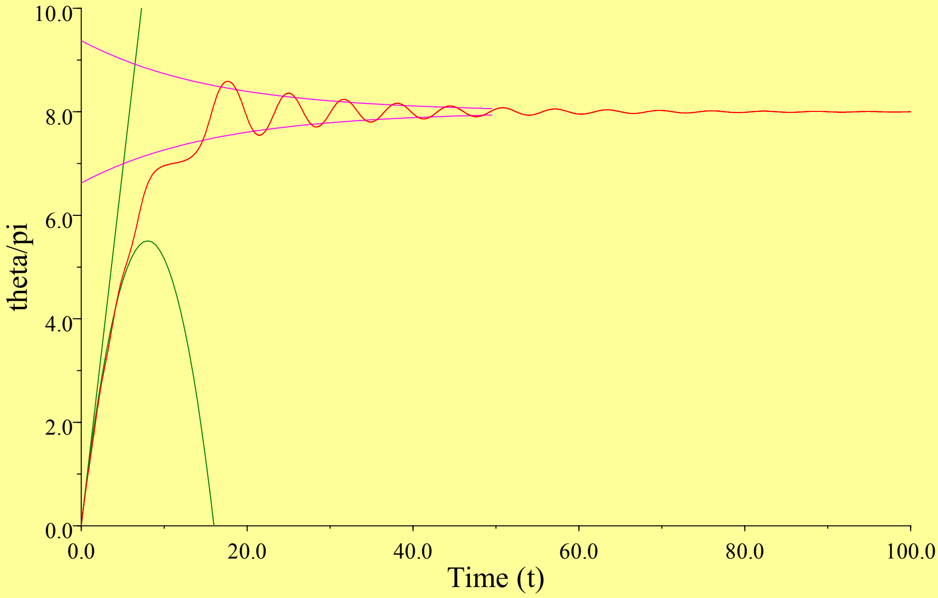
Late time $\mathbf{y} \sim e^{-kt}$ Linearised about stable fixed point.

Note: Amplitude of oscillations about mean approximation $\theta = v_0 t\,(1 - kt)$ increase as θ increases.

As $d\theta/dt$ decreases, then period of oscillation increases while still executing complete (but slowing) circuit.

At late time, tends towards linear behaviour around stable fixed point, exhibiting exponential decay as e^{-kt}.

In this case, pendulum just about reaches the top before falling back at the first velocity reversal. If initial velocity had been slightly higher, then would have just made it over the top to make one more complete circuit.



5.4.5 Energy

Let $$E = gL\,(1 - \cos\theta) + \tfrac{1}{2} v^2 \geq 0$$

be the total energy per unit mass (*i.e.* the sum of the potential energy $gL\,(1 - \cos\theta)$ and kinetic energy $\tfrac{1}{2} v^2$).

Now $$E = 0 \text{ at } (\theta, v) = (2n\pi, 0),$$

i.e. at the asymptotically stable equilibrium points.

Now
$$\begin{aligned}\dot{E} &= gL\sin\theta\,\dot{\theta} + v\dot{v} \\ &= \left(gL\sin\theta\right)\frac{V}{L} + v\left(-g\sin\theta - 2kv\right) \\ &= -2kv^2\end{aligned} \qquad \begin{aligned}\dot{\theta} &= \frac{v}{L} \\ \dot{v} &= -g\sin\theta - 2kv\end{aligned}$$

which is negative except at equilibria where $v = 0$.

Thus, as $t \to \infty$ the system **must** evolve towards one of its asymptotically stable equilibrium points.

This is an example of a theorem due to Liapounov. In this context, E is called a *Liapounov function*. It is more powerful than analysing the solutions in the neighbourhood of the equilibrium points because it is global. For example, it can enable one to determine the basins of global stability, which in the case of the damped pendulum is the entire phase plane.

5.5 Competing species

5.5.1 General ideas

Single species

We saw in §3.4.2 that, for a single species, when there was no difficulty finding a mate the population would be limited by the available food supply and that this could be modelled with a (nonlinear) logistic equation of the form

$$\frac{dy}{dt} = r\left(1 - \frac{y}{Y}\right)y,$$

with a single stable equilibrium at a population of $y = Y$. It was only through difficulty in finding mates that would lead to the $y = 0$ equilibrium being stabilised with extinction becoming possible.

What happens if there is more than one species?

Competition for food

Second species (B) competes for food needed by first species (A) in a manner similar to the self-competition seen in the logistics equation (§3.4.2). As we shall see, this leads to a second order nonlinear differential equation, and solutions of greater complexity.

The structure of the solution will depend on whether the inter-species competition is more or less important than the intra-species competition. If intra-species competition is more important then the solutions might look much like those for a single species. We shall concentrate on the other case.

Predator-prey relationship

If species B preys on species A we have a different form of interaction, which also leads to a second order equation.

Is it better to compete, or be prey? Competition can lead to one species becoming extinct, whereas being prey will not (in these simple systems) lead to extinction.

5.5.2 COMPETING VEGETARIANS

For a single isolated species X we can write

$$\dot{x} = x(r - \eta x), \qquad r > 0,\ \eta > 0,$$

where the self-limiting factor $-\eta x^2$ is a consequence of a shortage of food.

If we now have a second species Y that competes for the same food supply, then we will find the growth rate $r - \eta x$ is reduced by this second species, to say $r - \eta x - py$. Hence we can write

$$\dot{x} = x(r - \eta x - py), \qquad p > 0.$$

Similarly the second species has its growth reduced by the population of the first species, thus

$$\dot{y} = y(s - \mu y - qx), \qquad s, \mu, q > 0.$$

We are, of course, only interested in solutions with $x \geq 0$ and $y \geq 0$.

Note that η and μ may not be equal as species have different behaviours towards internal competition and different equilibrium populations. Similarly, p and q may not be equal as one species may dominate over the other in such interactions.

Critical points

The critical points are clearly

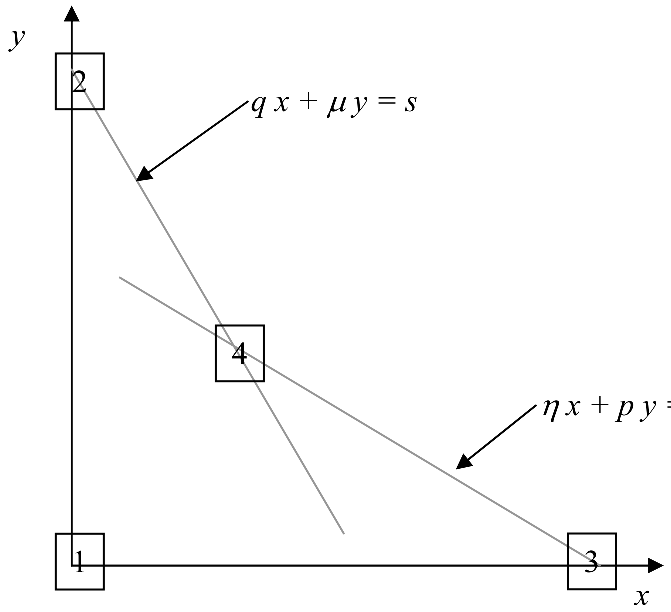
$$x = 0 \text{ or } \eta x + p y = r$$

and

$$y = 0 \text{ or } q x + \mu y = s.$$

This gives four solutions:

1. $x_1 = 0, y_1 = 0$
2. $x_2 = 0, y_2 = s/\mu$.
3. $y_3 = 0, x_3 = r/\eta$
4. $x_4 = \dfrac{-r\mu + sp}{\Delta}$, $y_4 = \dfrac{-s\eta + rq}{\Delta}$, where $\Delta = pq - \eta\mu$.



For the present, we shall assume that $\eta\mu < pq$, *i.e.* $\Delta > 0$. In this case the (geometric) mean of the self-limitation rate is less than the limitation imposed by competing species.

We shall also assume x_4, y_4 is in the positive quadrant, *i.e.*

$$s\,p > r\,\mu \text{ and } r\,q > s\,\eta.$$

Behaviour near fixed points

At the fixed (critical) points, let $x = x_i + u$, $y = y_i + v$.

$$\begin{aligned} \dot{u} &= u\left(r - \eta x_i - p y_i\right) - x_i\left(\eta u - p v\right) \\ &= u\left(r - 2\eta x_i - p y_i\right) + p x_i v \end{aligned},$$

$$\dot{v} = v\left(s - 2\mu y_i - q x_i\right) + q y_i u$$

Fixed point 1

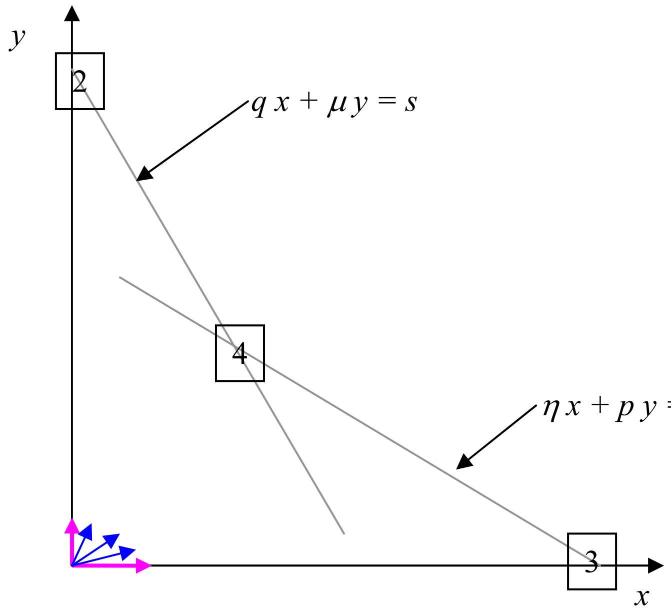
$x_1 = 0, y_1 = 0 \Rightarrow$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \dot{\mathbf{u}} = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}\mathbf{u},$$

giving real positive eigenvalues $\lambda = r,s$. Hence the fixed point at the origin is an unstable node.

Eigenvectors in x and y direction, so fundamental solutions

$$\mathbf{x}_{11} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{rt}, \quad \mathbf{x}_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{st}.$$



Fixed point 2

$x_2 = 0$, $y_2 = s/\mu$. $\Rightarrow$ $$\dot{\mathbf{u}} = \begin{bmatrix} r - 2\eta x_i - py_i & px_i \\ qy_i & s - 2\mu y_i - qx_i \end{bmatrix} \mathbf{u} = \begin{bmatrix} r - \dfrac{ps}{\mu} & 0 \\ \dfrac{qs}{\mu} & -s \end{bmatrix} \mathbf{u},$$

giving eigenvalues from

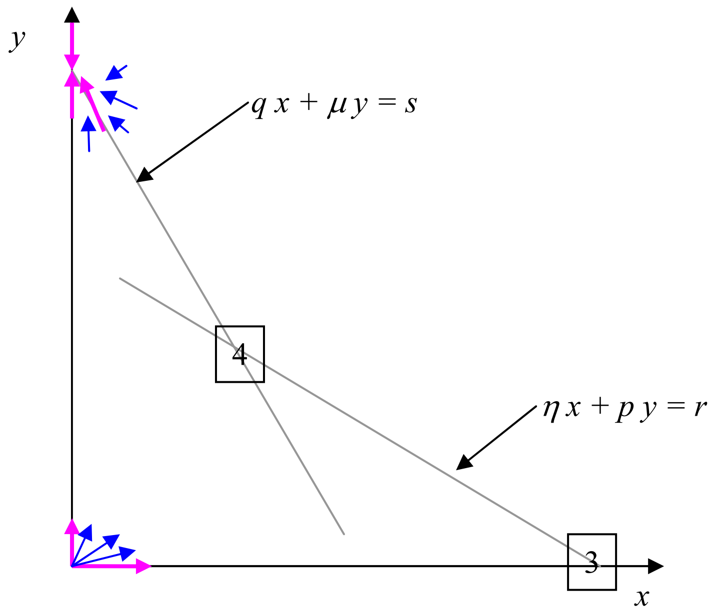
$$\begin{vmatrix} r - \dfrac{ps}{\mu} - \lambda & 0 \\ \dfrac{qs}{\mu} & -s - \lambda \end{vmatrix} = \left(\left(r - \frac{ps}{\mu} \right) - \lambda \right) (s + \lambda) = 0$$

are $\lambda = -s$ and $\lambda = r - ps/\mu < 0$ (since $x_4 > 0 \Rightarrow s\,p > r\,\mu$).

Thus both eigenvalues are real and negative, so point 2 is a stable node. The eigenvectors then give the fundamental solutions

$$\mathbf{x}_{21} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-st}, \quad \mathbf{x}_{22} = \begin{pmatrix} s + r - \dfrac{ps}{\mu} \\ \dfrac{qs}{\mu} \end{pmatrix} e^{\left(r - \frac{ps}{\mu} \right) t}.$$

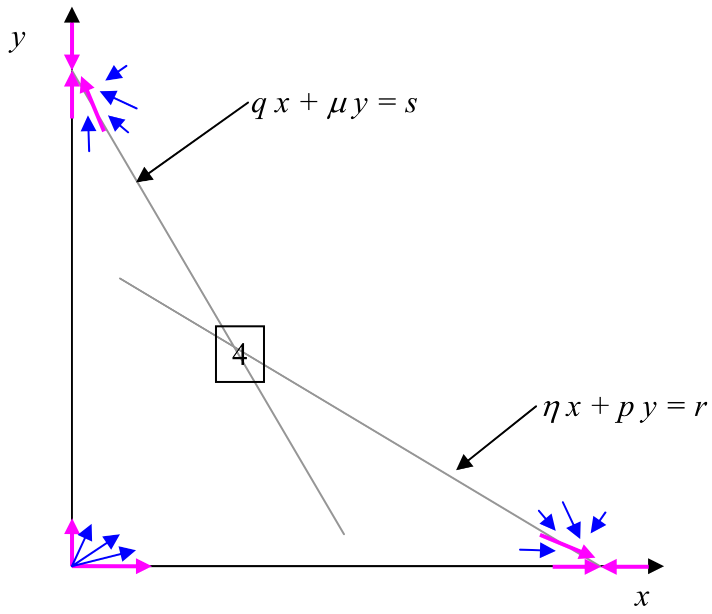
(We do not know the sign of $s + r - ps/\mu$.) If $s > ps/\mu - r$, then decay along $\mathbf{x}_{21}$ is faster than along $\mathbf{x}_{22}$ and the solution will approach the node along $\mathbf{x}_{22}$. (The opposite inequality leads to the reverse result.)



Fixed point 3

Analysis of this is identical to fixed point 2, but with the various variables and coefficients interchanged.

$$\mathbf{x}_{31} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-rt}, \quad \mathbf{x}_{32} = \begin{pmatrix} \dfrac{qr}{\eta} \\ s + r - \dfrac{qr}{\eta} \end{pmatrix} e^{\left(s - \frac{qr}{\eta}\right)t}.$$



Fixed point 4

$x_4 = \dfrac{-r\mu + sp}{\Delta}$, $y_4 = \dfrac{-s\eta + rq}{\Delta}$, where $\Delta = pq - \eta\mu$.

$$\Rightarrow \qquad \dot{\mathbf{u}} = \begin{bmatrix} r - 2\eta x_i - py_i & px_i \\ qy_i & s - 2\mu y_i - qx_i \end{bmatrix}\mathbf{u} = \frac{1}{pq - \eta\mu}\begin{bmatrix} -\eta(ps - \mu r) & p(ps - \mu r) \\ q(qr - \eta s) & -\mu(qr - \eta s) \end{bmatrix}\mathbf{u}.$$

Note that since this fixed point is in the positive quadrant then $ps - \mu r > 0$ and $qr - \eta s > 0$.

The algebra at this point becomes messy. However, we can get an indication of the general solution by considering the case where $ps - \mu r = qr - \eta s$, so that the fixed point lies on $x = y$ and the matrix simplifies to $\dfrac{s}{q+\mu}\begin{bmatrix} -\eta & p \\ q & -\mu \end{bmatrix}$. Taking further that $s = q + \mu$ and $r = p + \eta$ gives

$$\dot{\mathbf{u}} = \begin{bmatrix} -\eta & p \\ q & -\mu \end{bmatrix}\mathbf{u},$$

with

$$\begin{vmatrix} -\eta - \lambda & p \\ q & -\mu - \lambda \end{vmatrix} = \lambda^2 + (\eta + \mu)\lambda + (\eta\mu - pq) = 0,$$

and
$$\lambda = -\frac{\eta + \mu}{2} \pm \frac{1}{2}\sqrt{(\eta + \mu)^2 + 4(pq - \eta\mu)}.$$

Since we are looking at the case $\eta\mu < pq$, then the eigenvalues are real and of opposite sign.

$\Rightarrow$ fixed point 4 is a saddle point.

The corresponding eigenvectors are

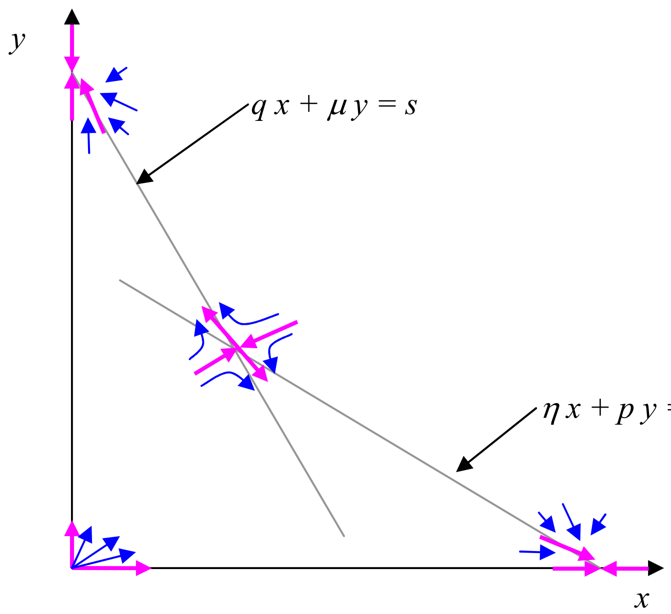
$$\mathbf{q}_{41} = \begin{pmatrix} \dfrac{(\eta-\mu)+\sqrt{(\eta-\mu)^2+4pq}}{2q} \\ -1 \end{pmatrix}, \quad \mathbf{q}_{42} = \begin{pmatrix} \dfrac{(\eta-\mu)-\sqrt{(\eta-\mu)^2+4pq}}{2q} \\ -1 \end{pmatrix},$$

for the negative (convergent) and positive (divergent) eigenvalues.

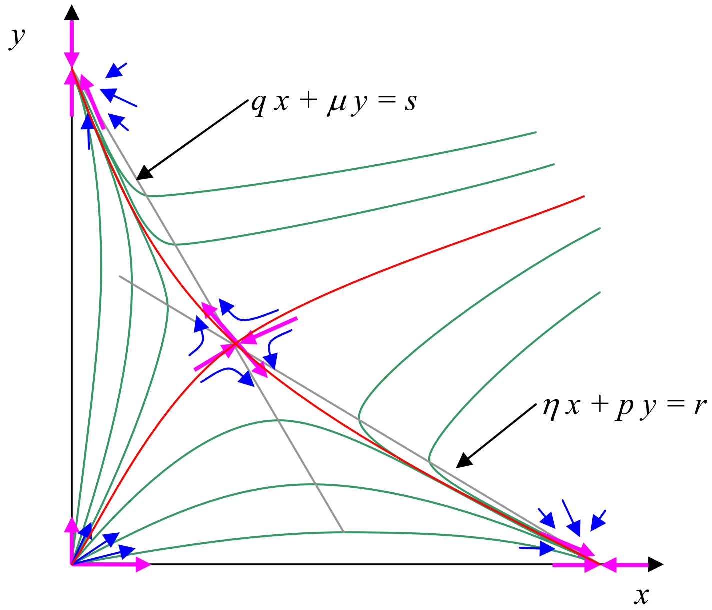
In the general case, after some messy algebra, we see that the eigenvalues are given by

$$\lambda = -\frac{1}{2(pq-\eta\mu)}\Big[\big(\eta(ps-\mu r)+\mu(qr-\eta s)\big)$$
$$\pm\sqrt{\big(\eta(ps-\mu r)+\mu(qr-\eta s)\big)^2+4(pq-\eta\mu)(ps-\mu r)(qr-\eta s)}\Big]$$

and even messier eigenvectors. However, a detailed knowledge of the eigenvectors is not required to complete a sketch of the phase plane.



Completing the solution



End of Lecture 23

Survival strategy for species X

It is important to remain below the separatrix passing through the saddle point. If conditions are allowed to rise above it, then solution will head off towards fixed point 2 and extinction. Key to this is the location of the saddle point.

Want
$$\frac{y_4}{x_4} = \frac{rq - s\eta}{sp - r\mu}$$

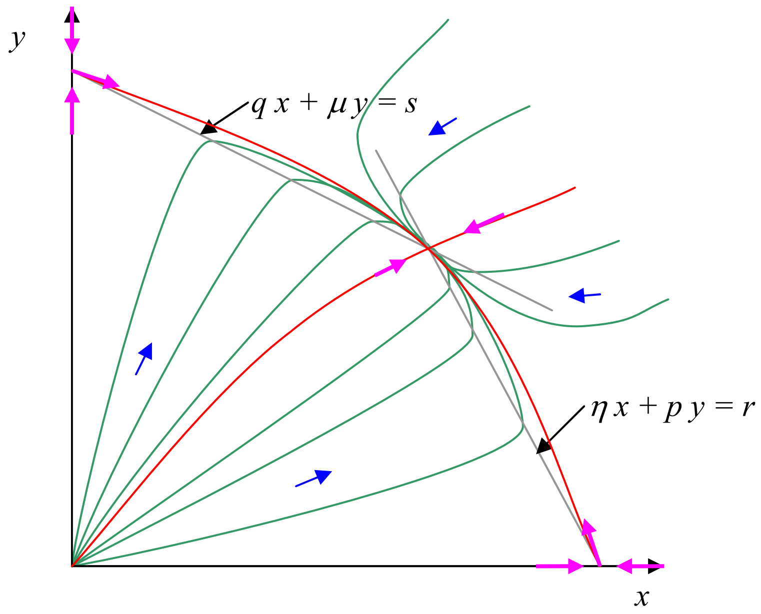
large so that likely to lie beneath the separatrix. If successful in this, want $x_3 = r/\eta$ large. Hence want r large and η small: lots of mating, and few fights.

Species X cannot influence s and μ, but can have some influence over p and q: want q large (win lots of fights) and p small (loose few). If species Y is stronger than you, then maybe it would be best to hide to minimise the pxy term.

A pacifist has $\eta = 0$ and $q = 0$. However, it may be very difficult to ensure members are not lost to the other species, so p is not likely to vanish. The obvious risk is than that species X has no control over the population of species Y, yet is strongly influenced by it. Could examine fixed points to show that must breed profusely in order to survive.

Less competitive species

If the competition between members of the same species is greater than that between members of the different species, *i.e.* $pq < \eta\mu$, then fixed points 2 and 3 become saddle points, and fixed point 4 becomes a stable node, and the two populations form a stable system.



The best strategy is, of course, to avoid direct competition by occupying a somewhat different ecological niche or a different ecosystem.

5.5.3 PREDITOR-PREY EQUATIONS

As we have seen, a species *X* with population *x*, left alone might evolve according to

$$\frac{dx}{dt} = rx, \qquad r > 0$$

if the self-limiting term of the logistic equation is ignored (*i.e.* no fighting, and no difficulty finding a mate).

Suppose species *Y* preys on species *X*. If there is no *X*, then the population of *Y* would decay as

$$\frac{dy}{dt} = -sy, \qquad s > 0$$

due to lack of food preventing enough individuals reaching sexual maturity to have the next generation.

Suppose members of *X* and *Y* meet at a rate proportional to their respective populations. eading to a loss of members of *X* at a rate *pxy*, and providing sufficient food to allow a birth rate of *Y* at rate *qxy*. (We again assume there is no difficulty finding mates.) Hence

$$\dot{x} = x(r - py), \qquad r, p > 0$$

$$\dot{y} = y(-s + qx). \qquad s, q > 0$$

These are the *Lotka-Volterra equations*. The equations are highly simplified, but still capture a broad range of relevant features.

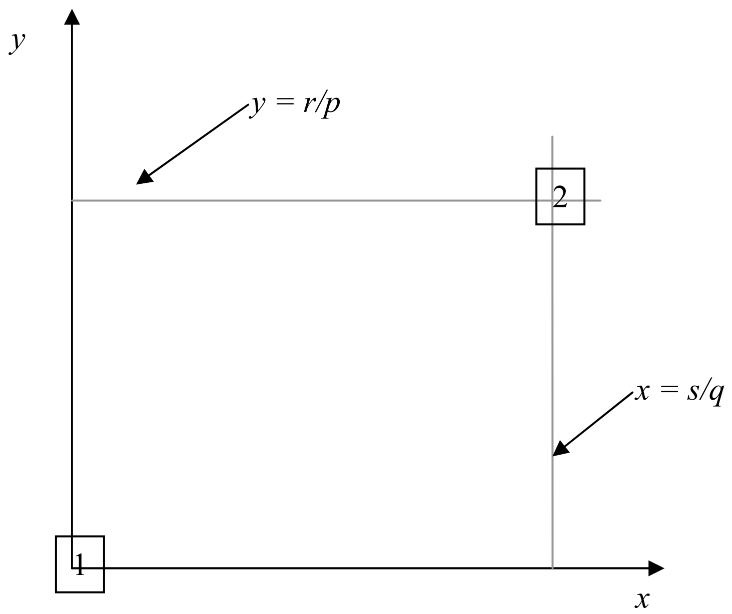
Critical points

$dx/dt = 0$ on $\quad x = 0$ and $y = r/p$,

$dy/dt = 0$ on $\quad y = 0$ and $x = s/q$.

Fixed point 1: $x_1 = 0, y_1 = 0$

Fixed point 2: $x_2 = s/q, y_2 = r/p$.



[Note that if we had included self-limiting terms due to fighting, then there would be additional fixed points on the x and y axes, but these would be at much higher populations.]

Linearising the equations about the fixed points

$$\dot{\mathbf{u}} = \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} r - py_i & -px_i \\ qy_i & -s + qx_i \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

allows us to explore the nature of the fixed points.

Fixed point 1

At $(x,y) = (0,0)$,

$$\dot{\mathbf{u}} = \begin{bmatrix} r & 0 \\ 0 & -s \end{bmatrix}\mathbf{u},$$

which has real eigenvalues $\lambda = r, -s$ with the corresponding eigenvectors (1, 0) and (0, 1). The resulting saddle point diverges along the x axis, and converges along the y axis.

Fixed point 2

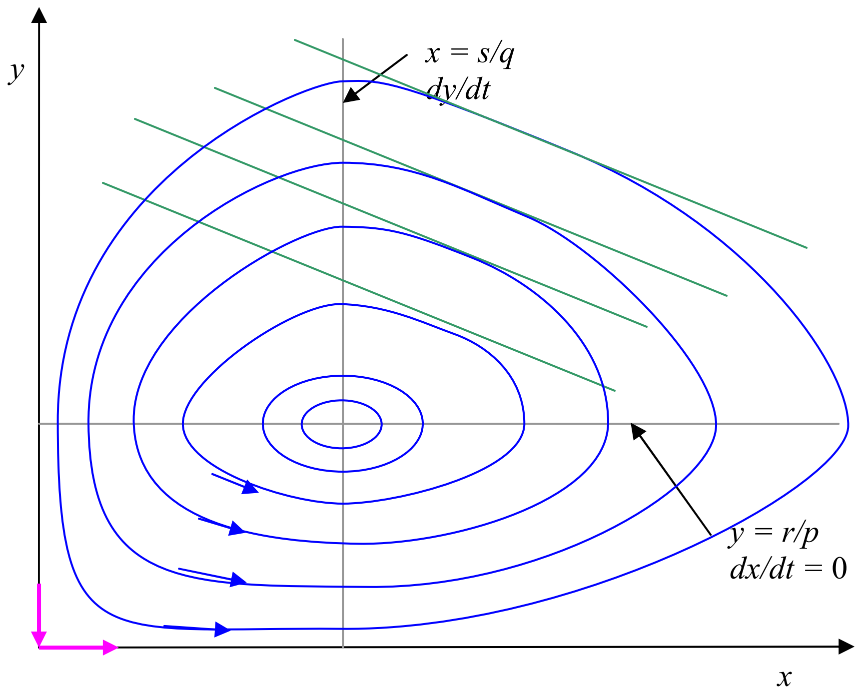
At $(x,y) = (s/q,r/p)$,

$$\dot{\mathbf{u}} = \begin{bmatrix} 0 & -ps/q \\ qr/p & 0 \end{bmatrix}\mathbf{u},$$

leading to $\lambda = \pm i\,(rs)^{1/2}$. These imaginary eigenvalues mean that fixed point 2 is a centre circle (in fact ellipses). Examination of the eigenvectors would show that the rotation is anticlockwise. However, we may see this more directly from the relationship with the saddle point.

From the original equations we can see that the trajectories are controlled by

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{y(-s+qx)}{x(r-py)} \to -\frac{q}{p} \quad \text{when } x \gg s/q \text{ and } y \gg r/p.$$



We can solve the equation for the trajectories explicitly as it is separable:

$$\left(\frac{r}{y} - p\right)\frac{dy}{dx} = -\frac{s}{x} + q\,,$$

$\Rightarrow$ $$r\ln|y| - py = -s\ln|x| + qx + const\,,$$

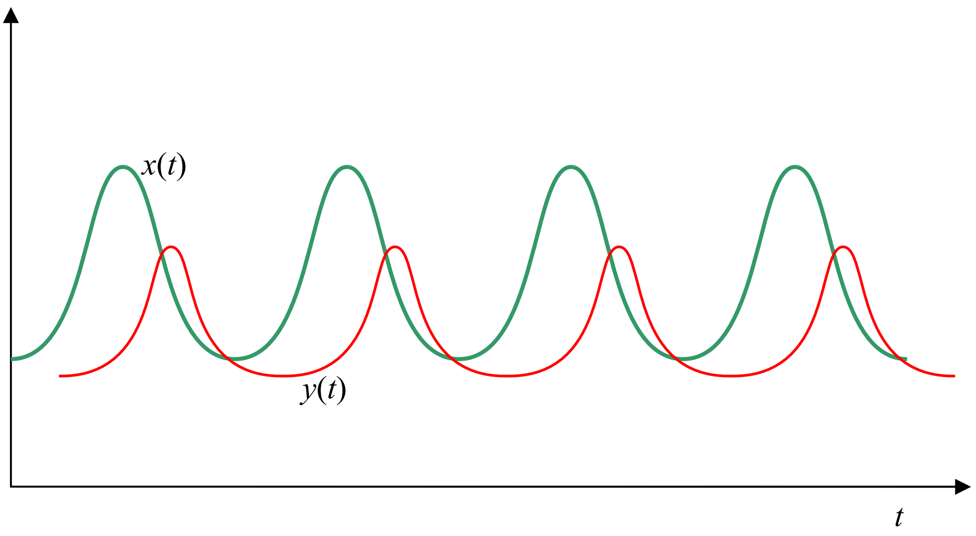
$\Rightarrow$ $$y^r e^{-py} = cx^{-s}e^{qx}\,, \qquad c = const$$

$\Rightarrow$ $$\psi(x,y) \equiv x^s y^r e^{-(qx+py)} = c\,.$$

At fixed y, F has a single maximum with respect to x.

At fixed x, F has a single maximum with respect to y.

$F = 0$ on x and y axes, and $F \to 0$ as $x \to \infty$ or $y \to \infty$.



Maxima and minima of each curve correspond to maximum slope of the other curve.

Behaviour

Near the centre point, the prey and preditor populations vary sinesoidally in response to one another with frequency $(rs)^{1/2}$, the geometric mean of the natural (exponential) growth and decay rates of x and y.

Prey population increases when there are few preditors. This increase later allows population of preditors to increase (more food), increasing the predation so driving down the prey population. A lack of food for the preditors then causes a decline in their population.

Near the centre, the phase lag is $\pi/2$ and the average population is the equilibrium population.

If the populations are further from the equilibrium, the populations may rise much further above the equilibrium than they fall below it.

Obviously, the model is deficient in that it allows the population to grow without limit in the absence of a preditor. We could correct this by adding a self-competing term as we have in the earlier models to have

$$\dot{x} = x\left(r - py - \eta x\right). \qquad r, p, \eta > 0$$

This lowers the $x_2 = s/q$ critical centre point from $y = r/p$ to $y = (r - \eta s/q)/p$ and changes it to an asymptotically stable point: either a spiral point or a node, depending on the parameters. Thus the population will tend towards a more stable solution, rather than executing violent cycles.

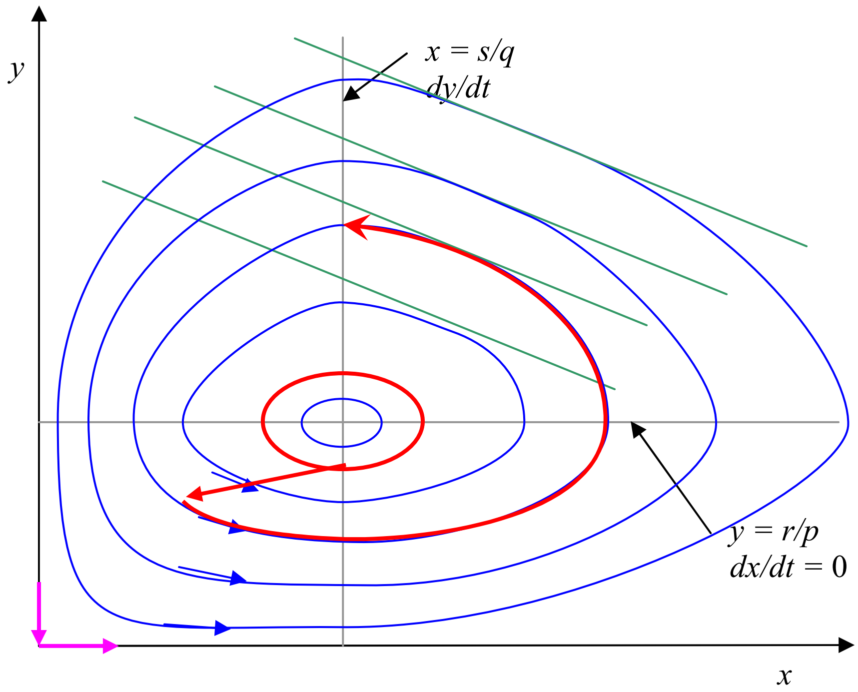
Effect of pesticide

An external attempt to control the population either preditor or prey may have the opposite effect to that intended.

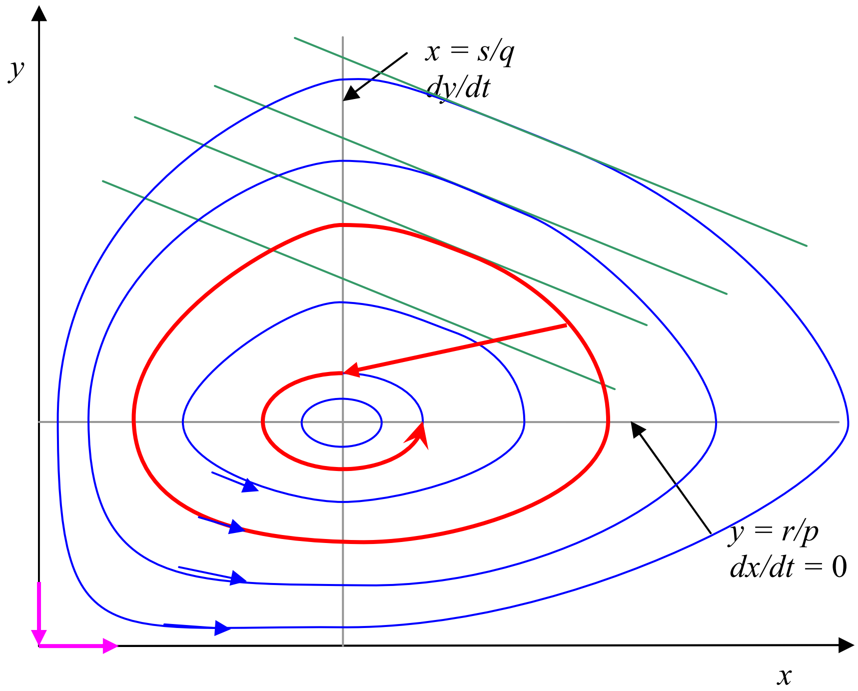
Consider the population of aphids, predated by ladybirds. If a gardener attempts to reduce the population through the use of a pesticide then the effect may be the opposite of what was intended.

Typically, pesticides do not discriminate between different species of insects, although have different potencies and will affect some fraction a of the prey and b of the predator. We shall assume the pesticide is instantaneous in its action and short-lived.

Introduction of the pesticide may take the solution either closer to or further from equilibrium, depending on when it is introduced. If it is administered when the population of aphids is increasing rapidly, then although it may kill off more aphids, the net effect on the aphids may be less significant than it is on the ladybirds. By killing of a relatively small number of ladybirds, the population of aphids may experience a more sustained period of population grown.



In contrast, the same pesticide introduced after the aphid population has peaked, but before the peak in ladybird population, may return both populations closer to equilibrium.



Continual application of pesticide, or a pesticide with a longer period of activity, will shift the equilibrium point. The net effect may be the same, with its introduction placing the populations on a trajectory further from equilibrium.

Is it better to live in a world of vegetarians, or to be the prey with no direct competators?

End of Lecture 24