The work in this dissertation covers the hydraulics of two-layer, buoyancy-driven exchange flows between two reservoirs connected by a channel of varying geometry. Over a wide range of reservoir conditions the flows are controlled in the sense that the exchange flow rate, velocity and interface profiles do not depend on the details of the flow and conditions within the two reservoirs. The control region is isolated from the reservoirs by a region of supercritical flow to each side. This prevents small amplitude long waves, which could communicate information about disturbances within the reservoirs, from propagating into the control region.

The work of previous investigators has been confined to the flow through three relatively simple geometries of rectangular cross-section. This dissertation reproduces their results using a new formulation of the hydraulics problem. The formulation is further extended to encompass a much larger variety of along-channel geometries and allow the effects of nonrectangular cross sections and system rotation (e.g., the rotation of the earth) to be explored and reported.

The theoretical results are applied to the tidal modulation of the exchange flow through the Straits of Gibraltar. Rotation and cross-channel geometry are found to have only a weak influence on the averaged exchange flow rate.

A series of laboratory experiments were performed to support the theoretical results for rotating rectangular channels. Very good agreement is achieved after a leading order correction for viscosity is made.
Two-layer Hydraulics

Maximal Exchange Flows

by
Stuart Bruce Dalziel
Sidney Sussex College

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at
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To
Sharon
Preface

The work presented in this dissertation was carried out between October 1985 and October 1988. Throughout this time I was a postgraduate student in the Department of Applied Mathematics and Theoretical Physics at The University of Cambridge. I am grateful to the Association of Commonwealth Universities for the Commonwealth Scholarship which supported me during my three years of study.

To the best of my knowledge the material presented here is original and my own work, except where explicit reference is made to other researchers. None of the work was done in collaboration with any other person. No part of this dissertation has been presented towards any degree at this or any other university.

I would like to thank my supervisor, Dr. P F Linden, for suggesting the experiments which lead to this work and his continuous encouragement and advice over the last three years. In particular I appreciate the way he ploughed through the various drafts of this dissertation without complaint. I am indebted to Mr G Lane-Serff for discussions concerning the airflow through doorways; these discussions spawned the hydraulic description of this flow presented in chapter 4.

Thanks are due to the technical staff, Messers D Cheesley, D Lipman, P O'Reilly and E Maclagan for building and altering the experimental rig to my ever-changing requirements.

In the preparation of this manuscript, I would like to thank Dr. J E Simpson for printing the streak and shadowgraph photos of chapter 9. I am deeply indebted to my fiancé, Sharon Kennedy, for nobly struggling through the piles of paper in search of the elusive typo's and spelling mistakes, and for her numerous helpful comments on my sentence structure. It is due to her constant support and encouragement that this work was completed, and that I remained reasonably sane.
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Introduction

1.1 Preamble

The subject of hydraulics has a long history. The flow of water in rivers and tidal channels is likely to have attracted man's attention long before the development of the mathematics ready to describe them. It would be surprising indeed if some prehistoric man or woman did not sit on the bank of a river and wonder why the flow, in two seemingly identical regions, differed so greatly. In natural single-layer flows it is common for there to be a gradual transition from a relatively deep, tranquil flow to one which is shallow and rapid. Similarly a shallow, rapid flow may suddenly evolve to one which is comparatively deep and tranquil, this second transition being accompanied by a region of intense turbulence.

The subject of this thesis is not the single-layer hydraulic problem outlined in the first paragraph, but its two-layer counterpart. Detailed analysis of this important problem has been largely neglected until recently (Armi, 1986). Nevertheless, as many of the concepts central to the single-layer problem also apply to two-layer exchange flows, it is worth giving a brief review of the single-layer problem. This is undertaken in section 1.2. Two-layer hydraulics is introduced in its historical context in section 1.3, and the effect of the Earth's rotation on the flow of both single- and two-layer systems is reviewed briefly in section 1.4. The final section in this chapter (section 1.5) outlines the overall structure of this thesis and previews some of the findings.

1.2 Single-layer hydraulics - a brief review

The possible existence of two flows of different character (one deep and tranquil, and the other shallow and rapid) is a fundamental feature of single-layer hydraulic problems. For any specified volumetric flow rate there are two alternate surface heights which yield the same momentum and energy transport.

page : 1
is possible due to the quadratic nonlinearity present in the momentum equation (and in its integrated form the Bernoulli equation): the rate at which momentum enters a control volume through a specific point is proportional to the square of the velocity through that point.

The fundamental assumptions we shall make are that the flow is incompressible and inviscid. Binnie (1989) showed that there is an exact analogy between the flow of water in an open channel and that of compressible gas through a nozzle; there appears to be no equivalent analogy for two-layer flows. We shall consider only incompressible fluids. Provided the Reynolds number is sufficiently large (greater than around 300, Barna, 1971; see section 2.1 for more details), viscous forces will have negligible effect on the flow, except in thin boundary layers adjacent to the rigid surfaces of the channel and, in two-layer flows, at the interface. For a large variety of such channel flows the pressure distribution within the fluid is simply hydrostatic. Formally this requires that variations in the geometry of the channel and variations in the depth of the fluid layer occur over a length scale much greater than the depth of the fluid. However it has been found (eg. Henderson, 1966) that the results obtained under these assumptions agree remarkably well with the flow in situations where they are clearly broken.

Unwin (1907; see Binnie, 1949) was the first to analyse the flow of water over a broad-crested weir under these assumptions. In the situation he examined (see figure 1.1), the height of the water surface downstream of the weir was lower than the crest of the weir. Using conservation of energy (conservation of momentum integrated along streamlines), as characterised by the Bernoulli potential

\[ G = \frac{1}{2} u^2 + \frac{p}{\rho} + g z, \]  

(1.1)

and conservation of mass (flow rate), it is possible to relate the flow over the weir with the flow upstream of the weir. In equation (1.1) \( G \) is the Bernoulli potential, \( u \) the along-channel velocity, \( p \) the pressure, \( \rho \) the density of the fluid, \( g \) the acceleration due to gravity and \( z \) the height above some datum. Typical flows of this type have an hydraulic jump downstream of the weir. The intense turbulence in such a jump produces a significant energy
Figure 1.1. Cartoon of a single-layer controlled flow over a broad-crested weir. Regions of supercritical (F > 1) and subcritical (F < 1) flow are marked. The wavy line represents an hydraulic jump.
Introduction

loss, which makes (1.1) invalid through the jump. It is therefore not possible to insist that \( G \) is the same in both upstream and downstream reservoirs. In order to close the system, Unwin hypothesised that the mass flow rate \( \rho q \) (\( q \) is the volumetric flow rate) over the crest of the weir was maximized, \( \text{i.e.} \)

\[
\frac{\partial q}{\partial h} = 0, \quad (1.2)
\]

where \( h = h_c \) is the height of the surface at the crest of the weir.

Binnie (1949), in explaining the analogy between the weir-flow and the flow of compressible gas through a nozzle, showed that the velocity over the crest of the weir, \( u_c \), is exactly equal to the velocity of long, small amplitude gravity waves in a stationary fluid layer of depth \( h \), \( \text{viz.} \)

\[
C_0 = \left( \frac{h \, g}{\lambda} \right)^{1/2}. \quad (1.3)
\]

Downstream of the weir crest the fluid accelerates, reducing in thickness in order to conserve mass. As \( C_0 \) decreases and \( u \) increases downstream, the long, small amplitude gravity waves, which would communicate information about any disturbances downstream of the weir crest, are unable to propagate upstream. Thus any disturbances on the downstream side of the weir are unable to affect the flow over the weir, provided the amplitude of the waves they produce remains small. Finite amplitude waves, such as those produced by placing a high barrier across the flow downstream of the weir crest, may be able to propagate upstream and fundamentally alter the flow.

Upstream of the crest of the weir the fluid velocity is less and the depth greater than at the crest, thus long, small amplitude gravity waves are able to propagate in both directions. Any disturbance to the flow on the upstream side of the weir will be felt by the flow everywhere. Binnie's (1949) analysis in terms of the phase speed of long, small amplitude gravity waves, and its associated meaning in terms of the propagation of information, has lead to hydraulic flows being discussed in terms of their Froude number.
The Froude number $F$ is the ratio of the velocity of a fluid particle to the velocity of long, small amplitude gravity waves. If $F < 1$, $C_0$ exceeds $u$ and so information about the cause of the waves may be communicated in either direction. The flow under these circumstances is called subcritical and corresponds to the deep, tranquil flow mentioned earlier in this section. When $F > 1$, the fluid flow will wash all the waves downstream, preventing information from being communicated to the upstream flow. The flow is now supercritical and appears shallow and rapid. If subcritical and supercritical flow exist simultaneously in different regions of the channel, then at some point the flow must be critical, i.e. the fluid particle velocity $u$ must exactly equal the wave phase velocity $C_0$ so that $F = 1$.

Flows which undergo an hydraulic transition, i.e. they pass from subcritical flow upstream to supercritical flow downstream of some specific feature, are said to be hydraulically controlled. They are controlled in the sense that the flow rate down the channel is a function solely of the upstream conditions (e.g. the height of fluid in a static reservoir) and the geometry of the control section (the section where $F = 1$). The flow downstream of the control section is unable to communicate any information concerning disturbances to it to the upstream flow.

Downstream of the sill crest the supercritical flow will eventually become subcritical again by passing through an hydraulic jump. Within the hydraulic jump energy is dissipated, and so the Bernoulli potential $G$ is not conserved, though the more fundamental concept of conservation of momentum still applies and enables the flow on one side of the jump to be computed from that on the other (e.g. section 3.2, Henderson, 1966). The position or height of the jump must initially be determined from some other criterion. Hydraulic jumps may be initiated by some source of momentum (e.g. a small but sharp change in the geometry of the channel such that it has a significant drag on the flow) or by the loss of energy from the fluid (e.g. due to viscous effects). We shall not consider the hydraulic jump in a single-layer system in any detail as it is of only peripheral interest to the work reported in this thesis; instead the reader should refer to any standard text on hydraulics (e.g. Henderson, 1966).

\[ F = \frac{u}{C_0} = \frac{u}{(h g)^{1/2}}. \]
Gill (1977) provides a unified approach to the mathematical structure underlying single-layer hydraulic-type problems. The features he noted will be reviewed more thoroughly in section 2.2 as his approach is central to that derived for two-layer problems in this thesis.

A vast quantity of civil engineering literature is available on classical hydraulics and so we shall not dwell on this problem. We shall, however, present a brief summary in section 1.4 of how rotation affects the single-layer flow. In the next section we introduce the ideas behind the hydraulics of two-layer systems.

1.3 Two-layer flows

The history of man's awareness of two-layer hydraulic flows is likely to be much shorter than the single-layer situation introduced in section 1.2. In most naturally occurring flows the position of an interface between two fluid layers is not obvious to a lay observer. Indeed the observer is unlikely to even realise that two fluid layers are present! Nevertheless some of the ideas associated with two-layer flows date back to the time of Aristotle, or earlier:

...and its [salty water] weight makes it sink below the sweet [fresh] water.

the salty element being heavy is carried down more into deep water

Aristotle, explaining the movement of saltier water within the Mediterranean Sea (p. 8-9, Deacon, 1971).

Of central concern in this thesis are two-layer flows characterised by the velocities in the two layers being in opposite directions, relative to the containing geometry. We shall term such flows exchange flows as the two reservoirs are effectively exchanging fluid along the connecting channel. In most of the discussion that follows we shall assume the flow is along some channel, of specified cross-section, connecting two large reservoirs such as is shown by figure 1.2. The lower (denser) layer will flow in one direction while the upper (lighter) layer will flow in the other.
Figure 1.2. Basic configuration of an exchange flow. The direction of the velocity within each layer is indicated by arrows. Regions of supercritical (F > 1) and subcritical (F < 1) flow are marked. The wavy lines represent hydraulic jumps.
By 1661 the possible existence of an exchange flow through the Strait of Gibraltar was realised, though at the time this was radical thinking with a large number of hypotheses being used to explain the reason for the Mediterranean not overflowing (chapter 7, Deacon, 1971). The greater density of the Mediterranean water (when compared with that in the Atlantic), due to a very high evaporation rate within the Mediterranean basin, was known by this time, as some of the hypotheses stated all the inflowing water was lost by evaporation! Presumably the existence of buoyancy-driven exchange flows through open doorways was well known (at least in colder climates) well before this through the associated cooling of a heated room, though the analogy between this and the Strait of Gibraltar does not appear to have been reported until von Waitz in 1755 (Deacon, 1985).

In 1681 Marsigli, after talking with local fishermen, observed the exchange flow through the Bosporus linking the Black Sea to the Strait of Gibraltar. He measured the densities of the water in the Black Sea and Mediterranean as well as the two layers within the Bosporus, and found that the heavier Mediterranean water flowed into the Black Sea beneath the outflowing Black Sea water (pp. 147-149, Deacon, 1971). To confirm his theory, Marsigli performed some simple experiments in a box with a barrier in the centre. One side of the box contained fresh water and the other half salt water. There were two holes in the barrier through which the water could flow (an illustration may be found in figure 5.1, Gill, 1982). Marsigli observed the flow was in opposite directions through the two holes.

The accounts of exchange flow of both von Waitz and Marsigli ignore the effects of friction. As pointed out by Armi & Farmer (1988), many subsequent authors attempted to include the effects of friction on the exchange flow through oceanic straits, failing to recognise the fundamental nature of the hydraulic control which limits the flow at high Reynolds numbers. Those authors who did consider the exchange flow an inviscid process followed the idea outlined in Yih (p. 136, 1965) that the rate at which potential energy is released by a two-layer flow, set up from rest, is equal to the rate at which the kinetic energy of the mean motion is increased. For flows in a channel of constant depth this approach may produce the correct answer, but in general is misleading as
the initial transients are dissipative (see section 3.3).

As with the single-layer hydraulic flows introduced in the previous section, the fundamental nonlinearity embodied in conservation of the Bernoulli potential allows a two-layer flow to take on more than one possible state, in a given cross-section geometry, for given prescribed values of the layer flow rates $q_1$ and $q_2$ and Bernoulli potentials $G_1$ and $G_2$. Throughout this thesis a subscript 1 will refer to the lower layer and subscript 2 to the upper layer. In particular, we shall show in section 2.4 and subsequent chapters that the flow may adopt one of three alternate states. The state which is realised in practice will be a function of the conditions elsewhere in the channel.

Over a wide range of reservoir conditions high Reynolds number (greater than around 300; p. 83, Barna, 1971), two-layer exchange flows are found to be controlled in the sense that the exchange flow rate, and details of the interface and velocity structure, is independent of the flow within the reservoirs over a large portion of the channel connecting the two reservoirs. Such flows are necessarily asymmetric (see section 2.4), the level of the interface in the dense reservoir being higher than that in the light reservoir. As we shall show in more detail in sections 2.3 and 2.4, such flows consist of a region of subcritical flow isolated from the reservoirs on each side by regions of supercritical flow. The terms subcritical and supercritical have a meaning analogous to that for single-layer flow in the previous section. If the flow is subcritical then long, small amplitude internal gravity waves may propagate in both directions along the interface. In contrast for a supercritical flow the waves may only propagate in one direction. The direction of propagation depends on the realisation of the flow, but for achievable flows this will be away from the subcritical region.

Stommel and Farmer (1953) were the first to analyse controlled two-layer flows in a channel of rectangular cross-section. They showed that the presence of an hydraulic transition at the mouth of an estuary limits the amount of mixing which can occur between the fresh and salt water. Wood (1970) studied two-layer flow through a contraction in width, considering both the case when the layers were moving in the same direction and when an exchange occurred with the layers moving in opposite
directions. When both layers were moving in the same direction, Wood found that the flow became internally critical at a virtual control (see section 2.4) upstream of the contraction.

Mehrotra (1973) analysed in more detail the exchange flow through a contraction and then (erroneously - see section 3.3) applied this analysis to the exchange flow over a sill. Arm1 (1986) pointed out the fallacy of Mehrotra's arguments and showed that the structure of two-layer hydraulic flows is much richer than supposed by earlier authors and demonstrated that the two-layer hydraulics problem could be written in quasi-linear form. From regularity conditions on the hyperbolic quasi-linear equations, Arm1 was able to demonstrate that the characteristics of the internal flow meet when the composite Froude number is unity. The composite Froude number $F$ is given by

$$F^2 = F_1^2 + F_2^2 = \frac{u_1^2}{(h_1 g')} + \frac{u_2^2}{(h_2 g')}.$$  (1.5)

with $u_i$, $h_i$ ($i = 1, 2$) the layer velocities and layer depths and $g' = 2g(\rho_1-\rho_2)/(\rho_1+\rho_2)$ is the reduced gravity; $F_1$ and $F_2$ are the internal layer Froude number (i.e. equation (1.4) with $g'$ replacing $g$).

Armi (1986) then cast the solution in the internal Froude number plane for a channel of rectangular cross-section: he characterised the flow at any section along the channel by the value of its two internal layer Froude numbers, and not directly by the geometry and interface position at that section. The solutions he found showed that, in general, the two-layer flow may adopt one of three different alternate states (two or more of these coincide when the characteristics meet with $F = 1$). If the flow is to be controlled in the sense that any subcritical region ($F < 1$) is bounded by regions of supercritical flow ($F > 1$), then at two locations along the channel the flow must be internally critical ($F = 1$). Only if the depth of the channel is constant (but the width varying), and the flow rates in the two layers are equal, will these two control sections (locations where $F = 1$) coincide and the analysis of Mehrotra (1973) can be applied. For the flow through channels of constant depth, Arm1 found that two control sections were present if the layer flow rates were not
equal. One control was positioned at the contraction (narrowest section of the channel), where it would be if the layer flow rates were equal, and the other upstream (with respect to the net flow) of the contraction.

The presence of the sill is felt directly only by the lower layer, in contrast to the flow through a contraction where variations in the channel width are felt equally by the two layers; as a result the two flows are fundamentally different. Armi (1986) examined the flow over a sill in a region of constant width joining two reservoirs of infinite depth. He showed that the flow is always critical at the crest of the sill and at the point where the channel widened into the dense reservoir. The investigation was supported by some experimental work; good agreement was achieved.

Armi & Farmer (1986) and Farmer & Armi (1986) developed further the hydraulic ideas of Armi (1986) for a rectangular cross-section, again using a Froude number plane formulation. Armi & Farmer (1986) considered in great detail the exchange flow through a channel of constant depth. They produced a detailed picture of how the controlled flow responds to a net barotropic forcing (i.e., the two layer flow rates are not equal) along the channel. In addition to looking in more detail at how the two-layer flow matches onto the reservoirs at each end, they gave an outline of how hydraulic control may be lost if certain conditions on the reservoirs are not satisfied.

Attention in Farmer & Armi (1986) was turned to extending Armi's (1986) work on the exchange flow over an infinitely high sill; they did not consider sills of finite height to resolve the question of why the flow over an infinite sill should respond so differently to the flow through a channel of constant depth. The work was further extended to give an outline of the flow through a combination of a sill and a contraction, suggesting this as a basic model for the Strait of Gibraltar.

The work in chapters 2 to 4 may be viewed as an extension to that of Armi and Farmer (Armi, 1986; Armi & Farmer, 1986; Farmer & Armi, 1986) for the classical (i.e., nonrotating) two-layer hydraulics problem. The fundamental difference is in the formulation of the problem in terms of a functional describing the flow (see section 2.2) instead of the Froude number plane approach.
of Armi and Farmer. The work also covers a much larger range of along-channel geometries (chapter 3), and considers a channel having a nonrectangular cross-section (chapter 5). The structure of this thesis, and how it relates to the work of previous authors is outlined more fully in section 1.5. First, in the next section, a brief summary is given of investigations exploring how the Earth's rotation alters the hydraulics of single- and two-layer flows.

1.4 The effects of rotation

When the length scale of a fluid flow is sufficiently large then the rotation of the Earth may alter the nature of the flow in a fundamental manner. In all our discussions we shall assume a right-handed coordinate system and northern hemisphere (counter-clockwise) rotation. One of the first people to analyse the effect of the Earth's rotation, at least in the oceanographic context, was Colin Maclaurin, in 1738 (p. 251, Deacon, 1971), who looked at the effect of rotation on the ocean tides. At a similar time Hadley was attempting to explain how the Earth's rotation could account for the trade winds (p. 189, Gill, 1982).

In terms of hydraulic flows, with one or two layers, whether or not rotation is important depends on the ratio of the Rossby radius of deformation to the length scale of the geometry enclosing the flow. As we have already stated, information about disturbances to the free surface or interface of an hydraulic flow is communicated to other regions of the flow by long, small amplitude gravity waves. For a rotating flow it is the Kelvin waves (the fastest moving modes) which are important. The Rossby radius of deformation is the distance free waves will travel before the rotation has had time to reverse the direction in which they are travelling relative to an inertial frame of reference. For a single-layer flow the wave speed is given by equation (1.3), and the time taken for the frame of reference to reverse its orientation, relative to an inertial frame, is $1/f$ ($f = 2 \pi \sin \theta$, where $\theta$ is the angle between the local vertical and the axis of rotation). Thus the barotropic or external Rossby radius of deformation is
As we shall show in more detail in sections 2.3 and 6.4, the phase speed of baroclinic or internal waves on the interface between two layers is a function of the ratio of the depths of the two layers, and of the total fluid depth. Any strict definition of the baroclinic Rossby radius, in the terms outlined in the previous paragraph, should include both these dependencies.

It is more convenient to consider the Rossby radius as the length scale for the effects of gravity and rotation to be of equal importance. In line with this, as the internal wave speed is of order $(D g')^{1/2}$, where $D$ is the total depth of the fluid, we shall define the internal Rossby radius as

$$R_{\text{int}} = (D g')^{1/2} / f. \tag{1.7}$$

In subsequent chapters we shall drop the subscript from $R_{\text{int}}$ as we shall be considering only internal flows. However, for the present section, we shall maintain the distinction between the external (barotropic) and internal (baroclinic) Rossby radii.

In channel flows, the important criteria when determining whether or not rotation plays a role is the ratio of the length scale(s) of the channel to the Rossby radius. In particular, if the along-channel geometry of the channel varies over a length scale much larger than the channel width (or depth), then the Burger number, the ratio of the channel width to the Rossby radius of deformation, characterises the importance of rotation. If the Burger number is large compared with unity, rotation dominates the flow. If it is small, then rotation has little influence. In chapters 6 to 10 we shall not discuss the flows in terms of the Burger number directly. Rather, in section 6.1, we introduce a system of nondimensional variables based on the internal Rossby radius of deformation, at the shallowest section, as the horizontal length scale. The nondimensional channel width at that section is therefore equal to the Burger number for the same section.

As fluid moves along the channel, the Coriolis force will try to deflect its movement to the right. However, the presence of the channel sidewall(s) will prevent this from happening. The level of
the fluid will build up against the right-hand wall (looking downstream with respect to the layer being considered), causing a cross-channel slope on the free surface or interface. The accumulation of fluid will grow until the pressure gradient associated with its slope balances the Coriolis force on the fluid. The fluid moving down the channel is said to be in geostrophic balance. Throughout the chapters dealing with the hydraulics of rotating flows we shall assume the flow is everywhere in geostrophic balance, though in section 7.3 we shall look at the adjustment process through which this may be achieved.

Whitehead, Leetmaa & Knox (1974) were the first to investigate the effect of rotation on hydraulically controlled flows. They looked at both single- and two-layer systems. Their formulation was very simple and did not include variations in the along-channel geometry, although they required the fluid to come from a stagnant reservoir of infinite depth (i.e. zero potential vorticity - see section 6.1). Their results were in good agreement with the experimental results they reported, at least provided the Rossby radius was greater than the width of the channel (see Gill (1977) for discussion of single-layer flows in a wide channel and section 9.3 of this thesis for discussion on two-layer flow through channels of a width greater than one Rossby radius). For the single-layer flow they utilised a maximal flow hypothesis, similar to that given by equation (1.2), to close the system. For nonrotating single-layer flows Binnie (1949) had shown that this maximal flow hypothesis was equivalent to requiring the flow to be critical (see section 1.2), and subsequent authors (e.g. Gill, 1977) have verified the approach for a single-layer rotating system.

In addition to assuming a maximal exchange flow hypothesis (the exchange flow is maximized), Whitehead et al. required a further closure assumption. They used the idea of Yih (p. 136, 1965) that the rate at which potential energy is released is numerically equal to the rate of increase of the kinetic energy of the mean flow. We shall show, in section 3.3, that this hypothesis is incorrect for channels in which the depth varies. In section 7.4 we show that for channels of constant depth, equating the energies may be applied to rotating systems only if the fluid layers have zero potential vorticity (as was the case for the work
Introduction

Section 1.4

of Whitehead et al.). Moreover, we contend that the formulation of the energy balance used by Whitehead et al. was incorrect, though the conclusions they reached remain the same.

Gill (1977) summarised the behaviour of single-layer hydraulic-like flows, giving great insight into the underlying mathematical structure of such problems. Using the formalism he developed (see section 2.2 for more details), he was able to explore the effects of rotation on controlled, single-layer flows of uniform potential vorticity in channels of rectangular cross-section (the channel geometry was allowed to vary along its length). The startling finding of Gill's analysis was that the ratio of the flux in the two upstream boundary currents (which form to supply fluid from the reservoirs when the system is rotating) could be specified arbitrarily. This upstream condition is needed in addition to the level of fluid in the upstream reservoir. For a nonrotating flow, only the fluid level is important. Thus Gill found that the steady-state controlled solution was not unique, but was instead a function of how the flow was set up (resolution of this feature of the single-layer flow is beyond the scope of this thesis).

Experimental work carried out by Shen (1981) supported Gill's (1977) theory. For subcritical flow over a submerged weir, Shen found that the flow rate down the channel was controlled by the height difference between the two reservoirs placing a limit on the cross-channel slope of the free surface. This finding is a precursor to the work of subsequent authors on geostrophic control (see section 6.5). In addition Shen extended the theory to zero potential vorticity flows in a channel of irregular cross-section, supporting the work with a range of experiments.

Borenäs & Lundberg (1986) extended Gill's (1977) uniform potential vorticity approach to include channels of parabolic cross-section. While we will not consider two-layer flows in a rotating channel of parabolic cross-section, the single-layer problem does introduce a feature which may be of concern to the two-layer problem in a rotating channel of rectangular cross-section. In particular, Borenäs & Lundberg found that under some conditions the velocity profile would change sign for a flow critical (\( F = 1 \)) at the section normally associated with hydraulic control. Any velocity reversal invalidates the implicit assumption
that all the fluid originates from the upstream reservoir, introducing the possibility of downstream effects due to advection and wave propagation within the region of velocity reversal. In Gill's (1977) rectangular channel this could be catered for by assuming the flow was separated from the left-hand bank; such a process can not occur in the parabolic cross-section. Borenäs & Lundberg suggested that a realised flow in such geometry may have a nonuniform upstream potential vorticity.

Pratt & Armi (1987) have analysed the hydraulic control of flows with a nonuniform potential vorticity in a channel of rectangular cross-section. They found that such flows retain many of the classical features of hydraulic control, viz. the existence of multiple alternate solutions, accessible only through branch points. These branch points occur only at specific topographic features (e.g. a contraction or a sill). At the branch points the flow is critical with respect to long, small amplitude gravity waves ($F = 1$) and the specific energy of the flow at the sidewall is a (local) minimum. They note the difficulty in prescribing the potential vorticity, especially when considering the closed streamlines which may occur in some flows.

Very little work, subsequent to Whitehead et al. (1974), has been done on how rotation affects two-layer hydraulic flows. Hogg (1987) investigated the effect of stratification on hydraulic control. He looked at two- and three-layer fluid systems where the velocities within the layers are in the same direction (i.e. not an exchange flow). The two-layer flows he considered were only partially controlled in that upstream flow is subcritical with respect to both internal and external modes; critical conditions were achieved only at the contraction. This work has little direct relevance to the two-layer exchange flows considered in this thesis and so will not be discussed in depth.

Wang (1987) presented a three-dimensional simulation of the surface outflow of a strait containing a two-layer exchange flow. The purpose of this work was to investigate the evolution of the gyre observed in the Alboran Sea on the Mediterranean side of the Strait of Gibraltar. This gyre is produced by the North African coast dipping southwards on a length scale much shorter than the Rossby radius of deformation (Whitehead & Miller, 1979). The flow exiting the Strait separates, only changing its direction over a
length scale of the same order as the Rossby radius. Wang's work is only of peripheral interest to the present study as it does not include details of the processes within the Strait. Furthermore, the treatment we shall present in this thesis requires changes in geometry to occur over a scale large relative to the depth of the fluid and the Rossby radius, if this is of the same order as the width of the channel.

In the next section we present a more detailed preview of the work reported in this thesis and how it relates to that of previous researchers.

1.5 Overview

The purpose of this section is to give an overview of how the various sections and chapters of this thesis are related, both to each other and to the work of earlier authors.

Chapter 2 introduces the formulation of the two-layer hydraulics problem, used throughout this thesis. The assumptions leading to the shallow water approximation, the nondimensionalisation used for the nonrotating channels, and the conserved quantities are given in section 2.1. Section 2.2 introduces the mathematical formalism explained by Gill (1977) for the single-layer problem, and derives the equations necessary to utilise it for two-layer flows in a channel of arbitrary (slowly varying) geometry. The relationship of this new formulation of the two-layer problem to the approach of previous investigators is shown in section 2.3 and the conditions necessary for critical flow to be achieved are outlined in section 2.4.

Chapter 3 applies the new formulation to the flow through channels of rectangular cross-section. Section 3.1 relates the hydraulic functional derived in section 2.2 to the rectangular cross-section geometry used in chapter 3. The flow through a channel of constant depth, when the flow in the two layers is equal, is analysed in section 3.2. This example illustrates how the hydraulic functional may be employed; the results of earlier workers (eg. Armi, 1986) are reproduced. Section 3.3 considers the flow over a sill of arbitrary height in the absence of a net flow along the channel. This section extends the analyses of Armi
(1986) and Farmer & Armi (1986) who considered only sills of infinite height. Asymptotic expansions are derived to describe how the flow varies with the height of the sill in addition to numerical evaluation of the exact solutions. The dissipation necessary when the flow is set-up from rest is examined.

On the basis of sections 3.2 and 3.3, an algorithm for solving two-layer controlled problems is presented in section 3.4. This algorithm is essentially that used throughout the remainder of the thesis. A detailed description of the conditions under which hydraulic control may be lost is given in section 3.5. It is shown that the controlled exchange flow is the maximal achievable flow. The ideas associated with loss of control are not entirely new, though the manner in which they are applied is an improvement on that employed by Armi & Farmer (1986) and Farmer & Armi (1986).

The effects of net barotropic forcing are introduced in section 3.6, reproducing the results of Armi (1986) and Armi & Farmer (1986). Bifurcations in the behaviour of the steady flow over a sill, due to the net barotropic forcing (i.e. whether the flow behaves like that over a sill without a net forcing or that through a channel of constant depth with net forcing), are discussed in section 3.7. In the limit of an infinite sill the results agree with those of Armi (1986) and Farmer & Armi (1986). Earlier authors have not found bifurcations in the behaviour of the flow as these are not apparent unless finite sill heights are considered. Section 3.8 corrects and extends the flow over a combination of a sill and a contraction originally introduced by Farmer & Armi (1986). The new work covers arbitrary sill heights and contraction widths, describing in detail the bifurcation structure discovered and the differences associated with the order in which the two geometric occur. The possibility of a simultaneous sill and contraction, where the depth and width of the channel vary simultaneously, is introduced in section 3.9. This new work helps explain some of the bifurcations observed in the sections earlier in the chapter.

A simple application of the flow described in chapter 3 to the buoyancy-driven air flow through doorways is attempted in chapter 4. The asymmetry of the position of the interface through the doorway is explained in terms of flow under an inverted sill. This work is new, though some of the ideas are due to discussions...
with Mr G Lane-Serff.

The work in chapter 5 is entirely new. Two-layer hydraulic theory is extended to analyse the flow through a channel of parabolic cross-section following the methodology outlined in section 2.2. The necessary expressions and conditions are introduced in section 5.1, while section 5.2 explores in detail the flow along a channel of constant maximum depth. This section is equivalent to a combination of sections 3.2 and 3.6. Section 5.3 is equivalent to sections 3.3 and 3.7 in that it explores the exchange flow over a sill of arbitrary height. A somewhat richer bifurcation structure is found for these parabolic channels than the rectangular channels of chapter 3. A brief summary of the affects of the nonrectangular cross-section is given in section 5.4. Application of this work is mainly oceanographic in nature and is delayed until chapter 10.

Chapter 6 serves as a detailed introduction to the hydraulics of two-layer rotating flows. The necessary equations, conservation relations and nondimensionalisation are introduced in section 6.1. Section 6.2 introduces the concept of an attached flow where there are two layers present over the entire width of the channel at a given section. The cross-channel structure of the flow is derived for flows with constant (positive) potential vorticity within each layer. Under some conditions the flow may be separated: the interface intersects the floor and/or top of the channel and two layers are present over only part of the channel width. The description necessary under these circumstances is described in section 6.3.

The rotating flow is cast in the functional form of the earlier chapters in section 6.4, a proof being given that solution branch points correspond to critical flow; thus hydraulic control is possible for rotating systems. Loss of the control mechanism due to reservoir conditions is discussed in section 6.5, along with the notion of geostrophic control for a submaximal flow. Geostrophic control is found to be a feature relevant only to single-layer systems where the flux in the two upstream boundary currents may be specified independently. It is not relevant to two-layer flows. Section 6.6 explores the limits of validity of the approach taken to hydraulic control and shows that the simple application of these ideas may cause the velocity in one of the
layers to change direction at the control section(s). A velocity reversal violates the fundamental assumptions of the flow. A possible resolution to this problem is offered.

Chapter 7 looks in detail at the exchange flow of fluid of constant and equal (i.e. in the two layers) potential vorticity along a rectangular channel of constant depth. As for the nonrotating channel of section 3.2, this flow is the most amenable to analysis. Section 7.1 gives the necessary equations and solves the hydraulic problem for an attached flow. The results are new, but agree with Whitehead et al. (1974) in the zero potential vorticity limit. A number of interesting features are found as the width of the channel (in terms of Rossby radii) is increased. These are explored more fully in section 7.2, along with the ideas of a zone of stagnant fluid which were introduced in section 6.6. Provided the channel is less than approximately one Rossby radius wide, the exchange flow rate and details of the flow are found to be nearly independent of the value of the potential vorticity.

Section 7.3 analyses the linear Rossby adjustment problem for the two-layer flow. This problem is found to be identical to the single-layer problem investigated by Gill (1976). Gill's solution, in terms of Poincare' waves which radiate a fraction of the original potential energy away to infinity, motivates section 7.4 which looks at the energetics of both the two-layer linear adjustment and two-layer hydraulic problems. Both are found to produce Poincare' waves under appropriate conditions and show that, while Whitehead et al. (1974) succeeded in producing the correct answer for the zero potential vorticity limit, they omitted a term in their energy balance.

Motivated by the mathematical simplicity of the zero potential vorticity limit, and the weak dependence on the value of the potential vorticity of the flows of chapter 7, chapter 8 restricts attention to zero potential vorticity flows. These new results cover a wide range of geometries and net forcings. Section 8.1 details the expressions and functional required for analysing attached flows, while section 8.2 does likewise for separated flows, finding that it is not possible to write an explicit expression for the functional when the flow is separated. Section 8.3 mirrors sections 3.2, 3.6 and 5.2 in that it explores the controlled solutions of flows along channels of constant depth.
The direct effects of rotation on the exchange flow rate are found to be moderated by net forcing, though details of the interface profile remain greatly affected.

The flow over a simple sill is investigated in section 8.4 in the absence of net forcing, asymptotic expressions being derived to explain how the flow varies with the topography. It is found that rotation decreases the asymmetry of the flow introduced by forcing the lower layer with the sill. Section 8.5 reintroduces net barotropic forcing and finds a rich bifurcation structure following some of the features found in the parabolic channels of chapter 5. A brief summary of the effects of rotation is given in section 8.6.

Chapter 9 describes the experimental work performed to support the theory for rotating channels. The first section introduces the experimental set-up and methodology in the four series of experiments. The first series of experiments - the flow through a channel of constant depth in the absence of a net barotropic flow - is analysed in sections 9.2 and 9.3. The first of these considers the flow when the Rossby radius is greater than the width of the channel, introducing a simple correction for the viscous Stewartson layers to account for the difference between the theory and experiments. Section 9.3 develops further the ideas of a zone of stagnant fluid introduced in section 6.6 in light of the observed behaviour at high rotation rates. This behaviour is analysed in terms of a simple model of the set-up process.

Sections 9.4 and 9.5 repeat the experiments of the first series in channels of different geometry: symmetric depth changes in section 9.4 and sills in section 9.5. The effects of a net barotropic flow are explored for a channel of constant depth in section 9.6. For all four series of experiments a good agreement between theory and experiment is obtained. This is summarised in section 9.7.

In chapter 10 the flow through the Strait of Gibraltar is analysed using the theory derived in this thesis. The model geometry of two sills and a contraction is introduced in section 10.1. Section 10.2 analyses the flow assuming rotation is not important using the hydraulic models of chapters 3 and 5, for rectangular and parabolic cross-sections respectively. Details of the quasi-steady response to tidal modulation are given in terms
of both the along-channel interface profile and the averaged exchange flow rate. It is found that the two models are in very close agreement. Section 10.3 repeats the analysis including the effects of rotation. In the Strait of Gibraltar the internal Rossby radius is approximately four times larger than the width of the channel. Under these circumstances rotation is found to make very little difference to the exchange flow rate, though it alters the along-channel profiles in a fundamental manner. Analyses are also made for an hypothetical series of straits, with the same geometry as the Strait of Gibraltar, but at different rotation rates. The purpose of this is to investigate more thoroughly how rotation affects such flows.

Section 10.4 reviews the observational results reported by Armi & Farmer (1988), and compares them with the current hydraulics model. The unsteady behaviour observed in the Strait due to an internal reservoir prevents the quasi-steady model from predicting the flow accurately. A simple extension to the quasi-steady model to account for the unsteady behaviour is suggested.

The work presented in this thesis is summarised in chapter 11. Conclusions are presented detailing the weak dependence of hydraulic flows on cross-channel geometry and rotation. A number of extensions and suggestions for further work are also included.
2 Approach

2.1 Problem and equations

Consider two large reservoirs connected by a channel of varying geometry. One reservoir contains mainly fluid of density \( \rho_1 \), and the second mainly of density \( \rho_2 \). We shall assume the fluids are Boussinesq (\( 0 < \rho_1 - \rho_2 \ll \frac{1}{2}(\rho_1 + \rho_2) \)). Suppose that initially \( t \leq 0 \) the two reservoirs are separated by a barrier at \( x = 0 \) as shown in figure 2.1a. If the barrier is removed at \( t = 0 \), the sharp density gradient at \( x = 0 \) will drive a flow along the channel. Provided any net barotropic forcing is not too strong (how strong is discussed in later sections), the denser fluid (\( \rho_1 \)) will flow underneath the lighter fluid towards the right in figure 2.1b. Similarly, the lighter fluid will flow towards the left. Assuming the densities of the fluids do not change and there are no geometric obstructions (i.e. regions of the channel with the bottom higher than the level of the fluid in the dense reservoir or channel top lower than the bottom of the light reservoir), the gravity current fronts will propagate into the reservoirs at each end (figure 2.1c).

After the fronts reach the reservoirs, the flow tends towards a steady state - at least so long as there are no significant instabilities and the reservoir conditions do not change too greatly. The transients the flow must pass through to reach a steady state may be dissipative (see section 3.3). The character of this steady state will depend on the relative importance of inertial, viscous and gravitational forces in addition to the channel geometry.

For a large number of such buoyancy driven flows the Reynolds number, based on the mean hydraulic radius (cross-sectional area divided by wetted perimeter - p. 91, Henderson, 1966), is sufficiently large (greater than around 300 - p. 83, Barna, 1971) for the viscous forces to be unimportant except in thin boundary layers near the channel surfaces and between the two fluid layers. In this thesis we shall consider primarily the inviscid internal flow. Compressibility effects are generally negligible in such flows, reducing the Navier-Stokes equations to the simpler
Figure 2.1. The set-up and final state of an exchange flow along a channel of varying geometry. Top: plan view of channel; (a) initial conditions \((t = 0)\); (b) propagation of fronts after release; (c) final steady state.
Euler equations:

\[ \frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i = -\nabla \left( \frac{p_i}{\rho_i} + g \mathbf{z} \right), \]

\[ \nabla \cdot \mathbf{u}_i = 0. \]  \hspace{1cm} (2.1)

The subscript \( i \) takes the values 1 and 2 representing the lower (denser) and upper (lighter) layers respectively. The velocity and pressure fields are given by \( \mathbf{u}_i \) and \( p_i \) with the local gravity \( g \) in the negative \( z \) direction. We shall use a right-handed coordinate system with the \( x \) axis along the channel such that the lower layer has a positive velocity and \( z \) is vertically upward.

We shall consider flows set-up from rest. Kelvin's circulation theorem requires that such flows remain irrotational. The only vorticity present is confined to thin vortex sheets at the interface and boundaries.

In a large number of real flows the geometry of the connecting channel varies over length scales much greater than the total depth of the channel, and hence the depth of any fluid layer. Provided that within the region of interest the interface height also varies only over length scales large compared with the channel depth (a feature which is observed in practice if there are no hydraulic jumps or bores), the shallow water approximation (the pressure field within a given fluid layer is hydrostatic) may be applied and the velocity is independent of \( z \) within a given fluid layer.

For convenience we nondimensionalise the equations with respect to the channel geometry at the shallowest section. Suppose that at this section the maximum depth is \( D_m \) and width \( b_m \). We define

\[ (x, y)^* = (x, y) / b_m, \]

\[ z^* = z / D_m, \]

\[ t^* = t \left( D_m g' \right)^{1/2} / b_m, \]

\[ (u_i, v_i, w_i)^* = (u_i, v_i, w_i) / \left( D_m g' \right)^{1/2}, \]

\[ p_i^* = p_i / \left( \rho_i D_m g' \right). \]
\[ \rho_i^* = \frac{2}{\rho_i} \left( \frac{\rho_1 + \rho_2}{\rho_1 - \rho_2} \right), \]  
\hfill (2.2)

where the variables with a superscript star (\( \ast \)) are dimensionless and \( g' \) is the reduced gravity \( (g' = 2 \left( \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)) \).

Dropping the superscript stars (\( \ast \)) and integrating the continuity equation over the layer depth \( h_i \) gives the shallow water equations as

\[
\begin{align*}
\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + v_i \frac{\partial u_i}{\partial y} &= - \frac{\partial}{\partial x} \left( \frac{\rho_i}{\rho_1 - \rho_2} \right) z, \\
\frac{\partial v_i}{\partial t} + u_i \frac{\partial v_i}{\partial x} + v_i \frac{\partial v_i}{\partial y} &= - \frac{\partial}{\partial y} \left( \frac{\rho_i}{\rho_1 - \rho_2} \right) z, \\
\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \frac{1}{h_i} \left[ \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x} + v_i \frac{\partial}{\partial y} \right] h_i &= 0. 
\end{align*}
\]  
\hfill (2.3)

Note that while \( w_i \) is small it does not follow that the \( z \) contribution to the continuity equation is negligible.

The shallow water equations of (2.3) may be integrated to yield Bernoulli's equation in the form

\[ G_i = \frac{\partial \phi_i}{\partial t} + v_2 \left( u_i^2 + v_i^2 \right) + \frac{\rho_i}{\rho_i} \left( \frac{\rho_1 + \rho_2}{\rho_1 - \rho_2} \right) + z. 
\]  
\hfill (2.4)

where \( G_i \), the Bernoulli potential, is conserved by a material particle and \( \phi_i \) is a velocity potential (such that \( u_i = \nabla \phi_i \)). Throughout the bulk of this thesis, we shall consider only the final steady state of flows (typified by figure 2.1c); the linear adjustment problem for rotating flows is covered in section 7.3 and may be thought of as one method through which the steady state is achieved. Thus we set \( \delta \delta t = 0 \) in equation (2.4).

The final assumption necessary in this analysis is that the flow is relatively straight in the sense that variations in the channel geometry along the length of the channel occur over length scales much greater than the width of the channel. As no mass can pass through the channel walls, the flow is essentially one dimensional and the velocity within a fluid layer constant across the channel width. The velocity field is given by \( u_i = (u_i(x), 0, 0) \) in this leading order approximation.

The pressure may be eliminated between \( G_1 \) and \( G_2 \) as (in the
absence of surface tension) it must be continuous at the interface given by \( z = H_0 + h \) (where \( H_0 = H_0(x) \) is the elevation of the channel bottom above datum and \( h = h(x) \) is some measure of the height of the interface above the channel floor - at this stage we do not require the channel floor to be horizontal). Thus

\[
\Delta G = G_1 - G_2 = \frac{1}{2} (u_1^2 - u_2^2) + H_0 + h
\]  

(2.5)

is constant everywhere two fluid layers are present (provided the basic assumptions hold).

Suppose the lower layer occupies an area \( S_1 = S_1(x) \) of the cross-section, and the upper layer \( S_2 = S_2(x) \). From continuity for this steady flow the layer flow rates,

\[
q_1 = S_1 u_1,
\]

(2.6)

are conserved. Provided the channel has rigid boundaries, or at least a rigid-lid approximation is valid (external Froude number much smaller than unity), the total cross-sectional area of the channel at a given \( x \),

\[
S = S_1 + S_2.
\]

(2.7)

is constant. We define the exchange flow rate, \( \bar{q} \), as

\[
\bar{q} = q_1 - q_2,
\]

(2.8)

and the net barotropic flow rate, \( Q \), as

\[
Q = \bar{q} + q_2.
\]

(2.9)

For the purposes of this study we shall consider \( \bar{q} \) as a dependent variable and \( Q \) as an independent (prescribed) variable. The reservoir conditions must be such that the net barotropic flow \( Q \) is provided by, for example, a difference in the free surface heights.

If the basic assumptions were to hold everywhere in the channel and reservoirs, we could use the flow in one reservoir to determine \( Q \), \( \bar{q} \) and \( \Delta G \), then trace this back along the channel to give the solution everywhere else. Generally such flows are not controlled as they are determined directly by the reservoir
Approach Section 2.1

Conditions. Flows which are subcritical (the terms subcritical and supercritical were defined in Chapter 1 with respect to information propagation) everywhere may be analyzed in this manner; if the flow becomes supercritical close to the "unknown" reservoir, the subcritical analysis may still be applied to most of the channel. However, if the flow is supercritical near the "known" reservoir, no mechanism exists by which information about the conditions in the reservoir, or changes to them, can be transmitted to the rest of the channel. Utilizing reservoir conditions as boundary conditions is no longer meaningful.

Even departures from the basic assumptions of the flow (such as introduced by hydraulic jumps) will not alter the flow through the region of channel isolated by a supercritical region (provided changes to the density of the layer flowing out of the reservoir are negligible). In general hydraulic jumps will occur in or near the reservoirs, allowing the controlled solution (which need not be unique, though frequently will be) to be matched onto a wide variety of reservoir conditions. Further details will be given in Sections 2.2 and 3.5.

Consider a flow set-up as in Figure 2.1. Suppose the channel is symmetric about \( x = 0 \). The interface height must tend towards that in the reservoirs as \( |x| \) becomes large so the flow is asymmetric (for \( Q = 0 \) in a rectangular channel of constant depth the interface will be antisymmetric) about \( x = 0 \). Thus we may deduce that the geometry of a given section must, under appropriate circumstances, be able to support at least two alternate solutions, conserving both mass and Bernoulli potential. Moreover, there must be at least one position along the channel where two (or more) of the solutions coincide. At such positions the flow may swap from one solution branch to another.

In the next section we review the formalism developed by Gill (1977) for describing the mathematical structure of single-layer hydraulic-type flows. We continue by deriving a functional embodying all the features of two-layer flows.
Gill (1977) in reviewing single-layer hydraulics noted the similarity of a wide class of fluid flow problems where the nonlinear inertial terms dominate the viscous effects. As such he introduced a general framework within which the structure of hydraulic-type problems could be cast. There are three essential features common to hydraulic-like flows:

i) the flow must be able to be specified by a single dependent variable, \( X \), whose x dependence is entirely implicit in terms of a set of geometric parameters, \( a_0, a_1, a_2, \ldots \), defining a smooth surface viz.

\[
J(a_0, a_1, a_2, \ldots; X) = \text{constant}; \quad (2.10)
\]

ii) the functional \( J \) is multiple valued for some range of \( a_0, a_1, a_2, \ldots \) in that there is more than one value of \( X \) satisfying \( (2.10) \);

iii) there is some sort of constriction in the sense that

\[
K = \left( \frac{\partial J}{\partial a_0} \right) (da_0/dx) + \left( \frac{\partial J}{\partial a_1} \right) (da_1/dx) + \ldots = 0.
\] (2.11)

As Gill showed, in general \( J \) is a surface in \( X, a_0, a_1, \ldots \) space; control sections represent the transition from one sheet of the surface to another. Different sheets meet along lines defined by

\[
\left( \frac{\partial J}{\partial X} \right) = 0.
\] (2.12)

Differentiation of \( (2.10) \) with respect to \( x \) shows \( (\partial J/\partial X) (dX/dx) = -K \) along such lines. Unless \( K = 0 \), \( dX/dx \) must be infinite and the solution breaks down; hence \( K \) must be zero along lines where sheets meet. Further differentiation with respect to \( x \) demonstrates that \( K = 0 \) must be a contraction rather than an expansion along the \( \partial J/\partial X = 0 \) lines. We shall reserve the term constriction to mean sections at which \( K = 0 \); the term contraction will be used to describe the geometrical narrowing of a channel and still to describe variations in the depth.

The other deduction Gill makes is that long waves must have zero phase speed, i.e., they must be stationary (and thus the flow critical), within the control section. This condition requires
that, in addition to (2.10),

\[ J(a_0, a_1, a_2, \ldots; \delta X) = \text{constant}, \]  

confirming (2.12). In the next section we shall look in more detail at how the functional is related to the phase velocities of long, small amplitude waves.

Clearly it is necessary to describe the present flow in functional form to utilise this framework. Provided the functional properly represents the flow (ie. solutions to \( J(\cdot) = \text{constant} \) satisfy the conservation of mass and Bernoulli potential requirements), the choice of the functional definition is arbitrary. A careful choice will, however, facilitate the understanding of the problem.

It is tempting to identify \( J \) with \( \bar{Q} \) and \( X \) with \( h \) so that (2.12) may be interpreted (loosely) as the maximal exchange condition traditionally used in two-layer hydraulics (eg. Whitehead, Leetmaa & Knox, 1974). However such a choice is not ideal as conservation of the Bernoulli potential introduces roots of a quadratic which may cause some ambiguity in the definition of \( J \). To remove this ambiguity we shall derive an alternative definition of the hydraulic functional.

Utilising the definitions of \( q_1, q_2, \bar{Q} \) and \( Q \), it is possible to write the difference in the Bernoulli potentials as

\[
\Delta G = \frac{2}{S} \left( \frac{S_1^2 + S_2^2}{S_1 S_2} \right) \bar{Q} \bar{q} - \left( S_1^2 - S_2^2 \right) \left( Q^2 + \bar{Q}^2 \right) + H_0 + H, \]

where the sectional areas, \( S_1 \) and \( S_2 \) are functions of \( h \) as well as the cross-sectional geometry. Thus we are able to write a functional relationship between \( \Delta G \) and \( h \) of the form

\[ \Delta G = f(\cdot, \bar{Q}, \bar{q}; h), \]

where \( \cdot \) represents the geometrical parameters necessary to describe the sectional areas \( S_1 \) and \( S_2 \) as functions of the position \( x \) and interface height \( h \).

We shall define \( X \) as the interface height \( h \) and the hydraulic functional \( J \) as the imbalance in the Bernoulli potential between some arbitrary interface configuration (described by \( h \)) and the
Bernoulli potential and the Bernoulli potential $\zeta = \Delta G$ of some specific realisation of the flow, viz.

\[ J(\xi, \zeta, \bar{q}, Q; h) = \zeta - f(D, H, b, Q, \bar{q}; h). \]  \hspace{1cm} (2.16)

Solutions to the flow conserving both mass and the Bernoulli potential difference are given by the roots of $J(\xi, \zeta, \bar{q}, Q; h) = 0$.

Note that $\zeta$ and $\bar{q}$ parameters whose values are unknown initially; they must be determined by considering the overall hydraulics problem. If the height of the interface $h = h_c$ is known at some control section $x = x_c$, where the flow branches from one solution sheet to another, the pair $\bar{q}$ and $\zeta$ have a known relationship through the requirements for $\partial J/\partial h = 0$ and $J = 0$ at that section. It is then possible for equation (2.16) to be written in the form required by Gill. The remaining task is to find the values of $\bar{q}$ and $\zeta$ so that $J(\xi; h) = 0$ is the physically meaningful solution.

By noting that equation (2.7) requires $\partial S_1/\partial h = -\partial S_2/\partial h$, equation (2.16) may be differentiated to give

\[ \frac{\partial J}{\partial h} = \mu \left[ \frac{2(S_1^2 - S_2^2)}{S_1^3 S_2^3} \frac{\partial \bar{q}}{\partial \zeta} \right] + \frac{(S_1^5 - 2 S_1^2 S_2^5 - 2 S_1 S_2^2 + S_2^3)(D^2 + q^2)}{S_1^2 S_2} \frac{\partial S_1}{\partial h}. \]  \hspace{1cm} (2.17)

Hydraulic flows are normally discussed in terms of information propagation as characterised by the composite Froude number for two-layer flows. It is therefore important to relate the current work (through equation (2.17)), which does not explicitly include the method of adjustment, with the classical treatment of the problem. We do this in the next section.

2.3 Froude number and criticality

In section 1.2 we discussed how information was communicated in single-layer flows by the propagation of long, small amplitude gravity waves. If such waves are able to propagate in both directions along the channel, the flow is subcritical. If they are able to propagate in only one direction, the flow is supercritical.
For a two-layer flow we must again consider the propagation of information by long, small amplitude internal gravity waves. We shall utilise the term subcritical to indicate flows in which such waves are able to propagate in both directions along the channel and supercritical when the waves may only travel in one direction. Traditionally (eg. section 3.2. Turner, 1973) the composite Froude number $F$ for a Boussinesq fluid is stated in terms of the layer Froude numbers $F_i$ as

$$F^2 = F_1^2 + F_2^2,$$

(2.18)

where, in dimensional terms,

$$F_1^2 = \frac{u_1^2}{(h \ g')},$$

$$F_2^2 = \frac{u_2^2}{[(D - h) \ g']}.$$  

(2.19)

Unfortunately this form hides the fundamental importance of the composite Froude number. Analysis of the small amplitude waves shows the phase velocities are

$$C_1, C_2 = \frac{(D-h) \ u_1 + h \ u_2}{D} \pm \left\{ \frac{(D-h) \ h}{D} \left[ 1 - \frac{(u_1 - u_2)^2}{D} \right] \right\}^{1/2},$$

(2.20)

in our dimensionless system. These are related to the composite Froude number by

$$F^2 = 1 + \frac{D}{(D-h) \ h} C_1 C_2.$$  

(2.21)

The expression for the composite Froude number given by (2.21) shows clearly the relationship between the wave phase velocities and sub/supercriticality. For supercritical flow $C_1$ and $C_2$ are of the same sign and $F > 1$. Subcritical flow gives $C_1 C_2 < 0$ and so $F < 1$. One or both of the phase velocities must vanish for critical flow ($F = 1$).

A number of authors have noted that for a single-layer flow critical conditions correspond to a minimum in the specific energy independently of the cross-section (eg. Henderson, 1966). By analogy with single-layer flows, the layer Froude numbers are related to the layer specific energies by
where the specific energies for the two layers are

\[ E_1 = \frac{1}{2} u_1^2 + h, \]
\[ E_2 = \frac{1}{2} u_2^2 + D_0 - h. \]  (2.23)

For our present two-layer functional we note that equations (2.16), (2.18), (2.22) and (2.23) imply

\[ F^2 = 2 - \frac{\partial E_1}{\partial h} - \frac{\partial E_2}{\partial (D_0 - h)} \]
\[ = 2 - \left( \frac{\partial}{\partial h} \right) (G_1 - G_2) \]
\[ = 1 - \left( \frac{\partial}{\partial h} \right) \left[ \frac{1}{2} (u_1^2 - u_2^2) + H_0 + h \right] \]
\[ = 1 - \frac{\partial J}{\partial h}. \]  (2.24)

Proof of this result is important as we shall utilise it (in a somewhat more general form) in section 6.4 to relate information propagation to the hydraulic functional for rotating channels. For completeness, we shall prove the generality of the relationship (2.24) between the composite Froude number and the hydraulic functional for nonrotating channels of arbitrary cross-section. We proceed by differentiating (2.16), with respect to \( x \) for a channel of constant cross-section or arbitrary shape, viz.

\[ \frac{\partial J}{\partial x} = (\frac{\partial J}{\partial h}) (\frac{\partial h}{\partial x}) \]
\[ = \frac{\partial \zeta}{\partial x} - u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial x} + \frac{\partial h}{\partial x} \]
\[ = 0. \]  (2.25)

As we are considering channels in an inertial frame where the flow is set-up from rest, the flow is irrotational and \( \zeta \) is a universal constant for the flow (i.e. it is the same everywhere along and across the channel; this condition will be relaxed in section 6.4 for rotating channels), and so \( \frac{\partial \zeta}{\partial x} \) vanishes. If the pressure term is eliminated from the two time dependent \( x \) momentum equations of (2.3),
\[
\frac{\partial}{\partial t} (u_1 - u_2) + u_1 \frac{\partial u_1}{\partial x} - u_2 \frac{\partial u_2}{\partial x} = -\frac{\partial h}{\partial x},
\]
(2.26)

A comparison with (2.25) shows \(\frac{\partial J}{\partial h}(\partial h/\partial x) = (\partial/\partial t)(u_1 - u_2)\).

For small amplitude travelling wave solutions we may write \(\partial/\partial t = -C_0 \partial/\partial x\) (if the waves were not of small amplitude then \(C_0\) would become a function of \(x\) and we would not be able to perform this analysis), and eliminate \(u_1, u_2\) from (2.26) using the continuity condition in (2.3) (with \(v_1\) set to zero) to give

\[
(D-h) h \frac{\partial J}{\partial h} - C_0 \left[(D-h) u_1 + h u_2\right] \frac{\partial h}{\partial x} + D C_0^2 \frac{\partial h}{\partial x} = 0.
\]
(2.27)

For a nontrivial along-channel structure, \(\partial h/\partial x \neq 0\), at least one of the two phase velocity solutions \(C_0 = C_1, C_2\) must vanish if \(\partial J/\partial h = 0\), and so \(F = 1\) from (2.21). The composite Froude number is related to the slope of the hydraulic functional by (2.24) regardless of the form of the cross-section of the channel; \(\partial J/\partial h < 0\) is equivalent to \(F < 1\) and corresponds to subcritical flow; \(\partial J/\partial h > 0\) is equivalent to \(F > 1\) and represents supercritical flow. At solution branch points \(\partial J/\partial h = 0\), \(F = 1\) and the flow is critical.

For critical and supercritical flow the direction of \(C_1\) and/or \(C_2\) is given by the sign of the second term of (2.27). Differentiation of (2.27) with respect to \(x\) \(\partial C_0/\partial x = 0\) and elimination of \(\partial u_1/\partial x\) using (2.3) continuity equations gives, for a nontrivial solution \((\partial h/\partial x \neq 0)\),

\[
0 = (D - h) h (D - 2h) \frac{\partial J}{\partial h} + (D - h)^2 h^2 \frac{\partial^2 J}{\partial h^2} + \\
+ [((D-h)^2 - 1)u_1 + (h^2 - 1)u_2] C_0 - [(D - h)^2 + h^2] C_0^2.
\]
(2.28)

When \(\partial J/\partial h = 0\), a comparison of the term in \(C_0\) with that of (2.27) shows that if

\[
\frac{\partial^2 J}{\partial h^2} = 0
\]
(2.29)

both phase velocities \(C_1, C_2\) must vanish. Physically this may be interpreted as follows: if \(\partial J/\partial h = 0\) and \(\partial^2 J/\partial h^2 = 0\) then the
three solution sheets, two supercritical and one subcritical, coincide. On one of the supercritical sheets, both phase velocities must be towards the dense reservoir, while on the other they must both be towards the light reservoir; the only solution satisfying these two requirements is that both phase velocities vanish where the two supercritical sheets meet.

Furthermore, if $\frac{\partial J}{\partial h} > 0$ and $\frac{\partial^2 J}{\partial h^2} < 0$, for a given solution to $J = 0$, both phase velocities will be directed towards the light reservoir. Similarly, if $\frac{\partial J}{\partial h} > 0$ and $\frac{\partial^2 J}{\partial h^2} > 0$, both phase velocities will be directed towards the dense reservoir.

In the next section we go into the requirements of hydraulic control in greater detail and explore how the hydraulic functional is related to the flow.

2.4 Critical points

While there are a wide variety of solutions to $J(h; h) = 0$ matching onto the whole range of reservoir conditions (with and without hydraulic jumps), we shall concern ourselves mainly with those which are controlled in a sense which will be defined later in this section. First consider how $J$ varies with $h$.

Figure 2.2 shows the typical forms of the hydraulic functional (for rectangular and parabolic cross-sectioned channels, with and without rotation) as a function of the interface height $h$. Each plot is for a given section, flow rates ($Q$ and $\bar{Q}$) and functional constant $\zeta$. Note that it is cubic-like (the single-layer hydraulic functional is quadratic-like) in appearance, though the actual order of $J$ in terms of $h$ is normally higher and not necessarily integer (as will be shown in sections 3.1 and 5.1 for rectangular and parabolic channels respectively). The functional $J$ will have either no or two (real) turning points and may have either one or three real roots. Some or all of the roots may coincide. When there are three distinct real roots (figure 2.2a), two will be supercritical (having $\frac{\partial J}{\partial h} > 0$), and one subcritical ($\frac{\partial J}{\partial h} < 0$). The subcritical root occurs at a value of $h$ intermediate between the two supercritical roots. If two roots coincide they are necessarily critical ($\frac{\partial J}{\partial h} = 0$; figure 2.2 b, c and d), while if there is only one real root it
Figure 2.2. Typical forms of the hydraulic functional. (a) Three distinct real roots; (b) three real roots, two coincident; (c) three real roots, two coincident; (d) three coincident real roots; (e) one real root; (f) one real root and no turning points.
must be supercritical (figure 2.2e and f).

In a controlled flow, moving from the dense to the light reservoir, the interface height initially traces the supercritical root with the greater value of \( h \) (marked "A" on figure 2.2a). Moving along the channel allows the functional surface to change so that this supercritical root eventually coincides with the subcritical root at a critical point or control section (these terms will be used interchangeably), such as that marked "AB" on figure 2.2b. Moving further towards the light reservoir, the interface height continues to change, but remains subcritical (like "B" on figure 2.2a) until it coincides with the lower supercritical root at a second control section ("BC" of figure 2.2c). The solution is able to switch again to a supercritical branch (e.g., like "C" on figure 2.2a) for the remainder of the channel into the light reservoir.

For some flows the region over which the subcritical root is appropriate may be vanishingly small and the solution is able to switch directly from one supercritical root to another at a unique control section, as suggested by figure 2.2d (triple root marked as "ABC"). In the previous section we showed that \( \delta J/\delta h \) and \( \delta^2 J/\delta h^2 \) both vanish at such sections.

Throughout the remainder of this paper we shall use the term controlled flow to mean a two-layer flow in which:

i) The flow consists of two supercritical regions (\( \delta J/\delta h > 0 \)), one on either side of a subcritical region (\( \delta J/\delta h < 0 \)), though this subcritical region may be vanishingly small.

ii) The interface height is defined continuously within the channel (or at least within a sufficiently large region of interest) and thus must pass through critical points (\( \delta J/\delta h = 0 \)) when changing from supercritical to subcritical flow (and vice versa). Two such critical (branch) points must exist, although they may coalesce to a single value of \( x \) if the subcritical region is vanishingly small.

iii) The two supercritical regions must represent different solution branches. In particular, between the dense reservoir and the subcritical region the relevant solution must have a value of \( h \) greater than the
corresponding subcritical root of $J = 0$ for that section (should such a solution exist). Similarly between the subcritical region and the light reservoir, the supercritical root must have a value of $h$ less than that of the corresponding subcritical root. This requirement may be formulated in terms of the sign of $\frac{\partial^2 J}{\partial h^2}$ as suggested in the previous section.

The first of these criteria is essentially that the flow in the subcritical region is isolated from the details of what is happening outside this region. Details of the flow, or disturbances to it, within the supercritical regions are not able to communicate information about themselves or the reservoirs into the subcritical region (the long, small amplitude, interfacial waves have both phase velocities the same sign and only propagate away from the subcritical region). In the absence of dissipation it is not possible to have a supercritical region between two subcritical regions as long wave disturbances generated within the supercritical region will remain trapped within it. Similarly disturbances within the bounding subcritical regions may propagate into the supercritical region and become trapped. Thus such a situation is unstable and would soon evolve into a subcritical flow.

The second criterion must hold in order that such a subcritical region is able to exist under the basic assumptions of the flow. For the critical point closer to the denser reservoir $\frac{\partial^2 J}{\partial h^2} > 0$ (i.e. it is a local minimum except when the two critical points coincide), while that closer to the light reservoir has $\frac{\partial^2 J}{\partial h^2} < 0$. Within the subcritical region the two phase velocities have opposite signs and are thus able to transmit information about any disturbances to both control sections and the supercritical regions. Thus, in order to fully determine the flow, we need consider only the flow within the sub-critical and critical regions. Hydraulic jumps and other discontinuities may occur outside this region without altering the flow or the flow rates within the subcritical region. Hydraulic jumps may therefore provide a mechanism for matching the solution onto that in the reservoir. Note that while the Bernoulli potential is not conserved through the jump, the energy losses in the contracting layer are likely to be an order of magnitude less than in the
expanding layer (Wood & Simpson, 1984).

The final criterion is essentially one of irreversibility. Benjamin (1968) noted that a condition of energy conservation may be an unjustifiable assumption for theoretical models of steady flows. In section 3.3 we demonstrate that the set-up and adjustment mechanisms for the flow in a rectangular channel containing a sill is dissipative, and hence irreversible.

In general the two control sections are distinct and may be distinguished by whether they are at a specific geometric feature or in a region of expanding channel. We shall use the term primary control to denote the hydraulic control fixed at a specific geometric feature for a large (typically all) range of net forcings. This feature will typically be the section of minimum channel depth and/or width and is given by \( x = x_c \). The other control, the exit or virtual control, will typically be at the start of, or within, a region of expanding channel width. The precise position of the virtual control, \( x = x_v \), may be determined by either a secondary geometric feature (such as the start of the expansion) or may be a function of the net forcing \( Q \) in addition to the channel geometry. If \( x_v \) is determined by a secondary geometric feature then it will remain fixed at that feature for some (possibly small) range of forcings, otherwise it will move continuously with \( Q \). In section 3.7 we shall distinguish more clearly between these two types of behaviour for the virtual control.

The term exit control will be reserved for when the virtual control is at the foot of a simple sill (see section 3.3 for example). In more complicated channel geometries it may not be clear which of the two controls is the primary and which the virtual control. When such ambiguities arise (e.g. in chapter 10) we shall label the controls according to their associated geometric features.

If we know the position of the two control sections, \( x = x_c \) and \( x = x_v \), then, by solving \( \delta J/\delta h = 0 \) and \( J = 0 \) simultaneously (at \( x_c \) and \( x_v \)), we may determine the interface heights (at \( x_c \), \( x_v \)), the exchange flow rate \( q \) and functional constant \( \zeta \). Unfortunately the positions of the two controls will not generally be known, complicating the solution procedure.

In principle equation (2.11) can be utilised to determine the
positions of the control sections. In general there will be at least two values of \( x \) which satisfy \( K = 0 \). If there are more than two such sections, \( x = x_1, x_2, x_3, \ldots \) then the control sections will be the pair which give the lowest exchange flow rate. In particular, suppose \( x = x_i \) for \( i = 1, 2, \ldots, n \) all give \( K = 0 \). The control sections are given by \( x_0 = x_j, x_v = x_k \), \( j \) and \( k \) both in the range 1 to \( n \), such that \( x_j, x_k \) yields the minimum value for \( \dot{q} \). Any other pair would result in it not being possible to trace the necessary solution branches everywhere in the channel (two of the roots of \( J = 0 \), one or both of which are required to trace the solution, would be imaginary for some sections) and the flow would not satisfy the conditions for hydraulic control. Note that \( j = k \) is included in this set, corresponding to the two critical points coinciding.

If the exchange flow rate is lower than that associated with a controlled flow (as outlined above), the flow may either be subcritical everywhere, or have one and only one transition from subcritical to supercritical behaviour. There is no mechanism for isolating such a flow from one or both of the reservoirs and so the flow is not fully controlled. We return to the problem of partially controlled flows in section 3.5.

A result of primary importance to the concept of hydraulic flows is that the controlled solution is maximal. The solution is maximal in the sense that the exchange flow rate is greater than that for any flow lacking two hydraulic transitions. Moreover any configuration yielding a higher exchange flow rate will not satisfy the basic assumptions as it will not be possible to match such a flow onto appropriate reservoir conditions.

Long (1956) analysed the linear stability problem for long wave disturbances to an inviscid shear flow between two layers bounded by rigid, horizontal plates. In terms of our present nondimensional notation he found the flow is stable if

\[
(u_1 - u_2)^2 < D.
\]  

(2.30)

For our hydraulic problem we shall ignore short wave instabilities (the flow is always unstable to these in the absence of viscosity and surface tension, but may be stable for finite amplitude disturbances if either of these effects are included) and concentrate on the long wave modes as these are the mechanism.
through which disturbances are communicated along the channel. For critical conditions the composite Froude number is unity. Noting that $u_1$ is negative for exchange flows and replacing $u_2$ with the composite Froude number using equations (2.18) and (2.19) yields

$$D |u_1|^2 - 2 D^{1/2} |u_1| - [D (F^2 - 1) - F^2 h] h > 0.$$  

(2.31)

For critical flow, $F^2 = 1$, equation (2.31) shows the flow is never unstable. Marginal stability ((2.31) equal to zero) may occur if $D^{1/2} u_1 = h$. For $F^2 < 1$, the inequality of (2.31) holds for all values of $u_1$ and $h$, so subcritical flows are always stable with respect to long wave disturbances. The bounding supercritical flow may be unstable if $D^{1/2} u_1$ lies in the range

$$h > (F^2 - 1)^{1/2} (D - h)^{1/2},$$

though we note that the supercritical nature of the flow may wash such disturbances a significant distance away from the subcritical region before they have grown to finite amplitude.

The shear layer at the interface between the two layers may be unstable to long wave disturbances only in the supercritical regions. As we have shown in this section, disturbances in the supercritical regions are not able to communicate any information to the subcritical region bounded by the control sections. Thus we conclude that so long as the amplitude of the instabilities does not become sufficiently large to propagate against the supercritical flow and into the control region, or to significantly alter the vertical density structure, the maximal flow is stable.

The formation of a hydraulic jump in a supercritical region is a consequence of the instabilities growing to finite amplitude and breaking. This will often be associated with a reduction in the Froude number, either through frictional effects or a geometric feature (e.g. a secondary sill - see chapter 10), which will lead to a piling up of the finite amplitude disturbances.

In the next chapter we shall apply these ideas of hydraulic control, using Gill's formalism, to the exchange flow through a rectangular cross-section. A range of simple geometries are
investigated, both with and without a net forcing $Q$, in order to gain some understanding of how the formalism of the hydraulic functional relates to real channel flows and of how the flow itself responds to the different geometric configurations. We shall return briefly to the requirements for hydraulic control to exist in section 3.5.
Channels with rectangular cross-sections

3.1 Rectangular geometry and problem definition

A typical channel of rectangular cross-section is shown in figure 3.1. At any position along the channel given by \( x \), the channel has a width \( b(x) \) and depth \( D(x) \) (constant over the width). The height of the channel bottom above some fixed datum is \( H(x) \). The height of the interface above the channel bottom is \( h = h(x) \). The longitudinal length scale of variations in the geometry is not important, so long as it is much greater than the depth of the fluid and the width of the channel; generally diagrams of the flow will be shown with compressed longitudinal and lateral length scales for clarity. The flow is nondimensionalised so that the channel depth and width, at \( x = 0 \), are both unity (i.e. \( D(x=0) = 1, b(x = 0) = 1 \)).

For convenience we put

\[
    h(x) = (\tfrac{1}{2} + A(x)) D(x),
\]

and note that \( \partial / \partial h = D^{-1} \partial / \partial A \) at constant \( x \) (\( D(x) \) is fixed for a given geometry). The dependent variable \( A = A(x) \) will be called the \textit{interface height coefficient}. When \( A = 0 \) the interface is at half the channel depth. The sectional areas, \( S_1 = S_1(x) \) and \( S_2 = S_2(x) \), occupied by the lower and upper layers respectively are

\[
    S_1 = D \cdot b \left( \tfrac{1}{2} + A \right),
\]

\[
    S_2 = D \cdot b \left( \tfrac{1}{2} - A \right),
\]

and the exchange flow rate and net barotropic forcing are given by equations (2.8) and (2.9) respectively. The hydraulic functional may be expressed in terms of \( A \) as

\[
    J(\cdot; A) = \Delta \sigma + \left[ \frac{1}{2 Db} \right]^2 \frac{A \left( \sigma^2 + \bar{v}^2 \right) - 2 \left( \bar{w} + A^2 \right) \bar{q}}{(\bar{w} - A^2)^2} - H - D \left( \tfrac{1}{2} + A \right),
\]

and its derivative
Figure 3.1. Definition sketch of a channel with a rectangular cross-section.
Suppose the interface height coefficient, $A$, at the control sections takes the values $A_c = A(x=x_c)$ and $A_v = A(x=x_v)$. Critical conditions require $\frac{\partial J}{\partial A} = 0$ when $x = x_c$, $A = A_c$ or $x = x_v$, $A = A_v$. Hence, if the interface height is known at one or the other control section, the exchange flow rate $\bar{q}$ can be evaluated from (3.4) as

$$\bar{q} = \frac{-a_1 + (a_1^2 - 4a_0a_2)^{1/2}}{2a_2},$$

where

$$a_2 = \hat{W} + 3A^2,$$
$$a_1 = -A(3 + 4A^2)Q,$$
$$a_0 = (\hat{W} + 3A^2)Q^2 - 4D^3b^2(\hat{W} - A^2)^3.$$

and $D$, $b$ and $A$ are set to their appropriate values at $x = x_c$ or $x = x_v$. The constant $\zeta$ (which is the Bernoulli potential difference at the control section) can then be evaluated from solving $J = 0$ for the same control section.

For the present rectangular geometry, we can write the constriction condition of equation (2.11) as

$$K = \frac{\partial J}{\partial D} \frac{dD}{dx} + \frac{\partial J}{\partial H} \frac{dH}{dx} + \frac{\partial J}{\partial b} \frac{db}{dx}.$$

Clearly any values of $x$ for which $dD/dx$, $dH/dx$ and $db/dx$ all vanish give $K = 0$, and are thus candidates for controlling the flow. It is incorrect however, to assume that only such points may be control sections: contributions from the partial derivatives of $J$ must also be considered.

Before considering the flow through a complicated geometry, it is instructive to relate this theoretical framework to the flow through a number of simple model geometries. In the next section we apply the theory to the flow through a channel of constant depth and varying width.
3.2 Constant depth rectangular channels with no barotropic forcing

The simplest of all two-layer flows possessing the necessary characteristics for hydraulic control is that along a channel of constant depth and bottom elevation \( D(x) = 1, \ H(x) = -\frac{1}{2}, \) say, but varying width. In this section we assume there is no net flow between the two reservoirs. Without loss of generality, we can model the width variations of any such channel by \( b(x) = 1 + x^2 \) as the along-channel length scale does not come into the equations. From the overall symmetry of the system we can expect the interface to behave in an antisymmetric manner, passing through \( z = 0 \) at \( x = 0 \). Furthermore, the control sections must be symmetrically placed about \( x = 0 \).

For such a channel equation (3.6) reduces to

\[
K = (\frac{\partial J}{\partial b})(\frac{db}{dx}),
\]

in which

\[
\frac{\partial J}{\partial b} = \frac{-1}{2D^2b^3} \frac{A \frac{q^2}{(H - A^2)^2}}
\]

and \( D \) is identically unity. Setting equation (3.7) to zero requires either \( \frac{db}{dx} = 0 \), which occurs at \( x = 0 \), or \( \frac{\partial J}{\partial b} = 0 \) which may occur at any \( x \) provided \( A = 0 \). Again symmetry could be utilised to determine that the required solution is that both controls are at \( x = 0 \) (where \( A \) also happens to be zero). Moreover we shall see in section 3.6 that, for constant depth channels such as this, \( \frac{db}{dx} \) must vanish at one control (the primary control), and \( \frac{\partial J}{\partial b} \) at the other (the virtual control). In the meantime we shall look at how to obtain this result in general.

Suppose one of the controls occurs at \( x = x_b \) where the width of the channel is \( b_b \). Equation (3.8) would then require \( A_b = A(x=x_b) \) to be zero. Consideration of the behaviour of \( J \) about this point (in \( A-J \) space) shows \( \frac{\partial^2 J}{\partial A^2} = 0 \) in addition to \( \frac{\partial J}{\partial A} = 0 \) (critical conditions). The solution to \( J = 0 \) is therefore a triple root (as in figure 2.2d). We can determine the exchange flow rate from equation (3.5) to be
If the hypothesis introduced in section 2.4, that the position of the two control sections is that pair producing the lowest exchange flow rate, is correct, then it is readily apparent from (3.9) that the control sections must occur where \( b_a \) is a minimum. This gives \( x_c = 0 \) as expected from symmetry arguments.

Proof of this hypothesis is straightforward. For the present channel, if (3.9) applies, setting (3.4) to zero shows the turning points in \( J \) are solutions of

\[
A^6 - \frac{3}{2} A^4 + \left( \frac{3}{16}(1 + \gamma) A^2 \right) - \left( \frac{1}{64}(1 - \gamma) \right) = 0, \tag{3.10}
\]

where \( \gamma \) is the square of the ratio of the channel width at the control section to that at arbitrary \( x \) (i.e. \( \gamma = (b_a/b)^2 \)). Differentiation enables us to show that these turning points shift with the channel width ratio \( \gamma \) as

\[
\frac{\delta A^2}{\delta \gamma} = -3 \left( \frac{12 A^2 + 1}{144 A^4 - 32 A^2 + 9 (1 + \gamma)} \right). \tag{3.11}
\]

Now at the control at \( x = x_b \), \( A = A_S = 0 \) and \( \gamma = 1 \). So \( \delta A^2/\delta \gamma \) is \(-1/24\). If \( \gamma \) were to increase (i.e. \( b \) become less than \( b_a \)) away from the control, turning points in \( J \) would cease to exist (they would require imaginary values for \( A \) as \( A^2 < 0 \)). To complete the proof we note that an absence of turning points indicates a unique supercritical flow which, when combined with the present channel geometry, must become critical again at \( x = -x_b \). Such a situation violates the conditions for controlled flow given in section 2.4 and so can not occur. Hence it is not possible for the control section to be anywhere other than at the narrowest point of the channel (\( x = 0 \)).

In passing, we note that this controlled flow maximizes the exchange flow rate through the channel in the sense that the realized exchange flow rate is the minimum value \( q_{\text{max}} \) takes anywhere along the channel where \( q_{\text{max}} \) is the local maximum (with respect to \( A \)) of equation (3.5) for the given section. This result is consistent with the maximal exchange hypothesis (e.g. Whitehead et al., 1974) traditionally used in solving two-layer hydraulic problems. However, any crude formulation of the maximal exchange
Figure 3.2. Plan, elevation and end views of maximal exchange flow through a rectangular channel of constant depth (no barotropic forcing). Both controls are at the point of minimum width (indicated by an arrow).
hypothesis, based on a turning point in \( \bar{q} \) obtained from (3.5),
will fail to yield the correct results in any but this simplest
flow situation. We shall demonstrate this for a simple sill in the
next section.

Figure 3.2 shows some of the details of the controlled flow
through this geometry. Such a maximal exchange flow may be matched
onto a wide range of reservoir conditions by the formulation of an
hydraulic jump in the supercritical region (Armi & Farmer, 1986).
The presence of a jump in the supercritical region leading to the
dense reservoir (as shown in figure 3.2) will not alter the
controlled nature of the flow, provided mixing between the two
layers has a negligible effect on the density of the lower layer
and the interface in the dense reservoir remains higher than that
at the control section. Similarly the interface height \( (H + h) \) in
the light reservoir must be lower than that at the control
section. In section 3.5 we shall look in more detail at the
reservoir conditions required for critical flow to be achieved.

Conservation of momentum for the two-layer system is not
sufficient to determine the height (or position) of an hydraulic
jump. Momentum is transferred between the two layers by a
combination of the nonhydrostatic pressure on the curved face of
the jump and mixing between the two layers. Thus it is not
possible to look at the two layers separately. Additional
knowledge is required (Wood & Simpson, 1981) either about the
changes in energy or the pressure distribution over the jump. Such
considerations are beyond the scope of this thesis.

3.3 Depth varying rectangular channels with no net barotropic flow

Consider a channel whose depth as well as width varies. In
order to isolate certain important features, we shall consider the
set of channels consisting of a region of constant width and
varying depth between two regions of constant depth and varying
width. Such a channel is depicted in figure 3.3. Throughout the
channel the top surface remains horizontal, so \( H(x) = H_0 - D(x) \).
For convenience we shall assume that the depth and width of the
channel at \( x = 0 \) are unity, with the depth increasing towards \( D_0 \)
as one moves towards \( x = -1 \). The width remains unity over this
Figure 3.3. Schematic diagram of a simple sill.
Rectangular channels

range. The manner in which the depth changes is unimportant so long as it increases monotonically towards $D_w$ away from $x = 0$, and is continuous. Note that continuous behaviour of $b$ and $D$ results in continuous behaviour of all derivatives as the along-channel length scale is of no importance; $dD/dx$ must vanish at $x = 0$ and $\pm 1$. For $|x| > 1$ the width of the channel increases monotonically as $|x|$ increases from 1, again the exact form being of no consequence so long as it is continuous. As in the previous section we set $Q = 0$.

A number of researchers previous to Armi (1986) (e.g. Mehrotra, 1973) solved flow in channels of this form by analogy with the flow through a contraction in width as discussed in the previous section. They were motivated in part by the empirical law of maximal exchange, and in part by the ability to treat the unique control section in isolation in the channels as discussed in the previous section. Unfortunately the flow over a sill, such as that analysed here, is not so straightforward.

Suppose for the moment that the analogy with the flow through a contraction is valid. The control sections would both be at the sill crest ($x = 0$; $K$ vanishes here), and the interface would be described by $A(x=0) = 0$ in order to produce critical conditions. By simultaneously solving the conservation of Bernoulli potential and conservation of mass, or equivalently finding roots of $J = 0$, it is possible to trace a supercritical solution towards the lighter reservoir. However, if we try to trace the other supercritical solution to match onto the denser reservoir, we find that it ceases to exist as soon as we move away from $x = 0$ (see figure 3.8). While the functional at the sill crest has three real roots to $J = 0$ (albeit simultaneous), as soon as the depth is increased, even infinitesimally (the width being held constant), the values of $J$ at the turning points in $J$ become of the same sign. This means two of the roots of $J = 0$ are imaginary, leaving only one supercritical root to trace. As such we are unable to match onto the conditions in the dense reservoir, even if an hydraulic jump were included.

Thus, while $A(x=0) = 0$ would produce the maximum exchange flow through the channel, the flow could never be realised. This suggests a modification to the naive statement of the maximal exchange theorem: the controlled flow is the maximum realisable
Figure 3.4. Variations in the form of the hydraulic functional as a function of $A$ at different locations over a sill. Both control sections have been assumed to be at the sill crest. Note that except at the crest, where the functional has a triple root, the functional has only one real root to $J = 0$. 
exchange flow through the channel. This maximum is dictated by the section or pair of sections through which the maximum flow able to pass is least. At any section the maximum flow able to pass may occur either at a turning point in \( \bar{q} \) (giving coincident control sections), or at a boundary to the portion of space representing realisable flows (giving two distinct control sections).

The constriction requirement for this channel may be written as

\[
K = \frac{\partial J}{\partial b} \frac{db}{dx} + \frac{\partial J}{\partial D} \frac{dD}{dx} + \frac{\partial J}{\partial H} \frac{dH}{dx}.
\]

by recalling that \( H = \frac{1}{2} - D \). The first thing to notice is that both \( \frac{db}{dx} \) and \( \frac{dD}{dx} \), and hence \( K \), vanish at \( x = \pm 1 \), so these three locations are prime candidates for hydraulic control. When \( |x| > 1 \), \( \frac{dD}{dx} \) vanishes although \( \frac{db}{dx} \) remains nonzero, and so any further controls in these regions of the channel would be due to \( \frac{\partial J}{\partial b} = 0 \). This situation was examined and rejected in the previous section for a contraction; as the same arguments apply here, both the primary and virtual controls must lie in the interval \( x_c, x_v \in [-1, 1] \).

For \( |x| < 1 \), additional possibilities are introduced by the non-geometric terms of (3.12), i.e.

\[
\frac{\partial J}{\partial D} + \frac{\partial J}{\partial H} \frac{\partial H}{\partial D} = \frac{-1}{2 \, D^3} \frac{A}{b^2 (1 - A)^2} \frac{Q^2}{(\frac{1}{2} - A)} - (\frac{1}{2} - A)
\]

vanishing. Eliminating the square of the exchange flow rate from (3.13) by using the conditions for critical flow, as given by (3.5), and setting the result equal to zero yields a quadratic in \( A \) with no real roots. Hence we are able to conclude that (3.13) never vanishes.

Thus there are only six possible combinations of the positions of the control sections, namely

\[
(x_c, x_v) \in \{(0, 0), (0, -1), (0, 1), (-1, -1), (-1, 1), (1, 1)\}.
\]

(3.14)

Some of these may be eliminated immediately: the \((0, 0)\) solution has been shown to be invalid while the \((-1, -1)\) and \((1, 1)\) solutions
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have a greater depth but the same width as $x = 0$, and thus would produce a greater exchange flow rate. Moreover, as the geometry is identical at $x = \pm 1$, this would require a triple-root of $J$ at both these points (or a flow unable to match onto the reservoir conditions) and so the $(-1,1)$ solution would be identical to the $(-1,-1)$ and $(1,1)$ solutions.

The two remaining solutions, $(0,-1)$ and $(0,1)$, differ through the need for the interface height to slope down towards the lighter reservoir. Let $x_c = 0$ so that the $c$ subscript indicates the primary control on the crest of the sill, and $x_v$ be either $-1$ or $1$ with the $v$ subscript indicating the virtual control at the channel exit. If $x_v = -1$, then $D_v(\frac{1}{2} - A_v) < D_c(\frac{1}{2} - A_c)$, while $x_v = 1$ would reverse the inequality. More fundamentally, the symmetry of the turning points in $J$ about $A = 0$ shown by equation (3.10), in which $A$ occurs only as $A^2$, requires that $A_c$ and $A_v$ are of opposite sign. Unfortunately it is not possible to obtain an explicit solution for the necessary simultaneous equations for a general value of $D_w > 1$, though the limits $D_w \to \infty$ and $D_w \to 1$ are accessible.

If we pose the asymptotic expansion

$$A_v = A_{v0} + A_{v1}/D_w + O(D_w^{-2}),$$

we are able to determine that $A_v \sim \frac{1}{2} - 0.35104/D_w$, $A_c \sim -0.12544$ and $A_v \sim 0.41598$ for large $D_w$. The behaviour of $A_v$ is consistent with the interface remaining essentially a constant height above the channel datum as $D_w$ is varied.

Similarly we may undertake an asymptotic expansion for $D_w \to 1$, obtaining

$$A_c \sim -\varepsilon/4 + \varepsilon^2/8 + O(\varepsilon^4),$$

$$A_v \sim \varepsilon/4 + \varepsilon^2/8 + O(\varepsilon^4),$$

$$\bar{q} \sim \frac{1}{2} \left[ 1 - (3/4) \varepsilon^2 + (3/4) \varepsilon^3 + O(\varepsilon^4) \right],$$

$$D_w = 1 + \varepsilon^3.$$  

(3.16)

Note the $\varepsilon^3$ term vanishes for $A_c$ and $A_v$. In both limits $A_v > A_c$, and so the virtual control is at $x_v = -1$.

For general $D_w$ the value of $A_c$ will lie between zero and $A_{\infty} = A_c(D_w^{-\infty}) = -0.12544$, while $A_v$ will be bounded between zero
and $H = 0.3510/D_w$. To gain a better understanding of how the flow is related to the hydraulic functional, figure 3.5 plots how the hydraulic functional varies as a function of $A$ for a number of different sections along a channel with $D_w = 2$. The symmetry of the channel means that the curves for a section at $-x$ are identical to those for the section at $x$. The two curves where critical roots exist (i.e., those corresponding to $|x| = 0.1$) are plotted as heavy lines. Curves for $|x| < 1$ are continuous lines while those for $|x| > 1$ are dashed. Note that as we move away from the sill crest, the value of $J$ associated with the local minimum in $J$ initially increases towards zero, reaching zero for the $|x| = 1$ curve. However, as the channel increases in width past $|x| = 1$, the value of $J$ associated with the minimum decreases again. Thus, as $x$ increases, it is possible to trace solutions of $J = 0$ from high values of $A$ to low values, passing from supercritical ($\partial J/\partial A > 0$) to subcritical ($\partial J/\partial A < 0$) and back to supercritical solution branches.

Figures 3.6 and 3.7 show plots of the exact solutions (evaluated numerically) for $A_c$, $A_v$ and $\bar{Q}$, along with the asymptotic values computed from (3.15) and (3.16). It is clear from both plots that only a small increase in the depth of the channel away from the sill crest drastically alters the flow. Most of the changes in the interface height over the sill crest occur when the channel deepens by only around 25% ($A_c$ is then approximately 83% of its value for $D_w \rightarrow \infty$), though the exchange flow rate takes somewhat longer to approach its large $D_w$ asymptotic value.

It is interesting to note how well (3.16) matches the exact value for $A_c$ even for $D_w$ much greater than that for which it is formally valid. The deviation is less than 10% for $D_w$ between 1 and 2.5, by which time the true value of $A_c$ is within 3% of its large $D_w$ asymptotic value. Unfortunately both the interface height at the virtual (exit) control and the exchange flow rate are not such good fits. However, if the asymptotic value for $A_c$ is used in (3.5), rather than an asymptotic approximation to this equation, a more accurate value for $\bar{Q}$ may be obtained.

The hydraulic results are in agreement with Armi & Farmer (1986) who looked at channels of constant depth ($D_w = 1$) and Farmer & Armi (1986) who investigated sills whose depth went to
Figure 3.5. Variations in the form of the hydraulic functional as a function of $A$ at different locations over a simple sill ($D_w = 2$). The flow is controlled. The curves corresponding to the two controls ($x = 0$ and $x = -1$) are plotted as heavy lines. Curves for $|x| < 1$ are solid lines while those for $|x| > 1$ are dashed lines. The values of $A$ at the two control sections are indicated by arrows. Note that for all sections $J = 0$ has three real roots (though two are coincident at each of the controls).
Figure 3.6. Interface height coefficients at the sill crest \((A_c)\) and the virtual (exit) control \((A_v)\) as a function of the channel depth away from the sill \((D_w)\). Solid lines indicate the exact solution; long dashes the \(D_w \to 1\) asymptotic expansion; short dashes the \(D_w \to \infty\) asymptotic expansion.

Figure 3.7. Variations in the exchange flow rate over a simple sill as a function of the channel depth away from the sill \((D_w)\). The solid line indicates the exact solution; long dashes the \(D_w \to 1\) asymptotic expansion; short dashes the \(D_w \to \infty\) asymptotic expansion.
infinity \((D_W \to \infty)\) before widening. Their results were equivalent to \(A_c = 0\) (for \(D_W = 1\)) and \(A_c = -0.125\) \((D_W = \infty)\); note that the second of these is not \(-1/8\)! As we have shown here, the infinite sill \((D_W \to \infty)\) limit is a very good model for \(D_W\) greater than around 1.5, thus validating Farmer & Arm's use of this limit.

It is of interest to consider whether or not the controlled flow over a sill is dissipative. To do this we consider the ratio of the rate of increase in kinetic energy \(\dot{E}_K\) to the rate of release of potential energy \(\dot{E}_P\) of a mutual intrusion in a channel of varying depth (for the calculation for a channel of constant depth see p. 136, Yih, 1965). The simplest channel containing the essential features of the hydraulically controlled flow over a sill is one with \(D(x) = D_w\) for \(x < x_V\) and \(D(x) = 1\) for \(x > x_C\) \((x_V < x_C)\). The depth decreases between \(x_V\) and \(x_C\) in a smooth, monotonic manner. In terms of the interface heights \(h_V\) and \(h_C\), and the exchange flow rate \(q\), the dissipation coefficient \(\mu\) is

\[
\mu = \frac{\dot{E}_K}{\dot{E}_P} = \frac{h_C^{-2} + (D_v - h_V)^{-2} + h_V^{-1} (D_v - h_V) + h_C^{-1} (D_c - h_C)^{-1} q^2}{2 D_c - h_C - D_v + h_V}.
\]

(3.17)

As the hydraulic flow over a sill is critical at \(x_C\) and \(x_V\), we can use the hydraulic solutions to determine \(\mu\). This is plotted in figure 3.8 as a function of \(D_w\). When \(D_W = 1\) the dissipation coefficient is unity, indicating the flow is energy-conserving. For \(D_W > 1\) \(\mu\) is less than unity, showing that dissipation must occur in setting up such a flow from dam-break (and a wide variety of other initial conditions). In the limit as \(D_W \to \infty\), \(\mu \to 0.6623\).

Only in the case of a channel with symmetry about a horizontal plane is it possible to equate the rate of gain of kinetic energy with the rate of release of potential energy \((\mu = 1)\). Similar arguments apply to the adjustment process from any arbitrary initial state.

In light of the discussion in this and the previous sections, in the next section we present an algorithm describing the solution process for a channel of arbitrary along-channel geometry.
Figure 3.8. Dissipation coefficient ($\mu$) for the flow over a sill as a function of the sill geometry ($D_w$). Set-up is from rest.
3.4 Solution algorithm

We are now in a position to state a general algorithm for solving two-layer hydraulic problems. This algorithm may be utilised for the nonrotating rectangular cross-section channels in the remainder of chapter 3 and, without further complexity, for parabolic cross-section channels which will be introduced in chapter 5. However, for the rotating systems which shall be covered by chapters 6 to 8, the solution process is significantly more complex if the two-layer region is separated from one of the channel walls as the hydraulic functional may not be written explicitly. Details of these additional difficulties are delayed until a more appropriate stage in this thesis.

Algorithm:

1) Guess the position of one of the control sections, normally the primary control $x_c$. For simple along-channel geometries this may be a point where $dD/dx$, $db/dx$ and $dH/dx$ vanish simultaneously.

2) Determine $h_{\text{max}}(x=x_c)$, the interface height which maximizes $\bar{q}_{\text{crit}}(x=x_c)$, where $\bar{q}_{\text{crit}}(x=x_c)$ is the value of $\bar{q}$ required to give critical conditions at the section $x=x_c$ for a given interface height $h$.

3) Guess the position of the second control section, $x_v$. This need not be at a geometric feature in the channel.

4) Determine $h_{\text{max}}(x=x_v)$.

5) Guess the interface height $h_c$ at $x=x_c$. If $x_v$ is closer to the dense reservoir than $x_c$ then $h_c$ must be less than $h_{\text{max}}(x=x_c)$. If $x_v$ is closer to the light reservoir, then $h_c > h_{\text{max}}(x=x_c)$.

6) Calculate the critical exchange flow rate $\bar{q}_c = \bar{q}_{\text{crit}}(h=h_c)$ and the value of $\zeta$ to give $J(\zeta;h_c) = 0$.

7) If $x_v$ is closer to the dense reservoir than $x_c$ then find the turning point $\delta J/\delta h = 0$ at $x = x_v$ with $h > h_{\text{max}}(x=x_v)$, otherwise the turning point with $h < h_{\text{max}}(x=x_v)$.

8) Determine the value $J_v = J(x=x_v)$ for the turning point calculated in step 7.
9) If \( J_V = 0 \) then go to step 10, else adjust \( h_c \) and return to step 6.

10) If \( \bar{Q}_c \), evaluated in step 6, is the minimum for all values of \( x_V \) then go to step 11, else adjust \( x_V \) and return to step 4.

11) If \( \bar{Q}_c \) is the minimum for all values of \( x_C \) the stop, else adjust \( x_C \) and return to step 2.

For a large subset of channels the position of the primary control \( x_C \) is obvious (e.g. the contraction of a channel of constant depth or the crest of a simple sill), though there may be more than one possible choice (such as the channels which will be introduced in section 3.8 and chapter 10), or even a range of possible values. Likewise, it is generally possible to restrict the range of sections over which we must search to determine the position of the virtual control \( x_V \). Only in complicated geometries will it prove necessary to do a search over all possible pairs of \( x_C \) and \( x_V \) to find the global minimum critical exchange flow rate.

### 3.5 Control flooding and submaximal flow

This section looks briefly at what happens when hydraulic control is lost at one or both of the control sections, and restates the reservoir interface height criteria outlined in section 2.4. Gill (1977) reviewed single-layer hydraulic theory and showed that over a wide range of conditions the surface height in the downstream reservoir could only be matched onto by the formation of an hydraulic jump in the supercritical region of the flow. Only if the surface height were greater than or equal to that associated with subcritical flow into the reservoir would there be a smooth transition from upstream to downstream reservoirs. In such a situation the flow would be subcritical everywhere.

For two-layer hydraulics the situation is essentially similar, although the addition of a second supercritical solution branch makes the picture a little more complex. The statement by Armi & Farmer (1986) that the interface height in the reservoirs can not be on the wrong side (i.e. lower for the dense reservoir or
higher for the light reservoir) of the interface height at the
virtual control is too simplistic. For this present discussion we
shall assume the interface height in the dense reservoir (i.e., the
left-hand reservoir) is greater than that in the less dense
reservoir (right-hand reservoir). The reverse situation may be
considered by reflection about the centre of the channel. Further,
our discussion will be confined to channels of the type utilised
in sections 3.2 and 3.3; the more general channel geometries
introduced later in this and subsequent chapters may be treated in
a similar manner.

Suppose a flow exists (with $Q = 0$) which is supercritical
everywhere along the channel with both wave phase velocities
towards the right-hand reservoir (phase velocities towards the
left-hand reservoir may be treated in a similar manner). Consider
the lower layer moving from the left-hand reservoir to the
right-hand reservoir. The overall symmetry of the geometry would
ensure the interface height at the entry to the right-hand
reservoir is the same as that at the entry to the left-hand
reservoir. An hydraulic jump cannot form in the left-hand
reservoir as the supercriticality of the flow would wash it into
the right-hand reservoir. If no hydraulic jump occurs within the
right-hand reservoir, then the interface heights are equal in the
two reservoirs and there is no potential energy available to drive
such a flow. On the other hand, if an hydraulic jump forms in the
right-hand reservoir, the interface height will be greater than
that in the left-hand reservoir. Such a situation is not possible
as it represents an increase in potential energy due to the flow.
Thus it is not possible for a flow to exist which is supercritical
everywhere.

Similarly, the inherent symmetry (about $X = 0$) of the
geometry ensures that any subcritical flow must also be symmetric
(i.e., the interface at the same height in the two reservoirs) as
subcritical flows cannot support stationary hydraulic jumps. As
no driving potential energy exists, the flow will everywhere be
zero.

We are therefore able to conclude that, in the absence of a
net forcing, if the two interface heights are equal the flow will
be zero everywhere. Moreover, if the interface heights in the two
reservoirs are not equal there must be an hydraulic transition at
some section along the channel. Similar arguments apply to channels containing a net barotropic flow.

Figure 3.9 shows how the interface profile may vary in a channel of constant depth. Solid lines represent supercritical solution branches and dashed lines subcritical branches. For all curves the flow passes through an hydraulic transition at the narrowest point in the channel (indicated by an arrow). The unique controlled solution, and the associated subcritical root, are shown by heavier lines than the other curves. If the interface height in the left-hand (dense) reservoir is above half the channel depth, then it may be matched onto only by an hydraulic jump forming from the associated supercritical flow (which gives the position of the interface closer to the top of the channel). Likewise, if the interface height in the right-hand (less dense) reservoir is below half the channel depth, it may only be matched onto by an hydraulic jump from the associated supercritical solution closer to the bottom of the channel.

On the other hand, if the dense reservoir interface height is less than half the channel depth, a subcritical solution exists which may match the interface onto the less dense reservoir via an hydraulic transition and hydraulic jump. Similarly, for interface heights in the less dense reservoir greater than half the channel depth, a subcritical solution exists which may pass through a transition and a jump to match onto the right-hand reservoir. Such solutions may be considered as partially controlled in the sense that they are isolated from the reservoir whose depth is further from the channel midpoint, but depend directly on the interface height in the other reservoir. Whether the partial control or the complete hydraulic control occurs depends on the relationship between the interfaces in the reservoirs and the height of the subcritical solution curve which corresponds to fully controlled flow (in figure 3.9 this curve is simply the half channel depth - shown as a heavy dashed line). If the dense reservoir interface lies above and the light reservoir lies below this curve, then the flow will be fully controlled; if not, then only partial control is possible. For constant depth channels this requirement is identical to that given by Armi & Farmer (1986); however in more complex geometries the two are no longer equivalent.

When the two controls of the fully controlled solution are
Figure 3.9. Interface profiles for flow through a channel of constant depth. The heavy lines denote the fully controlled (maximal) solution and its associated subcritical root. Light lines represent sub-maximal flows. Continuous lines indicate supercritical solution branches and dashed lines subcritical branches. See text for more details.

Figure 3.10. Interface profiles for flow over a simple sill. The heavy lines denote the fully controlled (maximal) solution and its associated subcritical root. Light lines represent sub-maximal flows. Continuous lines indicate supercritical solution branches and dashed lines subcritical branches. See text for more details.
separated, as occurs in the flow over a simple sill for example, the picture is somewhat more complicated. Figure 3.10 shows the various solution curves for flow over a simple sill, again with $Q = 0$. As in figure 3.9, supercritical solutions are shown as continuous lines and subcritical solutions as dashed lines. The fully controlled solution and its associated subcritical root, are given by heavy lines. As with the channel of constant depth, full hydraulic control requires the interface in the dense reservoir to lie above, and in the light reservoir to lie below, the heavy dashed line (subcritical solution corresponding to the fully controlled flow) within the corresponding reservoir; an hydraulic jump may then form to match the supercritical flow onto the reservoir height.

If the interface lies below the heavy dashed line in the dense reservoir, the exit control will be flooded and the flow will be subcritical everywhere to the left of the crest of the sill. As the lower layer passes over the crest, the flow undergoes a transition to supercritical flow into the light reservoir, isolating any subsequent hydraulic jump to match the reservoir interface height. In the extreme, if the reservoir interface height is below the crest of the sill, then no exchange flow will occur (when $Q = 0$).

When the interface in the light reservoir is above the heavy dashed curve, it is the sill crest control which is flooded. While the flow is critical at both $x = -1$ and $x = 1$ (due to the symmetry of the geometry), an hydraulic transition will occur only on the $x = -1$ side of the sill crest - the flow over the crest itself must remain subcritical in order to match onto the right-hand reservoir. The flow out into the dense reservoir is supercritical, isolating the necessary hydraulic jump from the partial control mechanism.

Note that in all cases when hydraulic control is lost at one of the two control sections, the interface height at the section where critical conditions are maintained has an associated value of $A$ further from the value of $A_{\text{max}}$ for that section (see next section for more details - here $A_{\text{max}} = 0$ as a consequence of $Q = 0$) than the value of $A$ for the fully controlled solution. Thus the exchange flow rate for all partially controlled solutions is less than that for the fully controlled solution, and so the fully
controlled solution represents the maximal exchange flow.

In this thesis we are concerned primarily with the fully controlled flow where the details of the reservoirs are isolated from the control mechanism by supercritical regions, and not the partially controlled solutions, discussed in this section, except as the limit of validity of the fully controlled flows. From here on we shall utilise the term controlled to mean the fully controlled flows. In section 6.5 we shall return to the loss of hydraulic control for a rotating channel and discuss the possibility of geostrophic control for two-layer flows.

### 3.6 Net barotropic flow in channels of constant depth

The introduction of a net barotropic flow through a channel of constant depth breaks the inherent symmetry of the channel geometry, therefore one would not expect the interface to be at half the channel depth at the narrowest point in the channel. However, a naive application of the maximal exchange hypothesis may lead one to believe the interface is positioned so as to maximize the exchange flow rate (equation (3.5)), requiring

\[ A_C = A_{\text{max}}(x=x_C=0) \]

where

\[ A_{\text{max}} = \frac{Q}{(2 D^{3/2} b)}. \]  

(3.18)

In addition to \( A_{\text{max}} \) being a solution to \( \partial J/\partial A = 0 \), it is a solution to \( \partial^2 J/\partial A^2 = 0 \), and so \( A_{\text{max}} \) corresponds to the two controls coinciding. By considering how the functional varies away from the contraction \( x = x_C = 0 \), we are able to note that two of the roots to \( J = 0 \) vanish (when \( A_C = A_{\text{max}} \) near \( x = x_C \) unless \( Q = 0 \), and so it would not be possible to trace the controlled solution to one of the reservoirs. Due to this, unless \( Q = 0 \), the two controls do not coincide and \( A_C \neq A_{\text{max}} \).

Consider equation (3.7). As in section 3.2 \( K \) vanishes with \( db/dx = 0 \) at \( x = 0 \), whereas \( \partial J/\partial b \) becomes

\[ \frac{\partial J}{\partial b} = \frac{-1}{2 D^2 b^3} \frac{A (\bar{q}^2 + Q^2) - 2 (\bar{q} + A^2) Q \bar{q}}{(\bar{q} - A^2)^2}. \]  

(3.19)

Setting (3.19) to zero and solving for \( A \), noting that we require \( A \)
to be finite as $Q \to 0$, informs us that the relevant solution is $A = A_B$, where

$$A_B = \frac{1}{2} Q / \bar{Q}. \quad (3.20)$$

Equation (3.20) is independent of $x$. Thus we are able to assert that at $x = x_C = 0$ there exists a supercritical root with $A = A_B$ and a critical root with $A = A_C$. Solving $J(A=A_B) = 0$, $J(A=A_C) = 0$ and $\partial J/\partial A = 0$, with $A = A_C$, simultaneously for $A_C$, $\zeta$ and $\bar{Q}$ will give us the conditions at the constriction control ($x_C = 0$). Subsequent solution of $\partial J/\partial A = 0$ with $A = A_B$ for $b_V$ gives the position of the virtual control at which $A_V = A_B$. We shall show shortly that $\bar{Q} < (2/3)^{3/2} < 1$ (for all values of $Q$ resulting in two flowing layers) so that equations (3.19) and (3.20) show that $|A_B| > |A_{\max}|$. This means the overall interface slope requires the virtual control to be upstream (with respect to the net barotropic forcing) of the contraction.

Since $A_C$ and $A_V$ must be on opposite sides of their respective values of $A_{\max}$ (the value for which a triple root would occur), we are able to assert that $A_C$ is bounded between zero and $\frac{1}{2} Q$, while $A_V$ is between $\frac{1}{2} Q / b < \frac{1}{2} Q$ and $\frac{1}{2} \text{sign}(Q)$ (where $\text{sign}(Q)$ is 1 if $Q > 0$, -1 if $Q < 0$ and zero for $Q = 0$). At both controls the interface is on the same sides of the midpoint of the channel, though is closer to this half-depth point at the primary control than at the virtual control.

As the net barotropic forcing is increased, both $|A_C|$ and $|A_V|$ will increase, taking the same sign as $Q$. Eventually, as $|Q|$ is increased, $|Q|$ will reach some value $Q_t < 1$ for which $|A_V|$ is $\frac{1}{2}$ and the corresponding layer thickness vanishes. Solving the necessary equations reveals that

$$Q_t = (2/3)^{3/2}, \quad (3.21)$$

at which point $|A_C| = 1/6$ and $b_V \to \infty$. If the strength of the net forcing is increased any further, the virtual control disappears and the flow becomes a single-layer flow with an overlying ($Q > 0$) or underlying ($Q < 0$) passive layer. The unique control is at the contraction, where the interface is then related to the net flow by
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\[ Q^2 = (\frac{\eta}{2} + |A_c|)^3, \]  

(3.22)

and is subcritical upstream and supercritical downstream of this point. Increasing the barotropic forcing to \( |Q| = 1 \) will cause the stagnant layer (upper layer if \( Q < 0 \) and lower if \( Q > 0 \)) to vanish from the contraction as well, removing all mechanisms for hydraulic control in the process. For \( |Q| > 1 \) the flow is simply that through a duct on the upstream side of the constriction, though it may regain a stagnant second layer downstream of the constriction if reservoir conditions allow.

Figures 3.11 and 3.12 show how the flow through this channel varies as a result of the net barotropic forcing. Notice in figure 3.11 the vanishing of \( q_1 \) for \( Q < -0.5443 \) (i.e. \((2/3)^{3/2}\)), and of \( q_2 \) for \( Q > 0.5443 \). This corresponds to \( b_V \to \infty \) and \( |A_V| \to \frac{\pi}{2} \) in figure 3.12. For larger values of \( |Q| \), figure 3.12 plots the width of the channel at the front where one of the layers vanishes, in addition to the width of the channel at the virtual control \( (b_V) \) when \( |Q| < 0.5443 \). Interface height coefficients at the primary and virtual controls are also plotted on figure 3.12 as and where appropriate.

These results are identical to those obtained by Armi & Farmer (1986) using a Froude number plane formulation of the problem. In the next section we shall analyse the response of the flow over a sill to net barotropic forcing. As with the \( Q = 0 \) limit, this flow differs in a fundamental manner from that discussed in this section.

3.7 Flow over a sill with net barotropic forcing

The response to net barotropic forcing of the flow over a sill (such as the sill illustrated in figure 3.3) is more complicated than that through a contraction. The flow may behave either like forced flow through a contraction or unforced flow over a sill, depending on how deep the channel becomes away from the sill crest, and on the magnitude and sign of \( Q \). The feature distinguishing these two types of behaviour is the position of the virtual control. We shall use the term contraction-like behaviour to describe flows in which the virtual control is upstream (with respect to the net barotropic flow) of the sill crest, positioned...
Figure 3.11. The effect of net barotropic forcing on the flow rates through a channel of constant depth.
Figure 3.12. The effect of net barotropic forcing (Q) on the flow through a channel of constant depth. Interface height coefficients at the contraction ($A_c$) and virtual control ($A_v$) are plotted as solid lines. The dashed line is the width at the virtual control ($b_v$) for $|Q| < (2/3)^{3/2}$, and the width of the front after stagnation of one layer when $|Q| > (2/3)^{3/2}$.
somewhere in the expanding region of the channel. In contrast
* sill-like behaviour* has the virtual control at \( x = -1 \) on figure
3.3, as would be the case if no net forcing were present. For
contraction-like behaviour the solution to \( K = 0 \) at the virtual
control is due to the term \( \partial J/\partial b \) vanishing (\( dD/dx = 0 \)), whereas
sill-like behaviour corresponds to both \( db/dx \) and \( dD/dx \) vanishing
simultaneously. The primary control is always at the sill crest.

Whether the channel, for a given geometry and net forcing \( Q \),
behaves like a sill or a contraction may be established by
evaluating the exchange flow rate for each type of behaviour. The
realised value of \( Q \) will be the lower of the two. In general the
equations must be evaluated numerically, the exception being for
the values of \( Q \) associated with one or the other of the layers
stagnating. Calculation of these values is carried out by putting
\( Q = \bar{Q} \) and \( AV = \bar{Q} \) into equations (3.3) and (3.5), and equating
for the two control sections. For \( Q > 0 \) this proves to occur at
precisely the same values as in the channel of constant depth,
namely \( Q = Q_t = (2/3)^{3/2} \) with \( A_c = 1/6 \). The value is independent
of the depth of the channel away from the sill crest. However if
\( Q < 0 \), the lower layer first stagnates when \( Q = -(2 \times 5/3)^{3/2} \),
corresponding to \( A_c = \frac{5}{2} - 2 \times 5/3 \) provided \( D_w < 3/2 \). This proviso
is recognising that for \( D_w > 3/2 \) the lower layer will vanish at
the sill crest before any virtual control at infinity. For both
\( Q > 1 \) and \( Q < -1 \) the flow at the sill crest will be single-layer.

The restriction that \( D_w < 3/2 \) for the lower layer to first
vanish at infinity is the same as the single layer hydraulics
result that the free surface/interface height over a weir is two
thirds that in the upstream reservoir. Moreover, it demonstrates
that contraction-like behaviour with \( Q < 0 \) is possible only if
\( D_w < 3/2 \). The vanishing of the lower layer at the sill crest for
\( D_w > 3/2 \) does not correspond to a sill-like solution as such a
solution would require the flow to remain critical at the foot of
the sill. Instead there has been a bifurcation to what we shall
term coincident behaviour with both controls at the sill crest and
\( A_c = A_{max} \). Note that since this bifurcation coincides with the
lower layer being brought to rest it does not introduce any new
features for the present rectangular channels. However it is of
direct importance in parabolic cross-sections (see section 5.3)
and rotating rectangular channels (see section 8.5). The
bifurcation from sill-like to coincident behaviour does not produce any jumps in the interface profile - the only jump is in the position of the virtual control.

While explicit solutions of the necessary equations are not possible, we are able to give bounds on the interface height at the sill crest and virtual control. In particular, when \( Q > 0 \) the interface height at the sill crest must be bounded between \( A_{co} \) and \( A_{\text{max}}(x=x_C) \), where \( A_{co} \) is the interface height in the absence of any barotropic forcing (\( Q = 0 \)) and \( A_{\text{max}} \) is given by equation (3.18). Similarly \( A_v \) is bounded between \( A_{vo} \) (the corresponding \( Q = 0 \) height in the same geometry) and \( \frac{1}{2} \). These bounds hold for \( Q > 0 \) regardless of whether the flow is sill-like or contraction-like in behaviour, the transition between the two being a continuous phenomenon (i.e. there are no jumps in the interface height or the position of the virtual control).

If the net forcing is negative, then the situation is somewhat more complicated. For sill-like behaviour (\( D_W > 3/2 \) or weak barotropic forcing), \( A_C \) will be bounded between the lesser of \( A_{co} \) and \( A_{\text{max}}(x=x_C) \), and \( -\frac{1}{2} \). At the virtual control, \( x_V = -1 \), \( A_V \) will be between \( A_{\text{max}}(x=x_V) \) and \( A_{vo} \). If \( D_W < 3/2 \) and the net forcing is sufficiently negative, contraction-like behaviour gives \( A_C \) in the range \( 0 \) to \( A_{\text{max}}(x=x_C) \), \( x_V > 1 \) and \( A_V \) in \( A_{\text{max}}(x=x_V) \) to \( -\frac{1}{2} \).

Figure 3.13 shows how the exchange flow rate varies with the net barotropic forcing for a number of different values of \( D_W \). Notice that the bifurcation from sill-like to contraction-like behaviour for \( Q < 0 \) with \( D_W = 1.1 \) (dashed lines) results in a sharp change in the slope of the flow rates. This is to be expected as \( \tilde{q} \) is essentially the envelope of solutions with the virtual control at \( x_V = -1 \) and those with \( x_V > 1 \). Such a discontinuity in the slope does not occur for the \( Q > 0 \) bifurcation from sill-like to contraction-like behaviour as the virtual control moves continuously with \( Q \), remaining at \( x = -1 \) until some threshold value is exceeded, then moving further from \( x = 0 \) with increasing \( Q \). As such, both the flow rates and the \( \tilde{q} \) are continuous at this bifurcation. For comparison \( \tilde{q}_{\text{max}} \), the value that \( \tilde{q} \) would take if \( A_C = A_{\text{max}} \) is also shown (dotted line) in figure 3.13.

The height of the interface, at the sill and virtual
Figure 3.13. The effect of net barotropic forcing on the flow rates over a simple sill. Solid lines for $D_w = 1$; long dashes for $D_w = 1.1$; dot-dash for $D_w = 1.5$. The exchange flow rate, if the two controls were to coincide at the sill crest, is also plotted (dotted line).
Figure 3.14. Variations in the interface height coefficient at the sill crest ($A_c$, solid lines) and virtual control ($A_v$, dashed lines) as a function of the net barotropic forcing. The value of $D_w$ (sill geometry) is marked on the $A_v$ curves. Bifurcations from contraction-like to sill-like behaviour, for $Q < 0$, are shown by vertical dashed lines. For $Q > 0$ the bifurcations correspond to sudden changes in the slope of $A_v$. 
controls, as a function of $Q$ for a number of different values of $D_w$ is shown in figure 3.14. Notice the discontinuities in both $A_C$ and $A_V$ at the bifurcation from sill-like to contraction-like behaviour for $Q < 0$. Even though the exchange flow rate is a continuous function of the net forcing, the interface profile is not. Some care should be exercised in considering the discontinuity in $A_V$ as a significant portion of this is due to the associated jump in $x_V$ from $x_V = -1$ to $x_V > 1$.

The $Q > 0$ bifurcation is marked by a sharp change in slope in the height of the interface at the virtual control, corresponding to its position becoming dependent on $Q$, though $A_C$ changes in a smooth manner.

A plot of $A_C$ and $A_V$ versus $D_w$ for a number of different values of the net forcing is shown in figure 3.15. The horizontal scale is logarithmic in order to expand the details for $D_w$ near unity while retaining the asymptotic limits for large $D_w$. This plot also shows the bifurcation from sill-like to contraction-like behaviour. When the flow is sill-like there exists a corresponding contraction-like solution which can be viewed as an unstable branch of the solution; likewise for contraction-like solutions there is an unstable sill-like branch. These unstable branches are shown on 3.15 (dashed lines) in addition to the realised (stable - solid lines) branches. Notice that for all values of $D_w$ there exists a sill-like branch which behaves in a manner qualitatively similar to that for a sill without forcing and taking the value of $A_{max}(x=0)$ in the limit $D_w = 1$.

For all values of the net flow rate $Q$ less than that for stagnation, the interface height over the sill is essentially that for $D_w \to \infty$ if $D_w$ is greater than around 2.5. At the virtual control the interface height coefficient $A_V$ deviates from $\frac{1}{2}$ like $D_w^{-\frac{1}{2}}$, reflecting a constant height of the interface above the channel datum.

Figure 3.16 is a phase diagram for this flow, showing the relationship between $D_w$ and $Q$ for which the bifurcation from contraction-like (the area beneath the $D_w$ curve) to sill-like behaviour occurs. The corresponding width at the virtual control is also plotted. Both curves are shown as dotted lines for $Q > 0$, to indicate the bifurcation does not produce a sudden change in the appearance of the flow, unlike the $Q < 0$ bifurcation. The
Figure 3.15. Variations in the values of $A_V$ (upper half of curve) and $A_C$ (lower half) as the sill height is altered. Curves are plotted for a number of different values of the net forcing $Q$ (as marked). The solid lines are (stable) realised solution branches. Dashed lines are unstable sill-like branches. Dotted lines are unstable contraction-like branches. Dot-dash lines indicate a bifurcation from contraction-like to sill-like behaviour.
Figure 3.16. Phase diagram for a simple sill showing the relationship between $Q$ and $D_W$ for the bifurcation from contraction-like (below $D_W$ curve) to sill-like behaviour (above $D_W$ curve). The width at the virtual control for the contraction-like solution, at this bifurcation, is also plotted ($b_V$ curve). Solid lines for $Q < 0$ indicate the bifurcation causes a jump in the values of $A_c$ and $A_v$; dashed lines for $Q > 0$ indicate no such jump occurs.
Figure 3.17. Changes in the interface height coefficients at the bifurcation point. The channel is a simple sill with $D_w$ such that for the given value of $Q$ the flow is at the behavioural bifurcation point. Thus the difference between curves (i) and (iii) and between (ii) and (iv) represent the jump in the interface height due to the bifurcation from sill-like to contraction-like behaviour. For $Q > 0$ (shown dashed) there is no jump.
corresponding values of the height parameters $A_{c}$ and $A_{v}$ are shown in figure 3.17, again as dotted lines for $Q > 0$.

In the limits $D_{w} = 1$ and $D_{w} \rightarrow \infty$, the present analysis agrees with the findings of Farmer & Armi (1986). The results for finite sill heights and the bifurcations in the behaviour of the flow are new. In the next section we shall determine how the influences of contractions and sills combine to affect the flow through a channel of more complex geometry.

3.8 Channels with a separated contraction and sill

A large number of oceanic channels are characterised by both a contraction and a sill, often spatially separated. It is therefore of interest to investigate how this combination alters the control mechanisms for the flow. In its simplest form, such a channel could be viewed as consisting of a single sill (unity depth at its crest) in a region of channel of constant width ($b = 1$) matching on to a single contraction (width $b = b_{n}$) in a region of constant depth ($D = D_{n}$). Away from the sill and contraction the depth is equal to that in the contraction and the width increases into the reservoirs at each end. Such a channel is pictured in figure 3.18. As with the earlier channels, the along-channel length scale is of little importance. The values $x = -2$ for the channel exit, $x = -1$ for the sill crest, $x = 0$ for the mid point, $x = 1$ for the contraction and $x = 2$ for the other channel exit have been introduced as a means of identifying certain regions of the channel. The figure shows the denser reservoir on the left (we shall identify this situation as CS1), though we shall consider also the reverse situation where the denser layer flows through the contraction before reaching the sill. This second situation will be identified as CS2. The two flows differ in a fundamental manner.

Note any additional sills/contractions would not fundamentally alter the flow as the flow will always be controlled by two and only two hydraulic transitions. A flow with a supercritical region, bounded on each side by subcritical regions, would not be stable as small amplitude waves originating in the subcritical regions would become trapped in the supercritical
Figure 3.18. Schematic elevation (top) and plan (bottom) views of a channel with a separated sill and contraction. Interface shown for CSI1 configuration - see text.
region and bring about a change in the flow. Only if one or more hydraulic jumps are present is it possible for there to be more than two sub/supercritical transitions. Thus the flow over multiple sills and/or through multiple contractions will have a region of subcritical flow bounded on each side by the control sections (this subcritical region may contain additional sills and/or contractions). As before, the control sections are determined by the need to minimize the exchange flow rate whilst maintaining critical conditions. The supercritical regions may extend all the way to the reservoirs from the control sections, or they may pass through a series of hydraulic jumps (the jumps are isolated from the control sections by the supercritical flow). If the flow passes through jumps there may be subsequent hydraulic transitions to restore supercritical flow over the outlying sills and/or contractions - these subsequent transitions do not affect or alter the hydraulic control of the solution. The ideas of multiple sills will be developed further in chapter 10 with reference to the flow through the Strait of Gibraltar.

By considering the roots of equation (3.12), it is clear that one of the control sections must be either at the sill crest \((x = -1)\) or the contraction \((x = 1)\). The second control may coincide with the first, be at the other geometric constriction, or be in one of the regions of expanding channel width \((x < -2\) or \(x > 1)\). The precise position will depend on the values of the contraction width \(b_n\), the maximum channel depth \(D\), and the net barotropic flow rate \(Q\). Some light can be shed on the positioning of the controls by considering the strength of barotropic forcing required to bring one of the layers to rest. Calculation of these is achieved in the same manner as for the simpler channels earlier in this chapter.

For \(CS1\) with \(Q > 0\), the upper layer first vanishes at the virtual control positioned at (negative) infinity \((b_v \to \infty)\). The required strength of the forcing will be the smaller of \((2/3)^{3/2}\) and \(b_n \left(2D_w/3\right)^{3/2}\), the former corresponding to the primary control being located at the sill crest \((x_C = -1)\) while the latter corresponds to the primary control at the contraction \((x_C = 1)\). For both cases the flow behaves like that through a contraction (the effective "narrowest" point being \(x_C = -1\) and \(x_C = 1\) respectively for the two possibilities) and the interface height
coefficient is $A_c = 1/6$. If the primary control is located at the same position for $Q = 0$, then it will be located at that position for all intermediate values of $Q$; if not, then a bifurcation in the flow behaviour must occur for some value of $Q$. The isolation of the geometric features does not allow the primary control to move continuously from one geometric feature to the other.

With negative forcing, the lower layer may first vanish at either the virtual control, with the contraction the primary control (analogous to the flow through a simple contraction), or at the sill crest (analogous to flow over a simple sill with $D_w > 1.5$). If the lower layer vanishes at the sill crest then the behaviour of the flow must bifurcate from sill-like behaviour with $x_v = -2$ to coincident behaviour. If $b_n \left(2D_w/3\right)^{3/2}$ is less than unity then the contraction will be the primary control with a net flow rate of negative this quantity, otherwise the sill dictates $Q = -1$. Again, if the primary control is in the same position for $Q = 0$, then it will be the primary control for all intermediate values; if not then a bifurcation in its position must occur for some value of the net flow rate.

Figure 3.19 plots how the interface height coefficients at the sill and contraction vary as a function of the maximum channel depth, $D_w$, for CS1 with a number of different values of the contraction width ($b_n$). There is no net barotropic forcing for the flows in this figure. For any given value of $D_w$ there exists a range of contraction widths for which the channel acts like a contraction or like a sill. In the limit of $b_n = 1$, contraction-like behaviour exists only for a vanishingly small sill, whereas $b_n \to 0$ requires $D_w \to \infty$ in order to attain sill-like behaviour. For all intermediate values there is a bifurcation in behaviour associated with changing the channel geometry.

Arguably CS2 is of greater physical importance as it is a basic model for the Strait of Gibraltar: the dense Mediterranean water flows through the Tarifa Narrows before dropping over the Camarinal Sill. Farmer & Armi (1986) utilised such a model with $D_w = \infty$. The analysis in this section extends their work to finite sill heights.

Analysis of the $Q < 0$ (the net flow is still from light to dense reservoir but passes over the sill first) forcing required to bring the lower layer to rest in CS2 introduces the additional
Figure 3.19. Variations in the height coefficients at the sill and contraction (as marked) as the depth, at the contraction ($D_w$) of a CS1 channel, is varied. The curves are for $Q = 0$; contraction widths ($b_n$) as marked. Solid curves indicate sill-like behaviour; dashed curves for contraction-like behaviour. Bifurcations in the behaviour are shown by dot-dash lines.
possibility of the lower layer first vanishing at the sill crest with the primary control at the contraction. Setting \( Q = -\tilde{Q} \) and \( A_V = -\mathcal{V} (x_V = -1) \) in equations (3.3) and (3.5) and solving \( J(x = x_c) = J(x = x_V) \) with critical conditions (i.e. (3.5) holds) at both \( x_c \) and \( x_V \) gives the required net forcing as

\[
Q = - b_n a^{3/2},
\]

where \( a \) is the root of

\[
a^3 - (3/b_n^2) a + 2/b_n^2 = 0. \tag{3.23}
\]

This value of \( Q \) must be smaller in magnitude than the \( Q = -b_n \left(2 D_w/3\right)^{3/2} \) required for the primary control to be at the contraction and the layer to vanish at infinity, or the \( Q = -1 \) required for vanishing at the sill crest alone. Figure 3.20 shows how the control mechanism for large \(-Q\) changes with the geometry. Region I is contraction-like behaviour and region II a combination of the contraction and the sill. Region III is the coincident behaviour of a simple sill which is able to occur only when no contraction is present \((b_n = 1)\). Equating \( Q \) of (3.23) with \( Q = -b_n \left(2 D_w/3\right)^{3/2} \), the value for contraction like behaviour, yields the curve

\[
b_n^2 = \left(27/4\right) \left(D_w - 1\right)/D_w^3. \tag{3.24}
\]

The solid line on figure 3.20 represents a portion of the curve given by (3.24) signifying the transition from region I to region II. The dashed line is also a solution to (3.24), but is not realised as it corresponds to a different solution of (3.23) with a corresponding value of \(|Q|\) greater than that for the realised solution to (3.23).

For \( Q > 0 \) the upper layer will vanish with \( Q \) the lesser of \((2/3)^{3/2}\) (for primary control at the sill and virtual control in the dense reservoir \( x_V \to \infty \)) and \( b_n \left(2 D_w/3\right)^{3/2} \) (for primary control at the contraction and \( x_V \to \infty \)). Again by considering the behaviour at \( Q = 0 \), in addition to that for large \(|Q|\), we are able to make predictions as to whether or not a bifurcation in behaviour exists.

As with figure 3.19, figure 3.21 shows how the interface height at the sill and contraction varies as a function of \( D_w \), for
Figure 3.20. Phase diagram for type of behaviour when lower layer is just brought to rest by the barotropic forcing $Q$ in CS2 channels. Region I has the virtual control in the reservoir and primary control at the contraction; region II also has the primary control at the contraction, but the virtual control is at the sill crest when the layer first vanishes; region III has the layer first vanishing at the sill crest (there is no contraction). The solid line separates regions I and II; the dashed line represents an unrealised solution of equation (3.24).
a number of different contraction widths $b_n$ in the absence of net barotropic forcing. For $b_n = 1$, the flow is simply that over a sill as given in section 3.3. The primary control is at the sill crest and the virtual control at the foot of the sill where the channel begins to widen (i.e. $x_v \in [0,2]$, the geometry being constant over this range for $b_n = 1$). For any contraction width $b_n < 1$, there exists a range of values of the depth at the contraction, $D_w$, for which the flow is controlled by the contraction. Clearly in the limit $D_w = 1$ contraction-like behaviour exists (shown by dashed lines) with both controls at $x = 1$. As $D_w$ becomes large (for nonzero $b_n$), there will be a bifurcation to sill-like behaviour (solid lines) with controls at $x = -1$ and 1.

It can be shown that for $D_w > 2$, the flow will always be sill-like, regardless of the value of $b_n$. Justification of this result is straightforward: contraction-like behaviour gives $A_c = 0$ at the contraction (when $Q = 0$); for $D_w > 2$ the sill effectively introduces a barrier preventing the lower layer from flowing on towards the light reservoir, forming instead an internal reservoir which will continue to fill with dense fluid until the fluid level reaches the crest of the barrier (sill) and can adopt critical conditions over it. At some stage during the filling process the fluid level will exceed that at the contraction control and flood it.

Figure 3.22 shows on the $D_w$-$b_n$ plane where bifurcations between contraction-like behaviour (to the lower left of the curves) and sill-like behaviour (upper right) are expected to occur for both CS1 and CS2. The difference in the two curves stems from the ability of the sill to flood the contraction for CS2 (if $D_w$ is sufficiently large) regardless of the width of the contraction.

Figure 3.23 shows how the exchange and layer flow rates vary as a function of the net barotropic forcing for CS2 with a number of different channel geometries. The depth at the contraction, $D_w$, is held at 1.25 (this value admitting the possibility of a bifurcation with no net flow), while the contraction width, $b_n$, varies from unity to 0.2. As expected, the flow rates decrease as the contraction gets narrower. The sharp change in the slope of the graph represents a bifurcation from sill-like to
Figure 3.21. Variations in height coefficients at the sill (lower curves) and contraction (upper curves) as the depth, at the contraction ($D_w$) of a CS2 channel, is varied. The curves for $Q = 0$ and a number of contraction widths ($b_n$ as marked) are shown. Solid lines indicate sill-like behaviour and dashed lines contraction-like behaviour. Bifurcations do not result in a jump in the interface height.
Figure 3.22. Phase plane showing the relationship between $D_w$ and $b_0$ for a bifurcation to occur with $Q = 0$. Channels below a curve have contraction-like and above the curve sill-like behaviour. The upper curve is for the denser layer flowing over the sill before through the contraction (CS1); the lower curve has the flow the other way around (CS2).
Figure 3.23. Variations in the flow rates in response to net barotropic forcing (Q) in a number of different contraction-sill geometries. All curves are for $D_w = 1.25$. Solid lines for $b_h = 1$; long; dashes for $b_h = 0.8$; short dashes for $b_h = 0.6$; dot-dash for $b_h = 0.4$; dot-dot-dash for $b_h = 0.2$. The lower layer flows through the contraction before going over the sill (i.e. CS2).
contraction-like behaviour. The value of $Q$ for which this bifurcation occurs decreases as the contraction gets narrower.

In the limit $D_w \to \infty$ the results presented here agree with those of Farmer & Armi (1986). The effects of finite $D_w$ and the associated bifurcation structure are new. In the next section we investigate how simultaneous changes in the channel width and depth can alter the control mechanism.

3.9 Simultaneous sills and contractions

The final class of rectangular geometry we shall deal with briefly is channels which possess a sill and a contraction concentric with each other. In particular, the narrowest and shallowest points of the channel coincide, at $x = 0$ say, both having unity values. Away from this geometric constriction, both increase monotonically towards their reservoir values. The flow through a contraction and flow over a sill considered in sections 3.2 to 3.6 are simply special limits of this situation. Farmer & Armi (1986) looked briefly at this problem but did not reach the correct conclusion about the flow: they stated that for $Q = 0$ both controls would be at the sill crest and $A_c$ (in our notation) would vanish. They did not study the effects of barotropic forcing through such a channel.

As with other channels, under most circumstances the present channel will exhibit two control sections, one of which is positioned at $x = 0$. For no net barotropic flow we would expect the virtual control to be somewhere upstream, with respect to the lower layer, of $x = 0$ as with an isolated sill. Suppose the geometry is described by

$$D(x) = S(b(x)), $$
$$H(x) = \frac{1}{2} - D(x). $$

The constriction in the hydraulic functional is therefore given by

$$K = \left[ \frac{\delta J}{\delta b} - \frac{\delta J}{\delta H} \frac{db}{dx} + \frac{\delta J}{\delta D} \frac{db}{dx} \right] db = 0, \quad (3.26)$$

which may, in principle, be used to obtain the equations required
to determine the position of the virtual control and the interface heights. As the necessary algebra will vary depending on the form of $b(b)$, and moreover the resulting equations require numerical evaluation, we shall only illustrate typical results with a simple example.

One of the simplest functionals relating the channel depth to the width is the linear relation

$$D(x) = b(b(x)) = 1 + \lambda(b(x) - 1). \quad (3.27)$$

The choice of $b(x)$ is arbitrary, so long as it is slowly varying and increases monotonically (or nearly monotonically) over a sufficiently large range of $x$ away from $x = 0$. The depth in channels with $\lambda = 0$ is constant, and are thus those of sections 3.2 and 3.6. The $\lambda \to \infty$ limit is similar to the simple sills of sections 3.3 and 3.7 with $D_w \to \infty$, though differs in that $dD/dx$ and $db/dx$ does not vanish for any $D \neq 1$. Thus the virtual control will be fixed by a combination of all the terms in equation (3.26). Channels with intermediate values of $\lambda$ have both their width and depth varying simultaneously, depth variations being more pronounced with larger values of $\lambda$.

Figures 3.24 to 3.26 show how the flow changes as the value of $\lambda$ is increased from zero in the absence of a net barotropic forcing. Figure 3.24 plots the interface height coefficient for the sill and virtual controls. The higher the value of $\lambda$ the closer to the channel bottom the interface becomes at the primary control. In the limit as $\lambda \to \infty$ the curve asymptotes to a value of $A_c$ slightly greater than the 0.12544 for an infinite sill. Even in this limit the flow retains some contraction-like character as can be seen from the plot of $b_x$ in figure 3.25. The $b_x$ curve reflects the movement of the virtual control away from the primary control as the channel becomes more sill-like and is essentially constant, but greater than unity, for $\lambda$ greater than approximately 2. The depth $D_x$ is related to $b_x$ through (3.27) and varies linearly with $\lambda$ for sufficiently large $\lambda$. Figure 3.26 shows how the exchange flow rate decreases from the $\bar{Q} = \frac{1}{2}$ for a contraction with the more sill-like behaviour at larger $\lambda$ (though remains greater than the infinite sill value of $\bar{Q} = 0.41598$).

The response to net barotropic forcing is akin to that in a channel of constant depth in that there is no bifurcation in the
Figure 3.24. Variations in $A_C$ and $A_V$ due to the sill strength $\lambda$ in channels with a simultaneous sill and contraction. There is no net flow along the channel ($Q = 0$).
Figure 3.25. Variations in the geometry at the virtual control due to the sill strength $\lambda$ in channels with a simultaneous sill and contraction. There is no net flow ($Q = 0$).
Figure 3.26. Variations in the exchange flow rate due to the sill strength $\lambda$ in channels with a simultaneous sill and contraction. There is no net flow ($Q = 0$).
behaviour of the flow. As Q is increased from zero, the virtual control moves towards the denser reservoir, shifting even for arbitrarily small values of Q. As with both constant depth channels and those containing a sill, the upper layer will first be brought to rest when \( Q = \left(\frac{2}{3}\right)^{3/2} \). At this value the virtual control is in the reservoir with \( b_v \) and \( D_v \) both tending to infinity for non-zero \( \lambda \) (\( \lambda = 0 \) would give \( b_v \to \infty \), \( D_v = 1 \)).

The front formed by the upper layer vanishing moves back towards the primary control (still at \( x = 0 \)) as Q is further increased, the relationship between geometry at the front (\( D_f, b_f \)) and barotropic forcing being

\[
3 Q^{2/3} - \left(\frac{Q}{D_f b_f}\right)^2 + 2 = 0, \quad (3.28)
\]

while \( Q < 1 \). Equation (3.28) is obtained from putting \( Q = \bar{q} \) into equations (3.3) and (3.5), equating \( J(x=x_c) \) with \( J(x=x_f) \) and eliminating \( A_c \) using (3.5). The upper layer vanishes at \( x = 0 \) when \( Q = 1 \). Stronger forcing pushes the front towards the lighter reservoir as

\[
Q^2 = D_f^3 b_f^2, \quad (3.29)
\]

with the flow critical at the front. In the single-layer region upstream (with respect to the net flow) of the front, sub/supercriticality has no meaning: long, small amplitude gravity waves cannot exist because of the rigid lid. Downstream of the front the flow is supercritical (\( F_2^2 > 1 \)).

As Q is decreased from zero, the virtual control moves towards the lighter reservoir. At small \( |Q| \) the virtual control is downstream (with respect to the net flow) of the contraction and shifts continuously towards the contraction (\( x = 0 \)) as \( |Q| \) is increased. The virtual control will always coincide with the primary control for some value of \( Q \leq 0 \). Decreasing Q further will push the virtual control further towards the light reservoir. Eventually for some value of \( Q \leq 1 \) the lower layer will be brought to rest. For \( \lambda = 0 \) this will occur with the virtual control in the light reservoir (\( b_v \to \infty \)), while for \( \lambda = \infty \) it will occur at the sill crest (\( b_v = 1, D_v = 1 \)). Intermediate values of \( \lambda \) lead to the layer first vanishing at finite values of \( D_v \) and \( b_v \). Solutions of \( \lambda = 0 \) with the lower layer at rest and \( Q \geq -1 \) relating Q to the
geometry at the front are given by

\[ 3 \frac{Q^{2/3}}{b_f^3} - \left( \frac{Q}{D_f b_f} \right)^2 - 2 D_f = 0. \]  

(3.30)

The front will first form when Q is a minimum; in particular when \( \frac{dQ}{db_f} = 0 \) which requires

\[ Q^2 = \lambda \left( \frac{D_f}{D_f + \lambda} \right)^3. \]  

(3.31)

Substituting into (3.30) and solving for \( b_f \) with a given \( \lambda \) (\( D_f \) being given by equation (3.27)) allows us to establish the position of the virtual control when the lower layer is just brought to rest. The relationship between channel geometry at the front and net forcing for \( Q < -1 \) is given by (3.29).

Figure 3.27 shows the flow rates as a function of Q for \( \lambda = 0, 1/2 \) and 2. Notice that for Q not too strongly negative the flow rates are decreased by non-zero \( \lambda \), while an increase in \( \lambda \) produces increased flow rates for strongly negative Q. This observation is similar to that for flows over a simple sill. The width \( b_v \) of the channel at the virtual control (or at the front if one of the layers has been brought to rest) is plotted in figure 3.28. Nonzero \( \lambda \) has comparatively little influence for \( Q > 0 \). For negative Q the virtual control does not shift out into the reservoir, but stays in the vicinity of the sill when \( \lambda \neq 0 \). The transition to only a single layer flowing (thin lines) causes a cusp in the curves (for \( \lambda = 0 \) this cusp is with infinite \( b_v \)). In the limit \( \lambda \to \infty \) the \( b_v \) curve would become a delta-function, centred on \( Q = (2/3)^{3/2} \), for \( Q \in [-1,1] \).

We expect the basic features of the flow to be qualitatively similar for any functional \( B(b) \), provided it describes a channel which behaves monotonically away from \( x = 0 \). However we do note that it may be possible for jumps in the position of the virtual control to occur with changing barotropic forcing, Q, in some geometries (e.g., those outlined in section 3.7).

In the next chapter we shall apply the hydraulic model of flow through a rectangular cross-section to the buoyancy driven movement of air through a doorway. While the geometry of this simple flow violates the slowly varying assumption introduced in chapter 2, a good qualitative agreement is nevertheless achieved.
Figure 3.27. The response of the flow rates to net barotropic forcing (Q) through a channel containing a simultaneous sill and contraction. Curves are plotted for sill strengths of $\lambda = 0$ (solid line), $\lambda = \frac{1}{2}$ (long dashes) and $\lambda = 2$ (short dashes).
Figure 3.28. The channel width at the virtual control (bv) as the result of net barotropic forcing (Q) through a simultaneous sill and contraction. Solid lines for $\lambda = 0$; long dashes for $\lambda = \frac{1}{2}$; and short dashes for $\lambda = 2$. The width of the front formed is plotted for values of Q which bring one of the layers to rest.
A further application of the theory to the flow through the Strait of Gibraltar will be delayed until chapter 10.
An application to airflow through doorways

In this chapter we shall show how the two-layer hydraulic theory developed in the preceding two chapters may be applied to a real-life situation. In the problem we consider, the geometric length scales are comparatively small, so the rotation of the Earth does not enter the dynamics. In chapter 10 we shall analyse the flow through the Strait of Gibraltar, where rotation may be important, utilising the nonrotating model of chapter 3 in addition to the nonrotating parabolic model, which will be introduced in chapter 5, and the rotating model of chapters 6 to 8.

The application we consider here is the steady state limit of the buoyancy driven flow of air through doorways. A number of workers have shown (e.g. chapter 6, Henderson, 1966) that hydraulics are relatively insensitive to departures from the slowly varying assumption introduced in chapter 2. Such departures do not alter the requirements for the flow to conserve both mass and Bernoulli potential, they only change the form of the expressions for these quantities. Qualitatively, such flows appear very similar to the slowly varying analysis, the main differences being in the velocity profiles away from the contraction or sill crest, and not the surface profile or flow rates. If the geometry varies sufficiently rapidly for separation to occur, then the flow may be considered equivalent (to a first approximation) to the flow through a slowly varying channel whose boundaries are the free streamlines.

As an example, consider a doorway between two sealed rooms (or one sealed room and an open space) such that \( Q = 0 \). One room contains warm, buoyant air and the other cool, dense air. When the door is opened the buoyancy difference will set up an exchange flow through the doorway. After a short time the initial transients will die away in the immediate vicinity of the door, and a steady flow will be established. Until recently most workers in this field have assumed that the rate at which potential energy is released by the exchange flow is equal to the rate at which the kinetic energy of the mean motion increases. This requires, in the absence of net flow, the neutral plane or interface (for a
two-layer approximation) to be positioned at the midpoint of the doorway (e.g., Shaw & Whyte, 1974; Steckler, Baum & Quintiere, 1984). In section 3.3 we demonstrated that this is not a general law for relatively straight hydraulic flows, and so we would not expect the flow through the doorway to be without dissipation. Thus the half-height assumption may not be justified.

If the doorway between the two rooms is the full height of the room containing warm air, the flow will be simply that of section 3.2 through a channel of constant depth, and the rate of release of potential energy will be equal to the rate of increase of kinetic energy. The two controls will coincide with the open doorway and the density interface will be at half the height of the doorway. If the ceiling in the room containing the cooler air is higher than the doorway (e.g., it is an external door), the warm air will accelerate upwards, forming an angled plume (Lane-Serff & Linden, 1988) as shown in figure 4.1a. Details of this process will not affect the control mechanism because of the isolating supercritical region just before the plume separates. The separated plume is in freefall and so the Froude number has no meaning over that region.

If the doorway is at the end of a corridor containing warm air (which may subsequently open out into a room of the same ceiling height at the warm end) such that the opening occupies the full width of the corridor, but not the full height, then the analysis is that of section 3.3. The ratio of the height of the doorway to the corridor height is $D_{w}$. Again, what happens in the region containing cold air is of little importance (so long as the ceiling is at least as high and the area at least as wide as the door). One control will be at the doorway, and the other where the corridor opens out into the room. The interface heights at these points and the exchange flow rate are given in by figures 3.7 and 3.8. A typical example of such a doorway is shown in figure 4.1b.

Variations in both the width and height of the warm room immediately away from the doorway are very common. Unfortunately, analysis of it is somewhat less straightforward. One naive approach would be to assume that the depth and width of the moving mass of air in the warm room increase by the same amount in every direction (i.e., the angle at which the current spreads out is constant for every direction). This corresponds to the channels of
Figure 4.1. Simple models for buoyancy-driven air flow through doorways: (a) doorway the full height of the warm room; (b) doorway the full width of a corridor leading to the warm room; (c) doorway into a large warm room.
section 3.9 with $\lambda = \frac{b_0}{2D_0}$ where $D_0$ is the height of the doorway and $b_0$ the width (the factor of one half is present because the depth is not able to increase symmetrically due to the floor of the room). For a typical single door $\lambda \approx 0.2$, placing the interface 2% up from the mid-point of the doorway. The exchange flow rate is 0.5% lower than that into a room the same height as the door. Increasing the width of the doorway (increasing $\lambda$) moves the interface further from the half-height point of the door, though will of course also increase the dimensional exchange flow rate (even though the dimensionless flow rate is decreased).

Full-scale experiments on fire induced flow through a doorway out of a large room have been performed by Steckler et al., (1984). They found that the neutral plane (the interface in the two-layer approximation) was displaced above the half-height point. The magnitude of the displacement depended on a combination of the aspect ratio of the doorway and the position and strength of the source of the fire. Figure 4.2 summarises their findings in terms of the aspect ratio $b_0/D_0$. Data for seven different door aspect ratios and three different fire positions away from the doorway (A, B and C in the data of Steckler et al.) are plotted. For comparison, the predictions of the hydraulic model of section 3.9 are also plotted. The solid line is for the naive $\lambda = b_0/2D_0$, and the dotted lines are for alternative relationships between $\lambda$ and the aspect ratio. Notice that the naive $\lambda = b_0/2D_0$ predicts the interface much closer to the mid-height level than was observed. The other curves are plotted to show that while the position of the neutral plane follows the same basic trend with the aspect ratio, the simplistic model is only a moderate predictor of the flow behaviour.

Recent two-layer laboratory experiments (Lane-Serff, pers. comm.) also suggest that the interface is displaced upwards by approximately 5% for a typical doorway ($b_0/D_0 \approx 0.4$) for flow into a wide room with a high ceiling. Lowering the ceiling causes the interface to be closer to the half-height point, while narrowing the room causes the interface to be further from the half-height point.

While the observed behaviour is qualitatively similar to the hydraulic model, a quantitative comparison shows considerable differences. The fundamental reason for this is that the
Airflow through doorways

Figure 4.2. Comparison of experimental data (Steckler et al., 1984) and simple hydraulic model for buoyancy-driven flow through doorways. Heat source at position A (triangles), B (squares) and C (crosses) - see Steckler et al.. The solid curve represents the naive model with $\lambda = \text{bo}/2\text{Do}$. The dotted curves are for other relationships between the door aspect ratio ($\text{bo}/\text{Do}$) and the sill strength ($\lambda$) for the combined sill and contraction model of section 3.9.
velocities near the doorway (unless the doorway is the full height of the room) are not two-dimensional, and the velocity profile is typically a function of $z$ within each of the two layers. Moreover the withdrawal process is likely to produce velocities throughout the entire depth of the warm room (observations suggest that the velocities are significant only in regions of the room where two layers are present - this is consistent with the hydraulic approach), rather than in a confined vertical region as suggested by the simplistic model used here. The withdrawal process would ensure the flow has a more sill-like character and weaker dependence on the aspect ratio than the model of section 3.9.

In addition, we are able to make some comments about the effects of a net flow of air through the doorway on the heat and mass exchange. Such a net flow may be due to natural ventilation (e.g. wind driven; buildings do not tend to be air-tight) or a consequence of the method of heating or cooling the room(s) concerned. Analysis of the limits for strong forcing such that one or the other layer is brought to rest shows that for a typical inverted sill-like doorway it is easier to keep cold air out (e.g. $Q = 0.544$) than warm air in ($Q$ may range from $-0.544$ to $-1$). If the aim is to keep a room with an open doorway warm, it may prove more economical to pressurise the room so that there is a net outflow of warm air at a rate $Q = 0.544$, rather than to allow an intrusion of cold air. Such an intrusion of cold air would require air to be heated at rate $2Q$ (typically in the range $0.208$ to $0.25$), a task more difficult (e.g. how do you selectively heat the new cold air?) than providing hot air through a duct at a rate of $0.544$. The arguments in favour of pressurising the room are particularly strong if the doorway is open only intermittently and it is necessary to prevent temperature fluctuations.

While the hydraulic model is of no use in modelling the performance of air-curtains (Howell, Van & Smith, 1976) to control the exchange through an open doorway, it does provide some suggestion as to where they should be placed. In particular an air-curtain in the region of nominally subcritical flow is likely to have a greater and more rapid effect than one isolated by a large expanse of supercritical flow as information is communicated in both directions without the disturbances first having to grow to finite amplitude.
The flow of air through rectangular windows and openings not close to any horizontal surface will result in the interface being at the midway point of the opening as the flow will be able to adopt a symmetric profile. The introduction of even a small asymmetry to the window frame may however result in a significant shift in the level of the interface in line with the results for channels containing a sill.

These results are not rigorous and we do not intend them to be interpreted as a true model of the buoyancy driven flow of air through a door. Rather it is hoped that they will provide some insight into the observations that the interface is not at half the height of the doorway (Steckler et al., 1984).
Channels with parabolic cross-sections

5.1 Parabolic channel geometry and equations

Channels may differ not only in their along-channel profiles, but also in their cross-sections. The rectangular cross-section we have considered so far in this thesis is merely one special case, arguably applicable in only a few circumstances. While in principle it is possible to treat any channel (even one in which the form of the cross-section changes along the channel) in the manner outlined in the previous sections, the analysis would be complex. Moreover, for some combinations of along channel geometry and net forcing we expect the results to be so close to those of rectangular cross-sections of the same area that such an analysis is not worthwhile. Instead we shall consider the flow through a channel whose nonrectangular cross-section may be analysed readily to determine how sensitive the flow is to the form of the cross-section.

In this chapter we shall consider the flow through channels having a parabolic cross-section with a flat, rigid lid. Such a section is arguably a much closer approximation to the majority of oceanic passages than the rectangular cross-section, especially as the results apply equally to a parabola sheared horizontally or any other distortion which conserves the width-height relationship over the range of heights the interface may occupy. The typical geometry for these channels is illustrated in figure 5.1; the insert shows an example of a sheared cross-section. At any given section, the maximum depth of the channel is \( D_0(x) \), the height of the lowest part of the channel bottom above the datum \( H_0(x) \) and the width at the upper (planar) surface \( b(x) \). The depth across an unsheared section is therefore given by

\[
D(x,y) = D_0(x) \left(1 - \left(\frac{y}{\xi(x)}\right)^2\right),
\]

where \( \xi(x) = \frac{1}{2} b(x) \) is the channel half width. Throughout most of the remainder of this thesis we shall use \( \xi \) and \( \frac{1}{2} b \) interchangeably; exceptions to this rule will be noted. As we are taking the upper boundary of the channel to be planar, the height of the channel bottom above the datum is
Figure 5.1. Definition sketch of a parabolic channel. The inset is of a sheared parabola.
Parabolic channels

Section 5.1

\[ H(x,y) = H_0(x) \left( 1 + \frac{(y - \xi(x))^2}{\Delta^2} \right). \]  
(5.2)

The height of the interface above the channel bottom is

\[ h(x,y) = D(x,y) - \tilde{A}(x) D_0(x) \]
\[ = D_0(x) \left( 1 - \tilde{A}(x) - \left( \frac{y - \xi(x)}{\Delta} \right)^2 \right). \]  
(5.3)

where

\[ y^2 - \xi^2 = (1 - \tilde{A}(x)) \xi(x)^2. \]  
(5.4)

defines the two-layer region (the interface extends from \( y = -\xi(x) \) to \( y = \xi(x) \)). Notice that \( \tilde{A}(x) = 0 \) corresponds to the vanishing of the upper layer, and \( \tilde{A}(x) = 1 \) to the vanishing of the lower layer.

The area of the cross-section, \( S(x) \), is

\[ S(x) = \left( \frac{4}{3} \right) D_0(x) \xi(x), \]  
(5.5)

which is the sum of the areas occupied by each layer, \textit{viz.}

\[ S_1(x) = \left( \frac{4}{3} \right) (1 - \tilde{A}(x))^{(3/2)} D_0(x) \xi(x), \]
\[ S_2(x) = \left( \frac{4}{3} \right) \left[ 1 - (1-\tilde{A}(x))^{(3/2)} \right] D_0(x) \xi(x). \]  
(5.6)

The areas \( S_1 \) and \( S_2 \) are equal when \( \tilde{A} = \tilde{A}_{\text{equal}} \), where

\[ \tilde{A}_{\text{equal}} = 1 - 2^{-2/3} \approx 0.37004. \]  
(5.7)

The hydraulic functional, defined by (2.16), is

\[ J(\cdot; \alpha) = \bar{Q} - 1 - \frac{1}{8} \left[ \frac{3}{4 D_0 \xi} \right]^2 \frac{4 \alpha^2 Q \bar{q} + (1 - 2\alpha) (Q + \bar{q})^2}{\alpha^2 (1 - \alpha)^2} \]
\[ - H_0 - \frac{D_0^{2/3}}{2 \alpha}. \]  
(5.8)

where

\[ \alpha(x) = (1 - \tilde{A}(x))^{3/2}, \]  
(5.9)

is introduced as a more convenient variable than \( \tilde{A}(x) \). Differentiating the functional with respect to \( \alpha \) gives
Note that equation (2.24) gives the composite Froude number, \( F \), as

\[
F^2 = 1 + \frac{\partial J}{\partial h} = 1 + (3/2 \, D_0) \alpha^{1/3} \frac{\partial J}{\partial \alpha}.
\]  

(5.11)

This is again equivalent to the sum of the squares of the layer Froude numbers, where the depth used for the rectangular channel has been replaced by the generalised depth: the area occupied by a layer divided by the width of the interface (Henderson, 1966).

Setting (5.10) equal to zero and solving for \( \bar{q} \) shows that the exchange flow rate for critical conditions is given by

\[
\bar{q} = -a_1 + \frac{(a_1^2 - 4 \, a_0 \, a_2)^{1/2}}{2 \, a_2},
\]

where

\[
a_0 = \left( 1 - 3 \alpha + 3 \alpha^2 \right) Q^2 - (8/3) \, D_0 \left( 4 \, D_0 \, \xi/3 \right)^2 \alpha^{8/3} (1 - \alpha)^3,
\]

\[
a_1 = 2 \left( 1 - 3 \alpha + 3 \alpha^2 - 2 \alpha^3 \right) Q,
\]

\[
a_2 = 1 - 3 \alpha + 3 \alpha^2.
\]

(5.12)

at \( \alpha = \alpha_c \) and \( \alpha = \alpha_v \).

As in section 3.1, we are able to state that the control sections require

\[
K = \frac{\partial J}{\partial D_0} \frac{dD_0}{dx} + \frac{\partial J}{\partial H_0} \frac{dH_0}{dx} + \frac{\partial J}{\partial b} \frac{db}{dx}
\]

(5.13)

to vanish. Thus, as before, sections where \( dD_0/dx \) and \( db/dx \) vanish \((dH_0/dx = -dD_0/dx \) for the channels described above) are candidates for control.

In this chapter we consider in detail only two basic along channel geometries, a third geometry will be introduced in chapter 10 with reference to the flow through the Strait of Gibraltar. The
next section covers the flow through channels whose maximum depth, Do, is constant and width is varying, while section 5.3 examines flow over sills analogous to section 3.7.

5.2 Flow through parabolic channels of constant depth

In this section we shall consider the flow in a channel whose maximum depth, bottom height and width vary in the same way as that outlined for the rectangular channel in section 3.2, the difference being a parabolic rather than rectangular cross-section. Due to the lack of symmetry of the parabolic cross-section about a horizontal plane, we do not expect the flow, in the absence of net barotropic forcing, to have the symmetries of a flow through the equivalent rectangular channel. Moreover, the lack of symmetry suggests that the two controls will not coincide when $Q = 0$.

Roots of equation (5.13) in this channel occur either when $\frac{db}{dx} = 0$ (which occurs at $x = 0$), or $\frac{\delta J}{\delta b} = 0$. The latter of these is the virtual control (with $\frac{\delta J}{\delta \alpha} = 0$) when $\alpha = \alpha_V$ where

$$\alpha_V = \frac{1}{2} \left( \bar{q} + q \right) / \bar{q}. \quad (5.14)$$

In the absence of a net barotropic component, $\alpha_V = \frac{1}{2} (\bar{A}_V = 0.3700)$ which is not equal to the value $\alpha_{\text{max}}(Q=0) = 0.48569$ ($\bar{A}_{\text{max}} = 0.38212$) that maximizes equation (5.12) at $x = 0$ and corresponds to the triple root solution. Thus, unlike the rectangular channel, the two controls do not coincide for $Q = 0$. Simultaneous solutions of $J = 0$ and $\frac{\delta J}{\delta \alpha} = 0$ show $\alpha_C = 0.47853$ ($\bar{A}_C = 0.38820$), $\bar{q} = 0.30604$ and $b_V = 1.00179$ ($b_C = 1$). The virtual control is displaced towards the denser reservoir.

Setting (5.14) equal to $\alpha_{\text{max}}(Q)$ and solving reveals that the two controls do coincide when $Q = -0.18113$ with $\alpha_C = \alpha_V = 0.47060$ ($\bar{A}_C = \bar{A}_V = 0.39498$) and $\bar{q} = 0.30806$.

As with rectangular channels, we are able to determine how strong the barotropic forcing must be in order to bring one or the other of the layers to rest ($\bar{q} = |Q|$). One control will remain at the contraction ($x_C = 0$) for all values of $Q$ resulting in a controlled flow. The virtual control moves along into the upstream (with respect to the net flow) reservoir as $|Q|$ is increased.
eventually forming a front when one of the layers is brought to rest. The strength of forcing required to bring one layer to rest and form a front is calculated in a manner analogous to that for rectangular channels in the previous chapter. Setting $Q = q$ in equations (5.8) and (5.12) along with $A_V = \frac{1}{2}$ and $b_V \to \infty$ and solving the resulting system shows that the upper layer first comes to rest when $Q = \left(\frac{3}{4}\right)^2 \left(\frac{2}{3}\right)^{3/2} \approx 0.30619$, $\alpha_c$ being $\left(\frac{3}{4}\right)^{3/2}$ which corresponds to $A_c = \frac{1}{4}$. Increasing the strength of the forcing further causes the upper layer to vanish from the contraction ($x = 0$) when $Q > \left(\frac{2}{3}\right)^{3/2} \approx 0.5443$.

Similarly setting $Q = -\tilde{q}$ and $A_V = -\frac{1}{2}$ allows us to calculate the strength of forcing required to bring the lower layer to rest. In particular, $Q = -(1/2)^{7/6}$, which corresponds to $\alpha_c = \frac{1}{4}$ or $A_c \approx 0.60315$. Increasing the strength of the forcing does not result in the lower layer vanishing at the contraction, although its thickness rapidly becomes very small. More specifically, when the lower layer ($Q = -\tilde{q}$ in equation (5.12)) is at rest, the net flow rate (equal to the exchange flow rate) is related to the interface coefficient $\alpha_c$ at $x = 0$ through

$$Q = -2 \left(\frac{2}{3}\right)^{3/2} (1 - \alpha_c)^{3/2} / \alpha_c^{1/6}. \quad (5.15)$$

Figures 5.2 and 5.3 show how the flow through this channel varies as a function of the net barotropic forcing. The first of these shows how the flow rates ($q_1$, $q_2$ and $\tilde{q}$) change. Notice the asymmetry about $Q = 0$ due to the asymmetric nature of the parabolic channel (solid line). The dashed curves are the flow rates through a constant depth channel with the same cross-sectional area and depth, but a rectangular cross-section. Two sets of scales are shown. Those marked parabolic represent normalisation of the area with respect to a parabolic channel with $D_0$ and $b$ both unity at $x = 0$ (i.e. the cross-sectional area is $2/3$). The second set marked rectangular represents a unity area for the rectangular cross-section (equivalent to the parabolic channel having a width $3/2$ times that of the rectangular channel).

It should be recognised that any quantitative comparison between the two cross-sections will be a function of the normalisation used. For example, an alternative normalisation would have been to have both channels with the same cross-sectional area and the same exchange flow rate when $Q = 0$. 


Figure 5.2. Variations in the flow rates due to net barotropic forcing through a channel of constant depth. Solid curve for parabolic channel with $D_0 = 1$, $b = b^*$ at contraction; dashed curve for rectangular channel with $D = 1$, $b = (2/3) b^*$ at contraction. The "parabolic" scales are for $b^* = 1$, i.e. normalised with respect to the parabolic channel. The "rectangular" scales are for $b^* = 3/2$, i.e. normalised with respect to the rectangular channel.
Figure 5.3. Variations in $\tilde{A}_c$ and $\tilde{A}_v$ due to net barotropic forcing through a channel of constant depth. Solid curves are for a parabolic channel, dashed curves are for $A_c$ and $A_v$ for a rectangular channel. See figure 5.2 for the meaning of the two "Q" scales. The scales for $\tilde{A}$ and $A$ are related to show the relative interface heights for the two channel types.
This approach has been rejected as the normalisation would be a function of the along-channel geometry.

Over most of the range of barotropic forcings, the exchange flow rate through a channel with a rectangular cross-section is greater (though less than 15% greater) than that through a parabolic cross-section of the same area. The upper layer is brought to rest with about a 16% weaker forcing for a parabolic channel. However, if the barotropic forcing is sufficiently negative ($Q < -0.2$ on the parabolic scale), the exchange flow through a parabolic channel becomes greater than that through a rectangular channel.

Two scales are also shown for the net barotropic forcing in figure 5.3, their meanings being the same as in figure 5.2. The two scales for the interface height coefficients correspond to the different relationship between $h$ and $A$ (rectangular channel) or $A$ (parabolic channel), viz. equations (3.1) and (5.3) for the rectangular and parabolic channels, respectively. On the plot the solid lines represent values in the parabolic channel, and the dashed curves those for rectangular cross-sections. Note that the curves for parabolic channels are similar to those in the rectangular channel (provided $Q$ is not too negative; the lower layer can never vanish at $x = 0$ for parabolic channels) but offset slightly towards higher interfaces and more negative forcings. These differences reflect the different vertical distribution of the cross-sectional area: for the rectangular channel equal areas for each layer are achieved with the interface at half the channel depth, whereas in the parabolic channel it must be approximately 63% of the way up from the bottom.

5.3 Flow over stills in parabolic channels

We now consider how the flow in a channel of parabolic cross-section responds to the presence of a sill. The channel geometry is essentially that of figure 3.3 except that the cross-section is replaced by a parabolic one, similar to that outlined in figure 5.1.

In the absence of any barotropic forcing, the flow behaves in a manner similar to that of the rectangular channel except that
for \( D_w = 1 \) the two controls do not coincide at \( x = 0 \). As we discussed in the previous section, this is due to the inherent asymmetry of the parabolic channel. However once the sill height becomes sufficiently large (\( D_w > 1.00017 \)) the virtual control becomes fixed at \( x = -1 \), the start of the expansion in channel width. Figure 5.4 illustrates how the interface height coefficients vary at the sill crest (\( A_c; x_c = 0 \)) and channel exit (\( A_v; x_v = -1 \)) in response to changing the maximum depth of the channel (\( D_w \)). The curves plotted in a solid line represent the values for a parabolic channel, and dashed lines for the equivalent rectangular channel. As in figure 5.3, there are two scales for the interface height coefficient due to their differing relationships with the interface height for the two channel types.

The upper layer is brought to rest for \( Q = 0.30619 \), the same value as found for contractions in the previous section. This is independent of how deep the channel becomes away from the sill. As with the rectangular channel the strength of negative forcing required to bring the lower layer to rest varies with the maximum depth, though unlike the rectangular channel, there will always be two layers present at the sill crest. Putting \( Q = -\tilde{Q} \) and \( A_v = -\tilde{A} \) with \( b_v = \infty \) and \( D_v = D_w \) into equations (5.8) and (5.12) shows the lower layer is first brought to rest with

\[
Q = - \frac{4}{3} \left( 2 \frac{D_w}{3} \right)^{\frac{1}{2}} \left( 1 - \alpha_c \right)^{3/2} / \alpha_c^{1/6},
\]

where \( \alpha_c \) is the root of

\[
1 - 3 (D_w - 1) \alpha_c^{\frac{3}{2}} - 4 \alpha_c = 0. \quad (5.16)
\]

The changes in behaviour between small and large negative barotropic forcing (when \( D_w > 1.00017 \)) requires the flow to undergo a bifurcation in the behaviour for some value of \( Q \). Clearly the larger the value of \( D_w \) the more negative \( Q \) must be in order to force contraction-like behaviour.

Figure 5.5 shows how the flow rates vary as a function of the net barotropic forcing for a number of different maximum channel depths. The dotted line represents some hypothesised flow where the height coefficient is equal to \( A_{\text{max}} \) (the value of \( A \) at which \( \partial^2 J / \partial A^2 \) vanishes and both controls are at \( x = 0 \); this solution is the local maximum of (5.12)). We shall term this the \( Q_{\text{max}} \) curve.
Figure 5.11. Variation in the interface height coefficients at the sill crest ($A_c$) and virtual control ($A_v$) as a result of changing the sill geometry $D_w$. Curves are shown for parabolic channels (solid lines) and the corresponding $A_c$ and $A_v$ for rectangular channels (dashed lines). Both are for $Q = 0$. 
Figure 5.5. Variations in the flow rate as a function of the net barotropic forcing over a sill in a channel of parabolic cross-section. Curves shown for $D_w = 1$ (solid line), $D_w = 1.125$ (long dashes), $D_w = 1.25$ (short dashes) and $D_w = 1.5$ (dot-dash). The dotted curve represents the flow if the two controls were to coincide (i.e. coincident behaviour). When the realised solutions adopt coincident behaviour they are shown dotted.
The $D_w = 1$ curve (shown as a solid line) touches the $q_{\text{max}}$ curve for $Q = -0.01811$, as stated in the previous section.

If $D_w$ is somewhere between unity and 1.107, then decreasing the net barotropic forcing from $-0.01811$ will cause the exchange flow rate to increase with sill-like behaviour occurring. Once the negative barotropic forcing is sufficiently strong, the flow will bifurcate to a contraction-like behavioural mode with a corresponding sudden change in the interface profile. The value of $Q$ at which this bifurcation happens is a function of $D_w$ and is discussed more fully below.

When $D_w$ is greater than 1.107, the behaviour as $Q$ decreases from $-0.01811$ is fundamentally different. Rather than the sill-like behaviour bifurcating to contraction-like behaviour, it undergoes an intermediate bifurcation to coincident behaviour. The bifurcations between sill-like and coincident behaviour, and between coincident and contraction-like behaviour are of a continuous nature, both in terms of the flow rates and the interface profiles (though not in the position of the virtual control). In the coincident behaviour zone, both controls are at the sill crest ($x = 0$) and the exchange flow rate is equal to $q_{\text{max}}$. This form of behaviour also occurred in the rectangular channels of section 3.7 when $D_w > 3/2$, though it coincided with the lower layer being brought to rest and so was of no significance.

The curves on figure 5.5 for $D_w = 1.125$ (long dashes), 1.25 (short dashes) and 1.5 (dot-dash pattern) all show this form of bifurcation structure for negative $Q$, with the $q$, $q_1$ and $-q_2$ curves all coinciding with their $q_{\text{max}}$ equivalents over a range of values of $Q$. The range of values of $D_w$ over which bifurcations to coincident behaviour can occur are discussed more fully below.

A bifurcation from sill-like to contraction-like behaviour occurs for values of $Q > -0.01811$ in an analogous manner to positive forcing over sills in channels with a rectangular cross-section. The bifurcation leads to a smooth change in the slope of the flow rate graph and a continuous change in the interface profile.

The solid lines of figure 5.6 divide the $Q-D_w$ plane into four regions representing the different types of flow behaviour. The two regions marked $i$ are combinations for which the flow behaves
Figure 5.6. Phase plane showing behaviour type for a simple sill with a parabolic cross-section. The solid lines indicate bifurcation points in the solution and divide the plane up into regions. Channels falling into region (i) exhibit contraction-like behaviour. In region (ii) sill-like behaviour is found, while region (iii) represents coincident behaviour. The dashed line gives the value of $b_v$ for the contraction-like branch at the bifurcation point.
like a contraction with the primary control at the sill crest and
the virtual control in the expanding region of the channel,
upstream (with respect to the net flow) of the crest. Region ii
represents combinations for which flows exhibit sill-like
behaviour, with one control at the sill crest \( x = 0 \), and the
other at the channel exit \( x = -1 \), upstream of the sill, with
respect to the lower layer. The final region, iii, are those
combinations of \( Q \) and \( D_w \) which support coincident flow behaviour.
The dashed line on the graph gives the width at the virtual
control for contraction-like behaviour at the value of \( Q \) where it
bifurcates to/from either coincident or sill-like behaviour. The
triple-point, where all three behavioural types give the same
flow, is \( D_w = 1.107, \ Q = -0.341 \) and \( b_v = 1.527 \) (for
contraction-like behaviour).

5.4 Summary of parabolic channels

In this chapter we have investigated the hydraulic flow
through channels of parabolic cross-section. The motivation for
this was three-fold: parabolic cross-sections are a more realistic
model than rectangular cross-sections of many naturally occurring
channel flows; it is important to determine how sensitive
hydraulics are to the shape of the channel cross-section so that
we may gain some idea as to whether a model geometry is "good
enough"; and the cross-channel depth variations introduce some
features which will be seen in the rotating channels presented in
chapters 6 to 9.

The results of this chapter suggest that on a qualitative
level the flow through a channel of parabolic cross-section is
very similar to that through the rectangular cross-sections of
chapter 3. Such qualitative agreement is likely to carry over to
other channel geometries. While any quantitative comparison is
hampered by the need for some normalisation of the results, due to
the different relationship between the cross-sectional area and
the channel width and depth, the overall trends with the
introduction of net barotropic flow and varying along-channel
geometry show a remarkable similarity between the two
cross-section types.
The main difference between the rectangular and parabolic cross-sections is characterised by the vertical distribution of their width. In the rectangular cross-section, an interface at half the channel depth will result in both layers occupying the same area. In contrast the interface must be approximately 63% of the way up from the channel bottom for the area to be occupied by the two layers to be equal (A for equal areas was given by equation (5.7)). It is therefore scarcely surprise that, while the response to net barotropic forcing in constant depth channels of rectangular cross-section is antisymmetric with respect to the forcing Q, the same does not apply to parabolic cross-sections. Moreover, we observed in figures 5.3 and 5.4 that the interface in the parabolic channel is further above the channel bottom than in the corresponding rectangular channel, a feature attributable to the different distribution of the cross-sectional area. An extreme example is that for negative net forcing it is possible to bring the lower layer to rest but not to remove it from the contraction (though its thickness may become arbitrarily small as Q is increased; in the corresponding rectangular channel the layer will vanish when Q = -1).

In the absence of net forcing there will be only one control section in a rectangular channel of constant depth. In contrast if the channel has a parabolic cross-section the virtual control will be separate from the primary control and be positioned slightly closer to the dense reservoir. Again this feature is a result of the asymmetry of the channel: the tendency for the layer velocities to be equal (A = 0.37004 > A_{max} = 0.38212, equation (5.7) and section 5.2) is satisfied by the virtual control, requiring the primary control to adopt a value greater than A_{max} (A_{c} = 0.38620). A weak negative forcing (Q = -0.18113) will allow the two controls to coincide.

Variations with the geometry of a simple sill were shown in section 5.3 to follow much the same trend for rectangular and parabolic channels once an allowance has been made for the offset of the interface towards the top of the channel when the cross-section is parabolic. While D_{W} is close to unity the interface height coefficients A_{c} and A_{v} vary as \((D_{W} - 1)^{1/3}\). When the sill is very high \((D_{W} \rightarrow \infty)\) the coefficient at the crest tends to a constant value while that at the exit control goes like \(D_{W}^{-1}\).
reflecting a constant height for the interface at that point.

For a forced flow over a sill we saw the bifurcation from sill-like behaviour to coincident behaviour become of much greater importance than in the rectangular channel (in the rectangular channel it occurred simultaneously with the lower layer being brought to rest). As we shall see in section 8.5, this behaviour also occurs in rotating rectangular channels; since it occurs in the nonrotating parabolic channel we are able to state that coincident behaviour is a function of the manner in which the area of a given layer changes as the height coefficient is altered and is not (directly) an artifact of rotation.

We delay applying this hydraulic theory for parabolic channels until chapter 10 so that we may compare the three different models of this thesis (i.e. nonrotating rectangular, nonrotating parabolic and rotating rectangular) for the tidally modulated exchange flow through the Strait of Gibraltar. In the next chapter we introduce the hydraulic formulation for rotating rectangular channels.
Hydraulic theory for rotating channels

6.1 Equations in the rotating frame

For a large variety of problems it proves convenient to take our frame of reference as stationary with respect to the containing geometry. If this geometry is moving in a non-inertial manner, then we must consider how the acceleration of the frame of reference interacts with the motion of the fluid relative to the frame. While it is possible to formulate the equations of motion relative to any arbitrary motion of the frame of reference (e.g., section 3.2, Batchelor, 1967), in this thesis we are concerned only with those modifications caused by the rotation about a stationary vertical axis.

For an homogeneous, incompressible, Newtonian fluid layer, whose coordinate system is rotating about some axis $\hat{k}$ fixed relative to some inertial frame, the Navier-Stokes equations may be written as

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla \left[ \frac{p}{\rho} + \mathbf{g} \cdot z \right] - \frac{f}{\rho} \hat{k} \times u - \frac{f^2}{\rho} \hat{k} \cdot (\hat{k} \times x) + \nu \nabla^2 u,$$

$$\nabla \cdot u = 0,$$  \hspace{1cm} (6.1)

where the (effective) rotation rate is $\Omega f$. While the effect of rotation is to introduce additional acceleration terms, these terms are traditionally written on the right-hand side of the equations of motion and treated as fictitious body forces.

The centrifugal force (term in $f^2$ in (6.1)) may be expressed in terms of the gradient of a scalar, and hence can be absorbed into the pressure term, yielding a system of equations which is independent of the position of the rotation axis. The other fictitious force is the Coriolis force which tries to turn moving fluid particles to the right (relative to the rotating frame) for $f > 0$. From here on, all discussion will pertain to this rotating frame unless otherwise specified.

The addition of rotation to the system introduces the time scale $1/\Omega f$, the inertial period, not present in the classical problem. If the flow system is able to support long, small
amplitude gravity waves (Kelvin waves), either on a free surface or internally between two fluid layers, then an additional length scale, the Rossby radius of deformation, is introduced to the problem. Physically the Rossby radius is the distance small amplitude gravity waves would travel in one inertial period (i.e., the time for the direction of their motion to be reversed relative to the inertial frame). If the Rossby radius is large compared with the physical dimensions of the flow situation, then the effect of rotation on buoyancy driven flows is negligible. In contrast, if the Rossby radius is small then rotation may dominate the fluid motion.

For the rotating hydraulics of this thesis we shall assume that the external or barotropic Rossby radius is much larger than the width of the channel; this condition is an extension to the idea, used for the nonrotating channels of the preceding chapters, that the external Froude number is very small. We are effectively saying that the channel has a rigid lid allowing no surface gravity or Kelvin waves. In contrast, as we are dealing with Boussinesq fluids, the internal or baroclinic Rossby radius of deformation,

$$Rr = \frac{(Dg')^{1/2}}{f}$$  \hspace{1cm} (6.2)

may be of the same order as the width of the channel. Thus rotation may introduce significant modifications to the flow.

For convenience, we shall nondimensionalise the equations using the inertial period as the typical time scale and the Rossby Radius of deformation as the typical horizontal length scale. For the nonrotating channels of chapters 2 to 5 we used the channel width and wave-crossing time as the basic length and time scales. Using the new description, the dimensionless variables (flagged with a superscript *) are related to their dimensional counterparts by

$$\begin{align*}
(x,y)^* &= \frac{f}{(D_m g')^{1/2}} (x,y) \\
z^* &= \frac{z}{D_m}
\end{align*}$$
where, as in section 2.1, $D_m$ is the minimum depth along the channel, and $i = 1, 2$ to represent the lower and upper fluid layers respectively.

This scaling is appropriate within the inviscid interior of either of the two fluid layers. We shall tackle the problem of thin (laminar) viscous boundary layers in relation to the laboratory experiments in section 9.2; in the meantime we shall neglect the viscous terms of (6.1).

Substituting (6.3) into (6.1) and dropping the superscripted asterisks (*) yields the dimensionless equations of motion (the rotating version of the Euler equations), viz.

$$
\frac{\partial u_i}{\partial t} + (u \cdot \nabla) u_i = - \frac{\partial}{\partial x} \left[ \frac{p_i}{\rho_i} + \frac{z}{\rho_1 - \rho_2} \right] + v_i,
$$

$$
\frac{\partial v_i}{\partial t} + (u \cdot \nabla) v_i = - \frac{\partial}{\partial y} \left[ \frac{p_i}{\rho_i} + \frac{z}{\rho_1 - \rho_2} \right] - u_i,
$$

$$
\frac{\partial w_i}{\partial t} + (u \cdot \nabla) w_i = - \frac{Rr^2}{D_m^2} \frac{\partial}{\partial z} \left[ \frac{p_i}{\rho_i} + \frac{z}{\rho_1 - \rho_2} \right],
$$

$$
\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \frac{Rr}{D_m} \frac{\partial w_i}{\partial z} = 0. \quad (6.4)
$$

As with the nonrotating equations, we may apply a shallow water model in which the pressure is hydrostatic within a given layer of fluid. The requirements for such a model to be applied in the classical limit are outlined in section 2.1. When considering rotating channels we must insist, in addition to these restrictions, that the internal Rossby radius of deformation is
much larger than the depth of the fluid, $Rr / Dm > 1$. Note that
the shallow water model requires that the horizontal components of
velocity (within a given layer) are independent of $z$, and that
this is fully consistent with the Taylor-Proudman theorem for
rapidly rotating flows.

We are able to integrate (6.4) along material particle
trajectories to obtain Bernoulli's equation in the same form as
given by (2.4). Again $G_i$ is conserved by fluid particles and, in
the steady state limit, $G_i$ is conserved along streamlines.

Rossby (1938a,b) was the first to show that the steady state
solution of any rotating, inviscid flow may be obtained directly
from its initial conditions without the need to consider the time
dependent set-up procedure, though the steady state so obtained
may not be unique. In doing so he introduced the concept of
potential vorticity, showing that, in the absence of dissipation
and sinks/sources of fluid, the potential vorticity is conserved
by fluid particles. Provided the depth $h$ of a fluid layer is
nonzero, it is possible to integrate the $z$ component of the
vorticity equation over the layer depth to show that

$$\frac{D\Pi_i}{Dt} = 0, \quad (6.5)$$

where $D/Dt$ is the material derivative ($D/Dt = \partial/\partial t + (\mathbf{u} \cdot \nabla)$) and $\Pi_i$
the potential vorticity of layer $i$. In the current
dimensionless system the potential vorticities of the two layers
are

$$\Pi_1 = \frac{v_1, z - u_1, y + 1}{h},$$

$$\Pi_2 = \frac{v_2, z - u_2, y + 1}{D - h}, \quad (6.6)$$

where $D$ is the dimensionless depth of the channel and $h$ is the
height of the interface above the channel bottom. Conservation of
potential vorticity is a generalisation to the rotating frame of
Kelvin's theorem that the circulation of an inviscid fluid is
conserved. It is easily shown that potential vorticity $\Pi$ and
Bernoulli potential $G$ are related by
Theory for rotating channels

\[
dG_i/d\psi_i = - \Pi_i(\psi_i),
\]

where the stream functions are defined by

\[
h_i \psi_i = \nabla \cdot (\hat{K} \psi_i).
\]  (5.7)

The basic assumptions of the shallow water equations are broken if either the potential vorticity or the Bernoulli potential are not conserved. Their behaviour is of particular interest in the presence of an hydraulic jump - a region representing dissipation due to viscous effects (enhanced by intense turbulence) and mixing, as well as the horizontal length scale being comparable with the fluid depth. Detailed discussion of these phenomena are beyond the scope of this thesis.

In order to make progress on this problem in terms of hydraulics, we must restrict our attention to the limiting steady state, and require the flow to be relatively straight (i.e. \( v_i \ll u_i \)). Some discussion on the time dependent initial transients is given in section 7.3, and of flows which are not relatively straight in sections 9.2 to 9.5.

When the flow is relatively straight, the \( y \) component of (6.4) further reduces to the geostrophic balance equation,

\[
0 = -\frac{\partial}{\partial y} \left[ \frac{\rho_i}{\rho_1} \left( \frac{\psi_i}{\rho_1 - \rho_2} \right) \right] - u_i, \]  (6.8)

which represents the balance between the Coriolis force and the cross-channel interface slope and pressure gradient. It is this feature which outlines the essential difference between a rotating channel and the classical limit, in which the interface is horizontal across the width of the channel.

In the next two sections we shall develop the descriptions necessary for attached and separated flows. An attached flow is characterised by two fluid layers being present over the entire width of the channel. In contrast, two fluid layers are present over only part of the channel width for a separated flow. The ideas will be brought together to produce the required functional formulation in section 6.4.
6.2 Attached flows

As stated in the previous section, some regions of the channel will contain two fluid layers over the entire channel width, while others may have the density interface intersecting the channel top and/or floor. In this section we consider only regions of the channel where two layers are present across the entire width of a section. We shall call the flow in such regions attached flow. Section 6.3 introduces the concepts associated with regions of separated flow where only a part of the channel width contains two fluid layers.

Figure 6.1 shows a typical channel section in a region of attached flow. The interface is banked up on the right-hand side (looking downstream with respect to the lower layer) of the channel as a result of the Coriolis component of acceleration. The channel extends from \( y = -\xi(x) \) to \( y = \xi(x) \). The channel half-width \( \xi = \frac{b(x)}{2} \) has been introduced to simplify notation. The two symbols can be used interchangeably (with the appropriate scaling) depending on which is the more convenient.

As in chapter 2, if there is no significant surface tension at the interface between the two layers, the pressure may be eliminated between the two Bernoulli potentials to give

\[
\Delta G = G_1 - G_2 = \frac{1}{2} (u_1^2 - u_2^2) + H + h.
\]

Similarly, eliminating the pressure from (6.8) yields

\[
0 = -\frac{\partial}{\partial y} (H + h) - (u_1 - u_2),
\]

with \( H = H(x) \) in channels with a rectangular cross-section such as are being considered here.

The two layer velocities may be eliminated from (6.9), using the potential vorticity relations (6.1#7), to give

\[
h_y y - (\Pi_1 + \Pi_2) h = -D \Pi_2.
\]

The apparent asymmetry with respect to the two reservoir potential vorticities is an artifact of how \( h \) has been defined.

While equation (6.10) may, in principle, be used for any
Figure 6.1. Typical cross-section containing attached flow.
distribution of potential vorticity, solution may be difficult. We shall restrict our attention to fluid layers of constant (uniform) potential vorticity, though this restriction will be relaxed somewhat in section 6.6 and some parts of chapters 7 and 9.

Integrating equation (6.10) for uniform layer potential vorticities gives

\[ h = D \left( A \sech \lambda \xi \cosh \lambda \eta + B \csch \lambda \xi \sinh \lambda \eta \right) + \frac{\mathcal{D}}{\Pi_1}, \]

(6.11)

where

\[ \lambda^2 = \Pi_1 + \Pi_2, \]

\[ \mathcal{D} = D \prod_1 \prod_2 / (\Pi_1 + \Pi_2), \]

(6.12)

and \( A = A(x), \quad B = B(x) \) are constants of integration. It is these constants which must be determined in order that the flow is fully specified. Note that in the classical two-layer problem of chapters 2 to 5, the horizontal cross-channel interface gave rise to only one constant of integration, \( A(x) \), characterising the height of the interface above the channel floor. The additional constant, \( B(x) \), introduced by rotation, characterises the cross-channel slope required by the interface to balance the Coriolis force. To distinguish between these two coefficients, we shall call \( A(x) \) the interface height coefficient and \( B(x) \) the interface slope coefficient. Note that in general the slope coefficient will be negative, corresponding to the current being banked up on the right-hand side of the channel (looking downstream with respect to the lower layer), though in order to show this, and for computational reasons, it is necessary to develop the expressions for both signs of \( B(x) \).

Equation (6.12) is valid only so long as two layers are present over the whole of the width of the channel. This restriction can be given in terms of the interface coefficients \( A \) and \( B \) as

\[ \mathcal{D} + D (A + B) > 0, \]

\[ \mathcal{D} + D (A - B) < 1, \]

(6.13)

for interfaces banked up on the right-hand side of the channel.
The two potential vorticity relations of equation (6.6) may be integrated, and the geostrophic balance (6.9) used to relate the associated constants of integration, to give the layer velocities as

\[
\begin{align*}
    u_1 &= (1 - B) \gamma + c - D \left( \frac{\Pi_1}{\lambda} \right) (A \text{sech}^2 \xi \sinh \gamma + B \text{csch} \xi \cosh \gamma), \\
    u_2 &= (1 - B) \gamma + c + D \left( \frac{\Pi_2}{\lambda} \right) (A \text{sech}^2 \xi \sinh \gamma + B \text{csch} \xi \cosh \gamma).
\end{align*}
\]

The constant of integration \(c = c(x)\) is also an unknown. Integrating the product of the layer velocity and layer thickness over the width of the channel yields the layer flow rates, \(q_1\) and \(q_2\), viz.

\[
\begin{align*}
    q_1 &= -2 \left( \frac{\Pi_1}{\lambda^2} \right) A B D^2 + 2 \left( \frac{1 - B}{\lambda} \right) B D t \xi - \\
    &\quad - \frac{2}{\lambda^2} B D + 2 \frac{c}{\lambda} A D t + 2 \frac{B}{\Pi_1} c \xi, \\
    q_2 &= -2 \left( \frac{\Pi_2}{\lambda^2} \right) A B D^2 - 2 \left( \frac{1 - B}{\lambda} \right) B D t \xi + \\
    &\quad + \frac{2}{\lambda^2} B D - 2 \frac{c}{\lambda} A D t + 2 \frac{B}{\Pi_2} c \xi.
\end{align*}
\]

where \(\xi = \text{asinh} \lambda \xi\).

As for nonrotating channels, the sum of these two flow rates is the net barotropic forcing.

\[
Q = -2 A B D^2 + 2 D c \xi.
\]

and the difference the exchange flow rate

\[
\bar{q} = -4 \frac{\Delta \Pi}{\Pi_1 + \Pi_2} A B D^2 + 4 \frac{t}{\lambda} A^2 B D^2 + 4 \left( \frac{1 - B}{\lambda} \right) B D t \xi - \\
\frac{4}{\lambda^2} B D + 2 \frac{t}{\lambda} A Q - \frac{\Delta \Pi}{\Pi_1 + \Pi_2} Q.
\]

The velocity constant \(c\) has been replaced in (6.18) by the net
barotropic forcing $Q$ using (6.17).

When looking at the effect of rotation on a channel of fixed (dimensional) width it may prove more convenient to talk about the flow in terms of the 'specific net forcing' and the 'specific exchange flow rate'. These are just $Q/b_c$ and $\bar{Q}/b_c$ respectively.

6.3 Separated flows

Under the appropriate conditions (e.g., the channel is approximately one Rossby radius in width at the control sections and/or the channel width or depth is much greater than that at the control sections), the slope of the density interface may be such that the interface intersects the channel top and/or bottom at some point over the channel width. In this section we shall consider an interface intersecting the channel floor at some point $y = \zeta(x)$. Such a situation is outlined in figure 6.2. As noted in the previous section, the slope coefficient will normally be negative (figure 6.2b), but it is necessary to consider $B > 0$ (figure 6.2a), both to be able to show this and to allow later numerical solutions to converge efficiently. Analysis of the interface intersecting the channel top may be treated in a similar manner. We shall not consider in detail flows in which the interface intersects both the top and floor of the channel simultaneously. For the flows discussed in chapters 7 and 8, the intersection with both top and bottom represents a reversal in the velocity in both layers and a break down of the basic assumption that the streamlines may be traced to the upstream reservoir. We shall cover this in more detail in the sections concerned within those chapters. The limiting case of the interface going from corner to corner of the channel may be treated as an attached flow.

For $B$ negative the region $-\xi < y < \zeta$ contains two layers and $\zeta < y < \xi$ contains only the upper layer. In the two-layer portion of the cross-section ($-\xi < y < \zeta$) equations (6.9) to (6.14) still hold as the flow must be in geostrophic balance and conserving potential vorticity. In the single-layer portion the lower layer does not exist. The interface height is given by
Figure 6.2. Typical cross-sections containing separated flows. (a): $B < 0$; (b): $B > 0$. 
Theory for rotating channels

\[ n = \begin{cases} 
2 + D \left( A \text{ sech} \xi \cosh y + B \text{ csch} \xi \sinh y \right); & -\xi \leq y \leq \xi \\
0 & \xi < y < \xi'
\end{cases} \]

(6.22)

where

\[ \xi = \frac{1}{\lambda} \ln \left[ -\frac{(S/D) + [(S/D)^2 - (A^2 \text{ sech}^2 \xi - B^2 \text{ csch}^2 \xi)]^{1/2}}{A \text{ sech} \lambda \xi + B \text{ csch} \lambda \xi} \right]. \]

(6.23)

Clearly the flow will only be separated if \(-\xi < \xi < \xi\). Integrating the potential vorticity relation for the upper layer for \(y > \xi\) shows \(u_2\) depends linearly on \(y\) in the single-layer region. The constant of integration may be eliminated by assuming the velocity in the upper layer is continuous at \(y = \xi\). The velocity of the lower layer is undefined for \(y > \xi\). Thus

\[ u_1 = \begin{cases} 
(1 - D) y + c - \\
- D (\Pi_2/\lambda) (A \text{ sech} \xi \sinh y + B \text{ csch} \xi \cosh y) & -\xi \leq y \leq \xi \\
- & \xi < y < \xi'
\end{cases} \]

Similarly, when the slope coefficient is positive, the flow is confined against the \(y = \xi\) wall and

\[ u_2 = \begin{cases} 
(1 - D) y + c + \\
+ D (\Pi_2/\lambda) (A \text{ sech} \xi \sinh y + B \text{ csch} \xi \cosh y) & -\xi \leq y \leq \xi \\
(1 - D \Pi_2) y + D (\Pi_2/\lambda)^2 \xi + c + \\
+ D (\Pi_2/\lambda) (A \text{ sech} \xi \sinh y + B \text{ csch} \xi \cosh y) & \xi < y < \xi'
\end{cases} \]

(6.24)

\[ n = \begin{cases} 
0 & ; -\xi < y < \xi \\
2 + D \left( A \text{ sech} \xi \cosh y + B \text{ csch} \xi \sinh y \right); & \xi \leq y \leq \xi'
\end{cases} \]

(6.25)

where this time
\[
\varsigma = \frac{1}{\lambda} \ln \left[ \frac{-(B/D) - [(B/D)^2 - (A^2 \text{sech}^2 \lambda \xi - B^2 \text{csch}^2 \lambda \xi)]^{1/2}}{A \text{sech} \lambda \xi + B \text{csch} \lambda \xi} \right],
\]

(6.26)

and the layer velocities are

\[
u_1 = \begin{cases} 
- & -\xi < y < \varsigma \\
(1 - \beta) y + c & -
\end{cases}
\]

\[
u_2 = \begin{cases} 
(1 - \beta) y + \frac{D (\Pi_2/\lambda) (A \text{sech} \lambda \xi \sinh \lambda y + B \text{csch} \lambda \xi \cosh \lambda y)}{\varsigma - y} & -\xi < y < \varsigma \\
(1 - \beta) y + c + \frac{D (\Pi_2/\lambda) (A \text{sech} \lambda \xi \sinh \lambda \xi + B \text{csch} \lambda \xi \cosh \lambda \xi)}{\varsigma - y} & \varsigma < y < \xi
\end{cases}
\]

(6.27)

As in the previous section, the layer flow rates may be determined by integrating the product of the layer velocity and layer depth over the channel width. In particular for \(B < 0\)

\[
q_1 = \int_{-\xi}^{\varsigma} u_1 h \, dy,
\]

\[
q_2 = \int_{-\xi}^{\varsigma} u_2 (D - h) \, dy + \int_{\varsigma}^{\xi} D u_2 \, dy,
\]

(6.28)

and for \(B > 0\)

\[
q_1 = \int_{\varsigma}^{\xi} u_1 h \, dy,
\]

\[
q_2 = \int_{\varsigma}^{\xi} u_2 (D - h) \, dy + \int_{-\xi}^{\varsigma} D u_2 \, dy.
\]

(6.29)

While these integrals may be obtained analytically, the resulting expressions are very complex and we shall delay stating them until they are needed (after a number of further simplifications to the
6.4 Hydraulic functional for rotating channels

The idea that an hydraulic flow may be represented by a single functional $J$, with special properties, was introduced in chapter 2. It was noted that, while the choice of such a functional is not unique, there are advantages in formulating it in terms of the difference in the Bernoulli potentials between the two layers. The same arguments apply when considering hydraulics in a rotating frame. The additional unknown coefficient, $B$, is determined by an additional equation (the geostrophic balance).

Due to the variations in the layer velocities across the width of the channel, it is not possible to write $\Delta G$ in the form of (2.14), although the more fundamental relation (2.5) remains true. We shall again define our hydraulic functional as

$$J(D,H,b,Q,\tilde{q},\tilde{\zeta};A) = \tilde{\zeta} - \Psi(u_1^2 - u_2^2) - H - h.$$  

(6.30)

As before, we treat the net forcing $Q$ as a known parameter and the two unknowns, the constant $\tilde{q}$ and the function $\tilde{\zeta}$, are chosen so as to produce the maximal hydraulically controlled solution. In general the streamlines in the two layers will not coincide over the length of the channel except at the channel walls. For computations, equation (6.30) will be invoked at the wall though this need not be the case by allowing $\tilde{\zeta}$ to be a function of $y$. More specifically, we shall choose the function $\tilde{\zeta}$ so that $J(\cdot) = 0$ is the solution of the flow anywhere across the channel. Differentiation with respect to $y$ of (6.30), using its basic definition (2.16) and noting the relationship between the Bernoulli function and potential vorticity given by (6.7), shows

$$\frac{\partial J}{\partial y} = \delta \tilde{\zeta}/\partial y - \delta G_1/\partial y + \delta G_2/\partial y$$
$$= \delta \tilde{\zeta}/\partial y - (dG_1/d\psi_1)(\partial \psi_1/\partial y) + (dG_2/d\psi_2)(\partial \psi_2/\partial y)$$
$$= \delta \tilde{\zeta}/\partial y + \Pi_1 \partial \psi_1/\partial y - \Pi_2 \partial \psi_2/\partial y$$
We shall choose

\[ \hat{\zeta} = \zeta - \Pi_1 \psi_1 + \Pi_2 \psi_2, \]  

(6.32)

where the constant \( \zeta \) is chosen in a manner similar to that for the nonrotating channels of chapters 3 to 5 and the stream functions \( \psi_1 \) and \( \psi_2 \) are zero at one of the channel walls. Note that \( \partial \hat{\zeta} / \partial x = 0 \) as the cross-channel components of velocity vanish everywhere (since they vanish at the channel walls). The basic solution algorithm given in section 3.4 is still appropriate, though is considerably more difficult to apply due to the need to calculate the additional coefficient \( B \).

The right-hand side of (6.30) may be evaluated using equations (6.11) and (6.14), if the flow is attached to both sidewalls. If \( A, B < 0 \) and the interface is separated from the \( y = \xi \) wall, then (6.30) is calculated from equations (6.22) to (6.24). Similarly, when \( A < 0, B > 0 \) and the interface is separated from \( y = -\xi \), equations (6.25) to (6.27) are appropriate.

The equations for separated flows with \( A > 0 \) may be obtained in the same manner as section 6.3. In all cases the slope coefficient \( B \) may be calculated by noting that \( \hat{\zeta} \) is constant along the channel and so, for given interface height coefficient, \( B \) is obtained from (6.18), (6.28) or (6.29) for attached and separated flows respectively. While such roots may not be unique, the correct one may be chosen by stipulating that \( J \) is continuous for all realisable \( A \) at a given section.

Gill's (1977) formalism shows that the control sections, where the interface solution may change between the supercritical and subcritical solution sheets, require \( \partial J / \partial \lambda = 0 \). In terms of the present functional we identify \( \lambda \) with \( A \) and note that \( B \) is a function of \( A \) through the conservation of mass requirement. Thus solution sheets meet along

\[ \frac{dJ}{dA} = 0. \]  

(6.33)

where the complete derivative \( d/dA \) is defined as
These lines pass through the control sections.

Analysis of the flow in terms of small amplitude internal gravity waves is possible using the approach outlined in section 2.3. Differentiation of \( J \) with respect to \( x \), and comparison with the \( x \) component of the momentum equation (noting that \( v_1 \) is zero over the entire cross-section as it must vanish at the walls) shows that

\[
\frac{\delta J}{\delta x} = \left[ \frac{\delta J}{\delta A} + \frac{\delta J}{\delta B} \frac{\delta B}{\delta A} \right] \frac{\delta A}{\delta x} - \frac{\delta J}{\delta A} \frac{\delta A}{\delta x} = \frac{\delta}{\delta x} (u_1 - u_2). \tag{6.35}
\]

The analysis proceeds in the same manner as in section 2.3 to produce

\[
(D - h) \frac{dh}{dA} - \left[ (D - h) u_1 + h u_2 \right] \frac{dh}{dA} C_0 + D \frac{dh}{dA} C_0^2 = 0. \tag{6.36}
\]

The complete derivative \( d/dA \) is given by equation (6.34) and our choice of \( \xi \) ensures \( dJ/dA \) is independent of \( y \). Note that the product of the two solutions to (6.36) varies across the channel width as

\[
C_1 C_2 = (D - h) \frac{dJ/dA}{dh/dA}. \tag{6.37}
\]

The conventional definition of the Froude number in terms of the wave speed, as given by equation (2.21), is not appropriate as \( dh/dA \) varies across the width of the channel. We suggest an alternative definition which is effectively the Froude number at the centre of the channel. This definition collapses to (2.21) in the narrow channel limit (\( \xi = \frac{1}{2} b + 0 \)):

\[
F^2 = 1 + \frac{D}{(D - h) h} \frac{dh/dA}{dJ/dA} C_1 C_2 = 1 + \frac{D}{dA} \frac{dJ}{dA}. \tag{6.38}
\]

As with nonrotating channels, \( dJ/dA < 0 \) corresponds to subcritical
flows with $F < 1$, $dJ/dA > 0$ to supercritical flows with $F > 1$ and $dJ/dA = 0$ to critical conditions at the solution branch points ($F = 1$). Further differentiation of (6.36) with respect to $x$ may be used to show that, for $dJ/dA > 0$, $d^2J/dA^2 > 0$ is required for sections between the dense reservoir and the subcritical region, and $d^2J/dA^2 < 0$ between the light reservoir and the subcritical region. In the remaining chapters of this thesis we shall discuss the sub/supercriticality of a flow in terms of $dJ/dA$ and not the Froude number directly; it should be remembered that the two descriptions are equivalent.

Using the approach outline above, it has not been necessary to determine explicitly the cross-channel structure of the fastest travelling modes: long, small amplitude, internal Kelvin waves. For completeness we shall derive their structure, but note that this is not important in terms of analysing the control process. We shall assume the basic state of the flow is that given in section 6.2. Perturbing the potential vorticity equations (6.6) and the geostrophic balance (6.9) around this basic state yields the system

\begin{align}
\Pi_1 h' & = - \partial u_1'/\partial y, \\
\Pi_2 h' & = \partial u_2'/\partial y, \\
\partial h'/\partial y + (u_1 - u_2) & = 0, \quad (6.39)
\end{align}

where $h'$, $u_1'$, $u_2'$ are the perturbed interface height and layer velocities. This system may be solved to yield

\begin{align}
h' & = D(A' \text{sech } \lambda x \cosh \lambda y + B' \text{csch } \lambda x \sinh \lambda y), \\
u_1' & = - D \Pi_1/\lambda [A' \text{sech } \lambda x \sinh \lambda y + B' \text{csch } \lambda x \cosh \lambda y] + c', \\
u_2' & = D \Pi_2/\lambda [A' \text{sech } \lambda x \sinh \lambda y + B' \text{csch } \lambda x \cosh \lambda y] + c', \quad (6.40)
\end{align}

where $\lambda'$, $B'$ and $c'$ are the perturbations to the interface height and slope coefficients, and the constant velocity introduced from integrating the potential vorticity relations. The second velocity constant was eliminated using the perturbed geostrophic balance. We may eliminate $c'$ by noting that the net barotropic flow rate is not affected by the waves, viz.
Theory for rotating channels  

\[ c' = D \left( A' B + A B' \right) / \xi. \]  

(6.41)

Note the similarity of (6.41) with (6.17) with \( Q = 0 \). The structure of the wave is identical to the structure of the basic flow, the exception being the absence of the distortion term \((1 - \xi) y\) from \( u_1' \), \( u_2' \) in equation (6.40).

Instability of the interfacial shear layer to long wave disturbances may be treated in the same manner as in section 2.4 where we showed that the flow is unstable only in the supercritical regions. We are therefore able to conclude that such instabilities will only alter the control mechanism if they grow to finite amplitude and are able to propagate against the supercritical flow and into the subcritical region, or they alter significantly the vertical density structure or potential vorticity of the flow.

We are considering flows in which the potential vorticity within a given layer is uniform, and so \( \delta \Pi_1 / \delta y \) (and \( u_1 \delta \Pi_1 / \delta y \)) vanishes everywhere for both layers. Linear stability theory (e.g. section 7.10, Pedlosky, 1979) suggests that such flows will be marginally stable to baroclinic modes. As a result, we assert that the maximal exchange flow is stable with respect to linear disturbances.

In the next section we shall discuss briefly the conditions necessary for hydraulic control to exist, and show that they are identical to those for nonrotating channels.

6.5 Hydraulic jumps and submaximal flow

The conditions necessary for an hydraulically controlled flow to exist in a nonrotating channel were outlined in sections 2.4 and 3.5. Even though the interface slopes across the channel the mechanism of flooding, by which an hydraulic control may be lost, is the same as for the nonrotating channels in section 3.5. In general an hydraulic jump in a supercritical region may represent a large shock to the expanding layer; however if the height of the interface after the jump is close to the level of the subcritical solution associated with the controlled solution (see section 3.5 for why this level is important), then the jump will be positioned...
close to the control and its amplitude will be small. Nof (1986) showed that potential vorticity generation is of $O(\alpha^2)$ for a shock amplitude of $O(\alpha)$, thus the potential vorticity of the two layers will be unaffected when the reservoir is close to flooding the control. Further, since the exchange and net flow rates are conserved (we do not expect intense mixing for a small shock) across the jump, the interface slope on the subcritical side of the jump will approach that on the supercritical side as the amplitude of the jump decreases. Hence, as with the nonrotating channels in section 3.5, we need consider only the interface height in the reservoir and not the details of processes within the hydraulic jump to determine whether the flow is fully or partially controlled.

A comparison with the loss of control in single-layer hydraulic flows is necessary as a number of investigators have suggested that rotation introduces the possibility of different control mechanisms, with lower flow rates, once hydraulic control has been lost. Shen (1981) looked experimentally at the effect of rotation on control of flow along a channel between two basins for a number of different channel geometries. He asserted that for a submerged weir (i.e. the crest of the weir or level of the channel floor in the contraction is lower than the height of the fluid in the downstream reservoir) the free-surface height of the current on the left wall (looking downstream) of the control section is equal to the fluid level in the downstream reservoir provided the flow was subcritical. The term *geostrophic control* has been introduced by subsequent authors (e.g. Whitehead, 1986; Nof, 1987a) to describe this situation. Shen justified his assertion with his experiments, and also noting that in Gill's (1976) linear adjustment problem the propagating Kelvin waves left the surface height of the left-hand wall unchanged downstream of the initial discontinuity. More recent work on the nonlinear Rossby adjustment in a channel of uniform cross-section (e.g. Hermann, Rhines & Johnson, 1988) suggests that the advection downstream of fluid initially upstream of the discontinuity alters the surface height across the entire channel; however we suggest, in light of Shen's experimental work, that variations in the channel geometry may allow Shen's assertion to hold.

Garrett & Toulany (1982) also developed the concept of
geostrophic control while analysing the sea level variability due to meteorological forcing in the Gulf of St. Lawrence. They noted that for a channel of constant width connecting two reservoirs, one of which was being periodically forced, the flow through the channel was not limited by friction but by geostrophy, if the frequency was sufficiently low. In particular, the height difference across the channel could not exceed the height difference between the two reservoirs. Whitehead (1986) attempted to unify the ideas of geostrophic control with those of hydraulic control, introducing a new parameter which looks like a Rossby radius of deformation based on the height difference between the two reservoirs, and not the total depth of the fluid.

In relating these ideas of geostrophic control to the present two-layer problem, we must first gain a better understanding as to why it may occur for the single-layer problem. Of fundamental importance is the discovery by Gill (1977) that the ratio of the fluxes in the two upstream boundary currents may be specified arbitrarily. In particular, the value of the stream function in the interior of the flow (\(\psi\) in Gill's notation) is an arbitrary constant. Gill's equation (5.13) (Gill's single-layer equivalent to our hydraulic functional) gives the relationship between the channel geometry, upstream parameters and flow variable as

\[
4 \hat{\psi}_i + 2 \hat{D}_\infty (\Delta + \hat{D} - \hat{D}_\infty) + (t \hat{D})^{-2} + t^2 (\hat{D} - \hat{D}_\infty)^2 = 0.
\]  

(6.42)

The geometric parameters \(\Delta\) and \(t = \tanh w\) represent the elevation of the channel floor and the width \((w)\) of the channel. The depth of the upstream reservoir is \(\hat{D}_\infty\) and the average of the depth of the fluid at the two channel walls for a given section is \(\hat{D}\) (the difference in the fluid height at the two walls, \(\delta D\), is related by \(\delta D = 1\), the height varying across the channel as the sum of cosh and sinh terms).

We note here that, at a given section (the geometry is specified by \(\Delta\) and \(t\)), for any arbitrary realisation of the flow \((\hat{D})\), the pair of upstream parameters \(\hat{D}_\infty\) and \(\hat{\psi}_i\) is not unique. It is therefore possible for a wholly subcritical flow to connect two reservoirs of identical geometry containing different fluid depths without the flow passing through an hydraulic transition. In particular, geostrophic control in the manner discussed by
previous authors is characterised by $\psi_1 = \frac{1}{2}$ in the upstream reservoir and $\psi_1 = -\frac{1}{2}$ in the downstream reservoir.

In contrast, the control mechanism for two-layer hydraulics does not leave any free parameters, and so it is not possible to specify the ratio of the fluxes in the two boundary currents. Thus there is no mechanism by which geostrophic control may be achieved: a subcritical solution has only one possible realisation at a given section along the channel and so will only be able to match onto the other reservoir if the fluid in that reservoir has the same height. Instead of geostrophic control, two-layer systems may be partially controlled in the manner outlined in section 3.5.

Nof (1987a) suggested a third possible control mechanism for single-layer flows, yielding a still lower flow rate. In particular he analysed the rate at which a front separating regions of fluid of different potential vorticites advances along a channel of infinite length. The rate of advance and the associated flux of a baroclinic intrusion is entirely dominated by vorticity. However, Nof also noted that for finite length channels the front will eventually advance along the whole length and loose its ability to control the flow. Hydraulic or geostrophic control would then be able to be set-up as the dominant mechanism. Due to the transient nature of this control mechanism we do not attempt to apply it to the two-layer system discussed in this thesis.

In the next section we shall turn our attention to other mechanisms by which hydraulic control may be lost, or at least modified, by the breakdown of the fundamental assumption that all streamlines originate in the upstream reservoir. We also introduce the idea of a zone of stagnant fluid as a possible resolution of the problem.

6.6 Traceability and stagnation

One of the key assumptions to the analysis of the present rotating system is that the potential vorticity is uniform throughout a given fluid layer. This would clearly be the situation if all the streamlines can be traced back to their upstream reservoir, where the potential vorticity is uniform, and that they do not pass through any region capable of modifying the
potential vorticity (e.g., hydraulic jumps).

We should note at this point that in an hydraulic jump the rate of dissipation in the expanding layer is an order of magnitude greater than that in the contracting layer (Wood & Simpson, 1986), so that fluid coming from its upstream reservoir, and which has not yet passed through the contraction, is likely to be little affected by any jump in the supercritical region. Moreover, the jump will represent a weak shock for the contracting layer, and so the potential vorticity of this layer will not be changed significantly by the jump (Nof, 1986).

Of more concern is the ability to trace the streamlines back to the upstream reservoir. Provided the fluid velocity is everywhere nonzero and of the same sign within each of the fluid layers, then this does not represent a problem. However, in wider dimensionless channels (higher dimensional rotation rates), this may not always be the case. If, at a given section along the channel, the velocity in one of the fluid layers falls to zero somewhere across the channel, then three possibilities exist for the fluid in the region of reversed flow: the fluid may have come from the upstream reservoir and was turned around somewhere further down the channel; the fluid may be trapped in the closed streamlines of a recirculating eddy; or the fluid may have come from the downstream reservoir.

If the first of these scenarios is valid, then it may still be possible to apply the hydraulics ideas presented thus far. Provided nothing untoward occurs (e.g., an hydraulic jump) along this streamline, we may still trace it back to the reservoir. In the second scenario we are not necessarily able to state the value of the potential vorticity. While the fluid may have started with the same value of \( \Pi \) as its corresponding reservoir, the long time-scales associated with the steady state will have given any arbitrarily small viscous effects time to significantly change the potential vorticity along the closed streamline. In the extreme case the fluid may come to rest. The final scenario of streamlines originating from the downstream reservoir removes all possibility of hydraulic control as there is a clear mechanism by which information can propagate nominally downstream effects to the control sections. Moreover, there is no a priori reason why the downstream reservoir (for a given layer) should have the same
potential vorticity as its upstream counterpart (any hydraulic jump adjusting the supercritical flow to the reservoir conditions may well be a strong shock for the expanding layer and so generate significant potential vorticity – Nof, 1986).

We need not be too concerned about the second scenario if it occurs only in the supercritical regions as it would be unable to alter the essentially controlled nature of the flow. What happens with such recirculating eddies will only be of interest if we are concerned with the complete along channel structure of the flow. However, if such streamlines pass through both of the control sections (frequently we will be able to apply the hydraulic analysis if a recirculating streamline passes only through the control closer to its upstream source reservoir) they may have a profound effect on the flow. The situation then appears to the controls as though the fluid originated from the downstream reservoir, and hence we shall treat it as such.

The third scenario of streamlines originating from the downstream reservoir can occur only if there is a velocity reversal at (both) the control section(s). However the arguments apply equally as well to recirculating streamlines passing through both controls. It will be shown in chapters 7 and 8 that such a reversal is able to occur only if the interface is separated at (both) the control section(s), and even this is not a sufficient condition. For the present discussion we will assume the truth of this statement.

Consider a series of geometrically similar channels whose minimum width is $b_c$ (the series differs only in the width at the narrowest point) and are controlled solely by the narrowest section. Suppose that the interface first intersects the channel floor with $b_c = \text{bsep}$, and that the velocity in the upper layer at the $y = b_c$ wall falls to zero with $b_c = b_{stag} > \text{bsep}$. For all channels with $b_c$ wider than $b_{stag}$ there will be a velocity reversal at some point in the single layer region, decreasing the exchange flow rate.

This reduction in the exchange flow rate is not in itself significant unless we realise that we can not assert that the value of the potential vorticity for the fluid from downstream is the same as that originating from the upstream reservoir. Moreover, the velocity reversal is contrary to the empirical idea
of hydraulically controlled flows maximizing $q$. These two facts lead us to hypothesise the existence of a zone of stagnant fluid in place of the reverse flow. Gill (1977) found such zones in his analysis of single-layer hydraulics. In doing so we eliminate the difficulties associated with streamlines starting in the downstream reservoir and the paradox of decreasing $\bar{q}$ with increasing channel width (instead, the exchange flow rate for channels wider than $b_{stag}$ will be independent of the channel width - such an idea is supported by the experimental work discussed in chapter 9).

Whether such a flow/vorticity distribution is either realisable or stable we are unable to say at present. As the potential vorticity gradient does not reverse sign, its stability seems likely (section 7.1, Pedlosky, 1979). To be realisable, the solution must be able to be extended continuously throughout the subcritical and supercritical regions, at least up to the point where the solution breaks down due to hydraulic jumps or similar dissipative regions. There are two main mechanisms through which the stagnant fluid may attain the necessary potential vorticity: the fluid may be at rest for all time (such as may occur if the flow were set-up from rest, e.g. a dam-break - see the discussion in section 9.3 on Margules fronts), or the small, but nonzero viscosity may be able to act over very large times to eventually bring the fluid to rest.

Figure 6.3 shows a cross-section and the associated velocity distribution for such a stagnating flow. For convenience, we centre the coordinate system on the nonzero velocity region so that it extends from $y = -\xi$ to $y = \xi$ (note that here $\xi < \frac{1}{2}b_c$; this applies only for the present section), the interface intersecting the channel floor at $y = \zeta$. The stagnant zone plays no dynamical role in the steady state, and thus we may replace the channel with one of width $2\xi$. The equations are therefore those of section 6.3, though the current width is defined implicitly to give stagnation as necessary. Note that this introduces an additional degree of freedom which may not be able to be modelled as an hydraulic flow.

We shall utilise these simple ideas of stagnating zones occurring in our discussion of nonzero potential vorticity channels in the next chapter. However, in chapter 8 we shall show that, provided the channel is narrower than one Rossby radius, the
Figure 6.3. Rectangular cross-section containing a *separated* flow in which a zone of stagnant fluid has formed. *Top*: interface profile; *bottom*: velocity profile.
flow does not undergo velocity reversal at either control section, and so stagnation will not occur. Finally chapter 9 will discuss these ideas in light of the experimental work presented there.
7 Uniform potential vorticity - a simple problem

7.1 Equations

The equations and ideas introduced in chapter 6 are unfortunately complicated. Although they may be solved numerically in their most general form, it is not clear that the results would be worth the effort involved. In this chapter we explore in some detail one special case of a flow with uniform potential vorticity: the flow through a channel of constant depth and varying width. For a large variety of oceanic flows we do not expect the conditions in the two reservoirs to produce radically differing potential vorticities, so taking the two potential vorticities as being equal,

$$\Pi_1 = \Pi_2 = \Pi,$$  \hspace{1cm} (7.1)

($\Delta\Pi = 0$) should capture all the essential features. We confine our attention to $\Pi \in [0,1]$ (i.e., the reservoirs are at least as deep as the channel). In addition, we shall assume there is no net flow through the channel,

$$Q = 0.$$  \hspace{1cm} (7.2)

In section 7.2 we shall relax the condition (7.1) slightly to allow for the formation of zones of stagnant fluid as outlined in section 6.6. Under these restrictions of no net flow and equal potential vorticity we shall show that the exchange flow rate and interface behaviour are only weakly dependent on the value of the potential vorticity. Motivated by this finding, in chapter 8 we shall look at zero potential vorticity flows in more general geometries and reintroduce net barotropic forcing.

As in the classical limit, the unforced flow in a channel of constant depth is the most easily understood and analysed. Again, we are able to use symmetry arguments (recall that the along channel length scale is not important and so the channel may be treated as symmetric and that we expect the controlled solutions to be antisymmetric) to arrive at many of the conclusions revealed by the hydraulic theory.
With these simplifications, the equations for the interface and layer velocities for an attached flow (equations (6.11) and (6.14)) become

\[
\begin{align*}
    h &= D \left( \frac{1}{2} + A \text{ sech} \xi \cosh \eta + B \text{ csch} \xi \sinh \eta \right), \\
    u_1 &= \left( 1 - \frac{1}{2} D \Pi \right) y + A B D / \xi - \\
    &\quad - \frac{1}{2} D \lambda \left( A \text{ sech} \xi \sinh \eta + B \text{ csch} \xi \cosh \eta \right), \\
    u_2 &= \left( 1 - \frac{1}{2} D \Pi \right) y + A B D / \xi + \\
    &\quad + \frac{1}{2} D \lambda \left( A \text{ sech} \xi \sinh \eta + B \text{ csch} \xi \cosh \eta \right).
\end{align*}
\]

(7.3)

(7.4)

Similarly for flow separated from the \( y = \xi \) sidewall, with \( A < 0 \), equations (6.22) and (6.24) simplify to

\[
\begin{align*}
    h &= \begin{cases} \\
        D \left( \frac{1}{2} + A \text{ sech} \xi \cosh \eta + B \text{ csch} \xi \sinh \eta \right); & -\xi \leq y < \xi, \\
        0, & \xi < y \leq \xi'.
        \end{cases} \\
    u_1 &= \begin{cases} \\
        \left( 1 - \frac{1}{2} D \Pi \right) y + c - \\
        - \frac{1}{2} D \lambda \left( A \text{ sech} \xi \sinh \eta + B \text{ csch} \xi \cosh \eta \right), & -\xi \leq y < \xi, \\
        0, & \xi \leq y \leq \xi.
        \end{cases} \\
    u_2 &= \begin{cases} \\
        \left( 1 - \frac{1}{2} D \Pi \right) y + \frac{1}{2} D \Pi \xi + c + \\
        + \frac{1}{2} D \lambda \left( A \text{ sech} \xi \sinh \eta + B \text{ csch} \xi \cosh \eta \right), & -\xi \leq y \leq \xi, \\
        \left( 1 - D \Pi \right) y + \frac{1}{2} D \Pi \xi + c + \\
        + \frac{1}{2} D \lambda \left( A \text{ sech} \xi \sinh \eta + B \text{ csch} \xi \cosh \eta \right), & \xi \leq y \leq \xi'.
        \end{cases}
\end{align*}
\]

(7.5)

(7.6)

where

\[
c = \frac{D}{2 \xi} \left\{ A B \left[ \frac{\cosh \lambda \xi \sinh \eta}{\cosh \lambda \xi \sinh \eta} + 1 \right] + \\
+ \frac{1}{4} A^2 \left[ \frac{\cosh^2 \lambda \xi - 1}{\cosh^2 \lambda \xi} \right] + \frac{1}{2} B^2 \left[ \frac{\sinh^2 \lambda \xi}{\sinh^2 \lambda \xi} - 1 \right] \right\}.
\]
Uniform potential vorticity Section 7.1

\[
- \frac{1}{2} A \left[ \frac{\cosh \xi - 1}{\cosh \xi} \right] - \frac{1}{2} B \left[ \frac{\sinh \xi - 1}{\sinh \xi} \right],
\]

(7.7)

is obtained from setting \( Q = 0 \), and

\[
\zeta = \ln \left[ - \frac{\sqrt{1 - (A^2 \sech^2 \lambda \xi - B^2 \csch^2 \lambda \xi) \left[ \frac{1}{2} + \frac{1}{2} \right]}}{A \sech \lambda \xi + B \csch \lambda \xi} \right].
\]

(7.8)

We shall not consider separated flows corresponding to positive values of \( B \) in this chapter.

The exchange flow rates are

\[
\bar{q} = 4 \left[ \frac{t}{\lambda \xi} A^2 D + \frac{1 - \frac{1}{2} D \pi}{\lambda t} \right] \text{ for attached}
\]

and

\[
\bar{q} = -D \left[ \begin{array}{c}
2 A \left[ \frac{c}{\lambda} \left( \frac{\sinh \lambda \xi}{\cosh \lambda \xi} + t \right) + \frac{1 - \frac{1}{2} D \pi}{\lambda^2} \left[ \frac{\lambda \xi \sinh \lambda \xi}{\cosh \lambda \xi} - \lambda \xi t - \frac{\cosh \lambda \xi}{\cosh \lambda \xi} + 1 \right] \right] + \\
+ \frac{1}{2} B \left[ \frac{c}{\lambda} \left( \frac{\cosh \lambda \xi - t'}{\sinh \lambda \xi} \right) + \frac{1 - \frac{1}{2} D \pi}{\lambda^2} \left[ \frac{\lambda \xi \cosh \lambda \xi}{\sinh \lambda \xi} + \frac{\lambda \xi \sinh \lambda \xi}{\sinh \lambda \xi} - 1 \right] - \frac{1}{2} D \right] - \\
- \frac{1}{2} \left( 1 - \frac{1}{2} D \pi \right) \left( \xi^2 - \xi^2 \right) - c \left( \xi - \xi \right) \end{array} \right],
\]

(7.10)

\((t = \tanh \lambda \xi)\) for attached and separated (with \( B < 0 \)) flows respectively.

Equation (6.30) defines the hydraulic functional for either rotating or nonrotating flows. For convenience we shall reference this to the \( y = -\xi \) wall. When the flow is separated from this wall we must evaluate the functional referenced to the \( y = \xi \) wall. Note that the functional achieved behaves identically in either case as
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$\Delta g$ differs between the two walls by $\frac{\partial}{\partial y} \Pi$ which is constant everywhere along the channel and is absorbed in the function $\xi$.

For values of the interface coefficients representing attached flows the hydraulic functional may be written as

$$J(\cdot;A) = \xi + [(1 - \frac{\partial g}{\partial y}) - \frac{\partial g}{\partial y} \frac{\partial^2}{\partial y^2} \frac{\partial}{\partial y} \frac{\partial^2}{\partial y^2}] (A + A \Pi) -$$

$$- H - D (\frac{\partial}{\partial y} + A - B), \quad (7.11)$$

where the slope coefficient is evaluated from the conservation of mass as

$$B = \frac{\frac{\partial}{\partial y} \Pi}{\frac{\partial}{\partial y} \Pi^2}.$$

When the functional represents separated flows it may be evaluated from

$$J(\cdot;A) = \xi + [(1 - \frac{\partial g}{\partial y}) - c] (A + B \frac{\partial}{\partial y}) -$$

$$- H - D (\frac{\partial}{\partial y} + A - B), \quad (7.13)$$

where $c$ is given in terms of $A$ and $B$ by (7.7), and $B$ is evaluated from (7.10) to conserve mass. Unfortunately it is not possible to write down an explicit solution for $B$; moreover, there will in general be more than one value of $B$ satisfying conservation of mass. The appropriate solution is selected by insisting that $J(\cdot;A)$ is continuously defined for all $A$ resulting in two layers somewhere in the cross-section.

In general, the functional will represent attached flows for some values of $A$, and separated flows for others, though under some circumstances it may be attached or separated for all values of $A$.

Any further discussion on the hydraulic functional when it represents separated flows is hampered through the need to evaluate it numerically. However, so long as the functional represents attached flows, we may continue the analysis analytically.

Suppose that, at some section $x = x_0$, the interface height coefficient is $A = A_0$. For the flow to be critical at this section, equation (6.33) must hold. Applying this to (7.11) and (7.12) reveals that the exchange flow rate must be given by
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\[ \bar{q}^2 = 16 \frac{D^3}{\lambda} \xi^2 \tau^2 \frac{1 - (1 - \omega D \Pi) \lambda \xi t}{[(1 - \omega D \Pi) \lambda \xi^2 - \tau \xi]/D - 3 \lambda A^2 \tau^2} \left[ \frac{t}{A^2} + \frac{1 - \omega D \Pi}{D \lambda t} \xi - \frac{1}{D \lambda^2} \right]^3. \]  \hspace{1cm} (7.14)

The position of the control section(s) may be ascertained in a manner identical to that of section 3.2. Recall from equations (2.11) and (3.7) that

\[ K = (\partial J/\partial \xi) (d \xi/dx) = 0 \]  \hspace{1cm} (7.15)

at control sections for a channel of constant depth and bottom elevation. This has solutions when either \( d \xi/dx = 0 \), which occurs at the narrowest point in the channel, or when \( \partial J/\partial \xi = 0 \) which occurs wherever \( A = 0 \). Note that \( A = 0 \) also maximizes (7.14) and gives \( d^2 J/dA^2 = 0 \) (i.e. the two phase velocities of small amplitude internal Kelvin waves vanish). Therefore at the control section(s)

\[ A_c = A_v = 0. \]  \hspace{1cm} (7.16)

The flow will be controlled by the section or pair of sections which minimize

\[ \bar{q} = 4 \left\{ \frac{D}{\lambda^5 \tau} [1 - (1 - \omega D \Pi) \lambda \xi t] \right\}^{\lambda \xi} \left[ t - (1 - \omega D \Pi) \lambda \xi \right]. \]  \hspace{1cm} (7.17)

(7.17) is obtained from (7.14) with \( A \) set to zero. In the classical limit, the equivalent expression to (7.17) was minimized by the narrowest section of the channel. However, once rotation is introduced this is no longer necessarily true. Differentiation of (7.17) with respect to \( \xi \) produces a zero for all positive values of \( \Pi \) for some \( \xi = \xi_{tp} \) less than or equal to that associated with the separation of the interface from both channel walls (if the channel is controlled at the section whose width is \( \xi_c \) then the interface will lie from corner to corner in the channel when \( \xi_c = \xi_{sep} \) since \( A_c = 0 \)). In the zero potential vorticity limit (\( \Pi = 0 \)) \( \xi_{tp} \) and \( \xi_{sep} \) coincide and are equal to \( 3 \lambda D^2 \Pi^2 \). As \( \Pi \) increases from zero, both \( \xi_{sep} \) and \( \xi_{tp} \) increase, though not at the same rate.
Figure 7.1. Variations with $\Pi$ in the channel width for separation (solid line), local maximum in $q$ (dot-dash line) and the width for an exchange flow rate equal to that at separation (dashed line).
Figure 7.1 plots how $\xi_{sep}$ and $\xi_{tp}$ vary as a function of $\Pi$ in channels of unit depth. The $\xi_{sep}$ line delimits regions of the parameter plane representing attached and separated flows when critical conditions prevail. The $\xi_{tp}$ line traces the local maximum of (7.17). The third line on the plot ($\xi_{same}$) indicates the width of a channel (with the same potential vorticity) having the same exchange flow rate as one of half-width $\xi_{sep}$ at the control section.

Suppose the minimum half width of the channel is $\xi_{min}$. Provided $\xi_{min}$ is less than $\xi_{same}$ the channel will be controlled by the narrowest section. If however $\xi_{min}$ lies between $\xi_{same}$ and $\xi_{sep}$ then the exchange flow rate required to produce critical conditions at the narrowest section is greater than that required to produce critical conditions away from this section where the width has increased to $\xi = \xi_{stag}$, for example. Thus the flow would not be controlled by the narrowest section, but instead by a pair of controls in the wider regions of the channel on either side of the contraction. The flow through the contraction would be subcritical with $A = 0$ (thus retaining the required symmetry).

In the next section we shall show that if a stagnation zone forms when the interface separates at the control section, then in channels falling between the $\xi_{same}$ and $\xi_{sep}$ lines, the flow will be critical for all channels whose half-width is greater than or equal to $\xi_{sep}$.

7.2 Variations with channel width and potential vorticity

Before discussing in detail how the exchange flow varies with the channel width and geometry, we shall note some of the features of the flow as it separates from one (or both) of the channel walls. In the previous section (equation 7.16) we showed that at the control section(s) the interface height coefficient vanishes. Therefore, in order that the interface just intersects the channel top and bottom ($h(y=\xi) = 0$ and $h(y=-\xi) = D$), the slope coefficient $B_c$ must be $-\frac{1}{2}$. If $B_c > -\frac{1}{2}$ the flow is attached and if $B_c < -\frac{1}{2}$ the flow is separated. As $u_1(y) = -u_2(-y)$ at the control section and the interface is an odd function (in $y$) about the midpoint of the channel, we need only consider what is happening in one of the two...
Substituting (7.1#16) into (7.1#12) gives the slope coefficient at the control section(s) as

$$B_c = -\left\{ \frac{t}{\lambda}(1 - y_2 \Pi) - \frac{t^2}{\lambda} \xi^2 \right\} D^{-2}.$$  

The solution of $B_c = -\frac{1}{2}$ (the interface just separating) may be used in conjunction with $u_1(y=-\xi)$ (equation (7.4)) to show that the velocities in the lower layer at the $y = -\xi$ wall and in the upper layer at the $y = \xi$ wall vanish as the interface just separates. Moreover, if we insist that the potential vorticity is exactly $\Pi$ everywhere in the channel, then, if the narrowest section is wider than that for separation to just occur, equation (7.4) shows the velocity will change sign within the two resulting single-layer regions (one on either side of a centrally placed two-layer core).

An alternative to this is the formation of stagnant zones of fluid in place of the velocity reversal, as was outlined in section 6.6. If $\xi_c > \xi_{sep}$ then one or two stagnant zones could form, their combined width being equal to $2(\xi_c - \xi_{sep})$. There is insufficient information in the steady state limit to determine where in the channel the two-layer region will be. However, if we look at the time-dependent set-up of the flow (such as is done for the linear limit in the next section, or from considering the dam-break problem in a wide channel - see section 9.3), we may assert that just on the dense reservoir side of the control section the two-layer region is against the $y = \xi$ wall, whereas on the light reservoir side of the control section the two-layer region is against the $y = -\xi$ wall. Moreover, within a region approximately one Rossby radius in length, the two-layer portion of the flow must cross from one side of the channel to the other (when $0 < b_c < 1$ the flow passes through an hydraulic transition which causes the flow to cross from one side of the channel to the other over a length scale determined by the along-channel geometry). As we shall explain more fully in section 9.3, the length scale is determined by the Rossby radius since the same channel-crossing process must occur regardless of the length of channel having the control section geometry provided the channel is greater than one Rossby radius wide (i.e. the channel-crossing
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is determined by the shorter of the geometric length scale $b_c$ and the Rossby radius).

Unfortunately our assumption that the flow is relatively straight breaks down if such a channel-crossing occurs, so we are not able to determine its details using the present analysis. However, to either side of the channel-crossing the geometry is still that of the control section, allowing us to determine the details of the flow elsewhere in the channel where the flow is relatively straight.

The validity of this approach for flows separated at the control section(s) will be discussed more fully in chapter 9 in light of the experimental results reported there. For the rest of this chapter, however, we shall assume the existence and stability of the stagnation zones and note some of the consequences.

If stagnation zones were to form in the manner outlined above for all channels with $t_c > t_{sep}$ (minimum width greater than 2 $t_{same}$), then the flow would behave exactly as though the narrowest section were only of half-width $t_{sep}$. The exchange flow rate would be given by equation (7.17) with $t$ set to $t_{sep}$. Moreover, the stagnation zone makes every section of the channel appear the same to the flow. The two-layer region will everywhere (except in the channel-crossing) be 2 $t_{sep}$ wide, and the interface will everywhere have the same shape and scale.

More fundamentally the flow will be critical, $\partial J/\partial A = 0$, everywhere throughout the channel (where two layers exist); $\partial^2 J/\partial A^2$ will vanish also, showing both long, small amplitude Kelvin wave phase velocities are zero. The concepts of hydraulic control can not be applied in their traditional sense. If the depth also varies somewhere along the channel then a supercritical flow may ensue and we may be able to call the flow hydraulically controlled, but that is beyond the scope of this thesis.

Figure 7.2 shows how the interface slope coefficient at the control section, $B_c$, (wherever this may be) varies as a function of the channel width ($b_c$) and potential vorticity ($\Pi$). For clarity, the value of $B_c$ for a separated flow is not shown, but will be $t_c/t_{sep}$ times its value at separation (from equation (7.3) with $A = 0$ noting that $h(y = t_{sep})$ must vanish). In the zero potential vorticity limit, $B_c$ decreases from zero linearly with $t_c$, reaching the value of $-\frac{1}{2}$ associated with separation when
Figure 7.2. Variations in the interface slope coefficient as a function of the control section width $b_c$ and potential vorticity $\Pi$. The region shaded green is not accessible to channels of constant depth—see text.

Figure 7.3. Variations in the exchange flow rate $\bar{q}$ (for critical conditions) as a function of the control section width $b_c$ and potential vorticity $\Pi$. Note the local maximum in the surface and the constant value of $\bar{q}$ for channels in which the flow is separated (region shaded pink). The region shaded green is not accessible to channels of constant depth—see text.
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\[ \xi_c = \frac{\Omega}{D} \] For higher values of potential vorticity, \( \xi_c \) decreases more slowly with \( \xi_c \), delaying interface separation.

Figure 7.3 shows how the exchange flow rate varies over the same parameter domain. There is comparatively little variation with \( \Pi \), the exception being the introduction of the local maximum in \( \tilde{q} \) for \( \Pi > 0 \). For fixed \( \Pi \), increasing the width of the control section from zero does not increase the exchange flow rate as quickly as would be the case if rotation played no dynamical role (the surface would be given by \( \tilde{q} = \frac{\Omega}{2} b_c \)). In the extreme, for separated flows, the exchange flow rate is independent of the channel width, whereas in the absence of rotation \( \tilde{q} \) would continue to increase linearly. Note that the region of the curve between the \( \xi_{\text{same}} \) and \( \xi_{\text{sep}} \) lines shown in figure 7.1 - coloured green in figure 7.3 - is not accessible in the channels of constant depth under discussion. That fraction of the surface is included here both for completeness and as it represents realisable flows in some other channel geometries (such as channels of constant width whose depth varies in such a manner as for there to be a horizontal plane of reflectional symmetry - such a channel was used in the experiments discussed in section 9.4).

Figure 7.4 plots the specific exchange flow rate \( \tilde{q}/b_c \) to show directly the effects of rotation on the exchange flow rate. A value of \( \frac{\Omega}{2} \) shows rotation plays no role. When \( \xi_c > \xi_{\text{sep}} \), the curve falls off as \( 1/\xi_c \).

The weak dependence of the exchange flow rate on the value of the potential vorticity is shown more clearly by figure 7.5 which plots \( \tilde{q}(b_c, \Pi)/\tilde{q}(b_c, 0) \). Over most of the parameter domain the value of \( \Pi \) changes the exchange flow rate by less than approximately 3.5%. The difference becomes larger only for \( b_c > 1 \), at which time the direct comparison used in this plot loses much of its significance as it is comparing either an attached flow with the separated zero potential vorticity limit, or simply the channel widths at separation.

This weak dependence of \( \tilde{q} \) on \( \Pi \) is very encouraging: it shows that the actual value of the potential vorticity is not important in determining the qualitative nature of the flow, at least so long as the minimum channel width channel is less than one Rossby radius. The only real effect introduced by nonzero \( \Pi \) is a slight variation in the control section width which will lead to
Figure 7.4. Variations in the specific exchange flow rate $\bar{q}/b_c$ (for critical conditions) as a function of the control section width $b_c$ and potential vorticity $\Pi$. Colour coding as for figure 7.3.

Figure 7.5. The effect of potential vorticity on the exchange flow rate. The surface $\bar{q}(b_c,\Pi)/\bar{q}(b_c,0)$ is plotted as a function of $b_c$ and $\Pi$. Colour coding as for figure 7.3.
separation of the interface. In terms of the overall physics of the flow, the analysis of parabolic channels in chapter 5 showed a remarkable qualitative agreement between rectangular and parabolic cross-sections in the absence of rotation. The gross features of the hydraulic control mechanism appear to be relatively insensitive to how they are achieved, thus the weak dependence of the flow on $\Pi$ found in this section may not be that surprising. Further, we note that the gross features of small amplitude Kelvin waves are little affected by the value of the potential vorticity, except in the fine detail of their cross-channel profiles.

The analysis leading to this conclusion is valid only for channels whose depth varies in a symmetric manner (though only constant depth channels have been discussed explicitly) and have no net flow through them. However, it is reasonable to expect that the same weak $\Pi$ dependence will carry over, at least in a more limited form, to more general variations in channel geometry and barotropic forcing. Moreover, it has recently been suggested (Armi, pers. comm.) for the single-layer case that the potential vorticity may not be uniform but instead adjust itself so as to maximize the exchange flow rate. If such an idea were to apply to the two-layer situation, then analysing the zero potential vorticity limit (the exchange flow rate for $\Pi \neq 0$ is generally less than that for $\Pi = 0$ when $b_c = 1$ in a channel of unity depth) is the appropriate thing to do. Nevertheless the naive introduction of a zone of stagnant fluid may not be appropriate; generalisation of this concept to allow for net barotropic flows is not obvious (there are also some difficulties applying it to sill flows as will be discussed in section 9.5).

Figure 7.6 shows the along-channel structure of the hydraulic flow for three different combinations of $b_c$ and $\Pi$. In all cases the velocity within a given layer has been assumed to be unidirectional everywhere along the channel: a zone of stagnant fluid forms, if necessary, to ensure this. Plots of both interface profiles and the velocity in the upper layer ($u_2$) are given. Separation from the sidewall is marked on the velocity plots by a solid line, the coloured (yellow) region of non-zero velocity corresponding to flow within a single layer region.

Even in comparatively narrow channels (e.g. $b_c = 0.2$ as shown in figure 7.6a) the interface may separate from the sidewalls away
(a) $b_c = 0.2; \Pi = 0.0$

Caption on next page.
Figure 7.6. Perspective view of along-channel structure of controlled flows in constant depth channels. A stagnation zone forms instead of a velocity reversal. Contraction width and potential vorticity as marked.

Continued on next page.
Control section(s) indicated by arrows. For each the upper plot is the interface profile, the central plot the velocity in the upper layer and the bottom plot approximate streamlines in the upper layer. Yellow colouring indicates regions where the flow is separated but $u_2 \neq 0$, and green where $u_2 = 0$. 

(c) $b_0 = 1.0; \Pi = 1.0$. 
from the control sections. This should be remembered when utilising classical two-layer hydraulics to analyse oceanographic exchange flows at low latitudes such as in the Strait of Gibraltar. Figure 7.6c is an example of a flow in which the control sections (indicated by arrows) have split and moved away from the narrowest section. The flow has a subcritical region through the narrowest section. Everywhere else in the channel the flow is critical \( \frac{dJ}{dA} = 0 \) with interface separation and velocity stagnation coinciding.

In chapter 8 we shall look at zero potential vorticity flows in much greater detail. Before we shall discuss some of the features of the adjustment/set-up problem and the associated energetics.

### 7.3 Two-layer Rossby adjustment in an infinitely long channel

In this section we shall extend Gill's (1976) analysis of the single-layer Rossby adjustment problem for an infinitely long channel, to include a second layer. While such a problem represents perturbations to a submaximal (i.e. not hydraulically controlled) flow, it does illuminate a number of important features.

Consider an infinitely long channel of uniform cross-section. Within the channel there are two inviscid, immiscible fluids of different densities, the one of lower density overlying the denser one. Suppose that at \( t = 0 \) the two fluids are at rest and that there is a small discontinuity in the height of the interface at \( x = 0 \), as shown in figure 7.7. The equations governing the flow are (6.4). We shall rescale all length scales, relative to those of earlier sections, by

\[
(\hat{x}, \hat{y}, \hat{z}) = \frac{D^2}{H (D - H)} (x, y, z) \tag{7.19}
\]

to remove the dependence on the basic interface height. Equations (6.4) are not affected by this rescaling. We shall drop the hats \(^\wedge\) from here on.

Suppose there exists a basic state such that the fluid is at rest and at equilibrium. Clearly such a state is given by
Figure 7.7. Definition sketch and initial state for linear adjustment problem.
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\( u_i = U_i = 0 \) with the fluid interface at \( \eta = H \) and the pressure \( p = P \) for an infinitely long channel of uniform rectangular cross-section, as shown in figure 7.7.

Linearising around this basic state

\[
\begin{align*}
    u_i &= U_i + u_i', \\
    p &= P + p', \\
    \eta &= H + \eta',
\end{align*}
\]

(7.20)

and requiring the depth of the fluid to be much less than the internal Rossby radius, yields the perturbation equations

\[
\begin{align*}
    \frac{\partial u_i'}{\partial t} &= -\frac{\delta (P'}{\delta x \rho_i} + \frac{\eta'}{\rho_1 - \rho_2} + v_1', \\
    \frac{\partial v_i'}{\partial t} &= -\frac{\delta (p'}{\delta y \rho_i} + \frac{\eta'}{\rho_1 - \rho_2} + u_1', \\
    0 &= -\frac{\delta (p'}{\delta z \rho_i} + \frac{\eta'}{\rho_1 - \rho_2}, \\
    \frac{\partial u_i'}{\partial x} + \frac{\partial v_i'}{\partial y} &= 0,
\end{align*}
\]

(7.21)

for \( i = 1, 2 \) at the interface. The pressure must be continuous at the interface (in the absence of surface tension), so \( p' \) may be eliminated between the \( i = 1 \) and \( i = 2 \) equations. Further, if the Boussinesq approximation is valid (i.e., \( \rho_1 - \rho_2 \ll \rho_1, \rho_2 \)), the system reduces to

\[
\begin{align*}
    \frac{\partial}{\partial t} (u_1' - u_2') - (v_1' - v_2') &= -\frac{\partial}{\partial x} \eta', \\
    \frac{\partial}{\partial t} (v_1' - v_2') + (u_1' - u_2') &= -\frac{\partial}{\partial y} \eta', \\
    \frac{\partial}{\partial x} (u_1' - u_2') + \frac{\partial}{\partial y} (v_1' - v_2') &= 0. 
\end{align*}
\]

(7.22)

From here on we write \( u = u_1' - u_2', \ v = v_1' - v_2' \) and drop the prime from \( \eta' \). The linearised potential vorticity equation
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(conserved for a point in space) may be determined to give the complete system as

\[
\begin{align*}
\frac{\partial u}{\partial t} - v &= - \frac{\partial}{\partial x} \eta, \\
\frac{\partial v}{\partial t} + u &= - \frac{\partial}{\partial y} \eta, \\
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \eta &= \Pi.
\end{align*}
\tag{7.23}
\]

This system is identical to that used by Gill (1976) for the linear Rossby adjustment of a single layer of fluid.

The initial conditions and boundary conditions for this flow are

\[
\begin{align*}
u(x, y, t=0) &= u_0 = 0, \\
v(x, y, t=0) &= v_0 = 0, \\
\eta(x, y, t=0) &= \eta_0 = \epsilon \text{ sgn } x,
\end{align*}
\]

where

\[
\text{sgn } x = \begin{cases} 
-1, & x < 0 \\
0, & x = 0, \\
1, & x > 0
\end{cases}
\]

and

\[
\begin{align*}
u(x, y, t) &\to u_0 \quad \text{as} \quad |x| \to \infty, \\
v(x, y, t) &\to v_0 \quad \text{as} \quad |x| \to \infty, \\
\eta(x, y, t) &\to \eta_0 \quad \text{as} \quad |x| \to \infty.
\end{align*}
\tag{7.24}
\]

By breaking \( u, v \) and \( \eta \) into their even and odd components, it is possible to write them as Fourier series in \( y \) (for details, see Gill, 1976), obtaining eventually

\[
\begin{align*}
u_{\text{ev}} &= \begin{cases} 
\epsilon \left( \text{sech} \frac{\mu y}{b} \cosh y + I \kappa \left( \frac{1}{2} \gamma^2 \right) \int_{0}^{\infty} \left( \frac{1}{2} \left( t^2 - \gamma^2 \right)^{3/2} \right) \cos \omega y \right), & |x| < t, \\
0, & |x| > t
\end{cases}
\end{align*}
\]

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\[ v_{ev} = \begin{cases} \frac{\epsilon (t^2-x^2)^{1/2}}{r + x} \sum_j \frac{\alpha_j}{(1 + m^2)^{1/2}} J_0[(1 + m^2)^{1/2} r] \cos mx \, dr, & |x| \leq t \\ 0, & |x| > t \end{cases} \]

\[ \eta_{ev} = \begin{cases} \frac{\epsilon x (t^2-x^2)^{1/2}}{r + x} \sum_j \frac{\alpha_j}{(1 + m^2)^{1/2}} J_0[(1 + m^2)^{1/2} r] \cos mx \, dr, & |x| \leq t \\ \epsilon \text{sgn} x, & |x| > t \end{cases} \]

where

\[ m = (2j + 1) \pi/b, \]
\[ \alpha_j = (-1)^j \frac{b}{(m b)}. \] (7.25)

Plots of streamlines and profiles may be found in Gill (1976), though note that streamlines for \( u, v \) do not necessarily correspond to those for \( u_1, v_1' \).

Consider the limit as \( t \to \infty \): the flow never attains a steady state throughout the whole of the \( x \) domain as the internal Kelvin waves continue to propagate for ever along the infinite channel. However, within an arbitrarily large region around the position of the initial discontinuity, the solution does become steady, tending towards

\[ \eta_{ev} = \frac{\epsilon \sum_j \frac{\alpha_j}{1 + m^2} M[(1 + m^2)^{1/2} x] \cos mx,} \]
\[ \eta_{od} = \epsilon \sinh y \text{ sech} \, \eta \, b. \]
\[ u_{ev} = -\epsilon \cosh y \text{ sech} \, \eta \, b. \]
\[ u_{od} = \epsilon \sum_j \frac{m}{1 + m^2} \alpha_j M[(1 + m^2)^{1/2} x] \sin mx \]

where

\[ M(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ -1 + e^x & x < 0 \end{cases}. \] (7.26)

At \( x = 0 \) these reduce to
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\[ \eta = \xi \sinh y \ \text{sech} \ \gamma b, \]
\[ u = -\xi \cosh y \ \text{sech} \ \gamma b. \]  \hspace{1cm} (7.27)

When the width of the channel was large, Gill found that in the steady state limit the flow takes place in currents, one Rossby radius in width, confined to the boundaries. The \textit{upstream} boundary current \((x \to \infty)\) is against the \(y = -\gamma b\) wall and the \textit{downstream} \((x \to -\infty)\) current against the \(y = \gamma b\) wall. At the position of the initial discontinuity, the current crosses from one channel wall to the other with the cross-channel velocity at \(x = 0\) tending towards unity away from the walls.

For the present two-layer flow, if the flow is symmetric (the average interface height is half the depth of the channel), then

\[ u_1(y=\xi) = -u_2(y=-\xi) \]

and

\[ u_1(x=0,y) = -\frac{\gamma}{2} \xi \ \text{sech} \ \frac{\gamma}{2} b \cosh y, \]
\[ u_2(x=0,y) = \frac{\gamma}{2} \xi \ \text{sech} \ \frac{\gamma}{2} b \cosh y. \]  \hspace{1cm} (7.28)

The boundary current and channel-crossing structure will be essentially the same as in Gill's single-layer flow. Note that (7.27) and (7.28) give the interface height and velocity in exactly the same form as equations (7.3) and (7.4) with \(\xi \ \text{sech} \ \gamma b\) replacing \(\lambda \ B \csch \ \lambda \xi\), \(A\) set to zero and \(\Pi\) set to unity (though remember the rescaled length scales). Thus the small wave solution, that the linear problem discussed in this section represents, yields a velocity and interface profile of the same form as the hydraulic solution at the position of the initial discontinuity. We therefore expect some of the features of this linear solution to apply, at least qualitatively, to the hydraulic solution.

7.4 Energetics

A well-known feature of adjustment problems in rotating flows is that the amount of potential energy released by the adjustment process is greater than the increase in the kinetic energy of the mean flow. As this occurs in full time-dependent inviscid problems, the difference can not caused by dissipation (as was the case for the flow over a sill in section 3.3), but rather it is...
due to the generation of internal Poincare' waves which radiate some of the released energy to infinity (Gill 1976). Gill's analysis, adapted for two-layer flows in the previous section, employs the notion of Poincare' waves explicitly when representing the flow in terms of its Fourier components. It is of interest to see not only how the wave energy radiation is carried over to the linear Rossby adjustment of a two-layer system, but also to gain some qualitative measure of the generation of Poincare' waves in hydraulically controlled flows.

We shall define the energy partition \( \mu \) as the ratio of the rate of increase in kinetic energy \( \dot{E}_k \) of the mean flow to the rate at which potential energy is released \( \dot{E}_p \),

\[
\mu = -\frac{\dot{E}_k}{\dot{E}_p}
\]  

(7.29)

Both rates of change of energy will be calculated as the flow in the central region approaches a steady state. While the definition of \( \mu \) here is the same as that given in section 3.3 (equation 3.17), the reason for its departure from unity is different. In section 3.3 \( \mu \neq 1 \) was the result of energy dissipation, whereas here it is the radiation of energy by Poincare waves.

Choosing the appropriate frame of reference is important when discussing the energy of a system; the results and conclusions may be very misleading if the frame chosen is incorrect. For rotating flows it is necessary to choose a frame fixed with respect to the rotating geometry. The results in any other frame will be a function of the distance from the axis of rotation whereas in section 6.1 we showed that that the fluid motion does not know of the position of the rotation axis. Moreover, the zero energy state must be carefully chosen. We contend that the flow pattern which would be present (to obtain the specified potential vorticity) if baroclinic (stratification) effects were not present, should be considered the zero kinetic energy state. Such a flow must, in general, exist in order that the initial conditions possess a potential vorticity equal to that of the final state and specified by \( \Pi \). The energies released or gained are obtained by supposing such a flow exists before switching on baroclinic effects.

We shall consider first the energetics of the linear Rossby adjustment problem. In the previous section we derived the full
time dependent solution in terms of the Poincare' waves which radiate to infinity a fraction \( 1 - \mu \) of the potential energy released. For this linear problem the rates at which potential energy is released and kinetic energy of the mean motion is gained are

\[
\text{Ep} = \int_{-\xi}^{\xi} \frac{\varepsilon}{\rho_1 - \rho_2} \left\{ \rho_1 (u_1 - \bar{u}) \frac{\varepsilon - \eta}{\varepsilon + \eta} H + \right. \\
\left. + \rho_2 (u_2 - \bar{u}) \frac{\varepsilon + \eta}{\varepsilon - \eta} H \right\} dy, \quad (7.30)
\]

\[
\text{Ek} = \int_{-\xi}^{\xi} \frac{\rho_2 (H + \eta)(\varepsilon - \eta)(u_1 - \bar{u}) - (H - \eta)(\varepsilon + \eta)(u_2 - \bar{u})}{\varepsilon^2 - \eta^2} \\
\left\{ (u_1 - \bar{u})^2 (H + \eta) + (u_2 - \bar{u})^2 (H - \eta) \right\} dy. \quad (7.31)
\]

The flow present in the absence of baroclinic effects for this linear problem is that of the basic state (initial conditions), namely \( \bar{u} = 0 \). The velocities \( u_1, u_2 \) and interface perturbation \( \eta \) are given by (7.27) and (7.28). Unfortunately the integrals in (7.31) must be evaluated; the results will be presented to the leading order in \( \varepsilon \) with \( H = \frac{\rho_2}{\rho_1} \) in figure 7.9. Before considering the variations with the channel width shown in this figure, we will derive the relevant expressions for the hydraulic limit.

When considering hydraulically controlled flows of the type described in sections 7.1 and 7.2, the equations for \( \text{Ep} \) and \( \text{Ek} \) take the simpler form

\[
\text{Ep} = \int_{-\xi}^{\xi} \frac{\rho_2}{\rho_1 - \rho_2} \left\{ \rho_1 (u_1 - \bar{u}) \eta + \rho_2 (u_2 - \bar{u})(D - \eta) \right\} dy, \quad (7.32)
\]

\[
\text{Ek} = \int_{-\xi}^{\xi} \frac{\rho_2 (u_1 - u_2)(u_1 - \bar{u})^2 \eta + (u_2 - \bar{u})^2 (D - \eta)}{\rho_1 - \rho_2} dy. \quad (7.33)
\]

The zero energy state will here be given by
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\[ \mu = (1 - D \Pi) y \]  \hspace{1cm} (7.34)

as the layers to each side of the initial discontinuity occupy the whole channel depth and must possess the same potential vorticity as the reservoirs. Equations (7.32) and (7.33) may be evaluated analytically to give equation (7.29) as

\[ \mu = \frac{\left[ b_c \right]^2 \xi^2 - \lambda \xi/t + 1} {1 - \frac{1}{2} \lambda \xi/t} \]  \hspace{1cm} (7.35)

where \( B \) is given by (7.18) for an attached flow (we are not considering separated flows in this section).

Figure 7.8 shows how the energy partition \( \mu \) varies as a function of both the channel width and potential vorticity. As \( b_c \rightarrow 0 \), all of the potential energy released goes towards increasing the kinetic energy of the mean flow. This is to be expected as Poincaré waves are not supported in the nonrotating limit.

As the potential vorticity approaches zero, the energy partition also approaches unity. With \( \Pi = 0 \) the length scale associated with the Poincaré waves \((1/\lambda)\) is infinite, so even though the channel is of nonzero (rotational) width, Poincaré modes are not supported. Thus the assumption used by Whitehead, Leetmaa & Knox (1974), that all the potential energy released goes towards increasing the mean kinetic energy of the mean flow in a channel of uniform depth, was correct. While their definition of the zero energy state was incorrect (they did not take into account the flow \( \mu = \gamma \) that would be present if there were no baroclinic effects), the symmetry/linearity of the \( \Pi = 0 \) problem enabled them to obtain the correct answer anyway.

For nonzero values of potential vorticity and channel width, some of the potential energy released is radiated away from the control section by Poincaré waves. Increasing either \( b_c \) or \( \Pi \) increases the proportion of the energy in the wave modes and hence decreases \( \mu \). The results plotted in figure 7.8 have been derived under the assumption that \( \mu \) becomes independent of the channel width once the flow has separated and a stagnation zone has formed. This effectively takes the basic state (described by \( \mu \)) as stagnant fluid within the stagnation zones, even though this
Figure 7.8. Variations in the energy partition $\mu$ with the channel width $b_c$ and potential vorticity $\Pi$ for hydraulically controlled flows. Regions representing separated flow are coloured pink. The regions in green are not accessible to channels of constant depth - see section 7.2.
Figure 7.9 shows a comparison between the values of \( \mu \) obtained for the hydraulically controlled problem (solid lines, from equation (7.35) - potential vorticity as marked) and the solution for the linear Rossby adjustment problem (dotted line, equations (7.29) to (7.31)). The attached hydraulic equations have been assumed to be valid for all channel widths, even though this is clearly not the case. The linear problem solution shows essentially the same features as the hydraulic problem solution (up until unity channel widths; after this the hydraulic solution is no longer valid due to separation), supporting the idea that Poincare' waves act as a sink for potential energy even in controlled channels.

Due to the importance of nonlinearities in the hydraulic solution, we do not expect the linear Rossby solution to agree precisely with any given value of \( \Pi \) in the hydraulic problem. However, figure 7.10 shows a remarkably close agreement with \( \Pi = 0.35 \).

In the next chapter we shall develop the ideas of hydraulic control in rotating channels further, although we will restrict attention to the \( \Pi = 0 \) limit. We have shown in the first two sections of this chapter that the \( \Pi = 0 \) limit captures all the essential features of constant potential vorticity flows through channels of constant depth. The weak dependence of these features on \( \Pi \) motivates us to explore further the mathematically more attractive zero potential vorticity limit. We note that the energetics are a relatively strong function of the potential vorticity, but suggest this is of comparatively little importance when compared with the gross features of hydraulic flows.
Figure 7.9. Comparison of energy partition $\mu$ for the linear Rossby adjustment solution (dotted line) and the hydraulic solution (solid lines) as a function of channel width. Potential vorticity for hydraulic solution as marked.

Figure 7.10. Comparison of the energy partition $\mu$ between the linear Rossby adjustment solution (dotted line) and the hydraulic solution (solid line) with $\Pi = 0.35$. Note the very close agreement over the plotted range of $b_c$. 
In chapter 7 we investigated how nonzero potential vorticity affects the flow through a channel of constant depth. We found that the value of $\Pi$ makes very little difference to the resulting flow, at least for channels narrower than a Rossby radius (i.e., unit width). We shall use this finding as justification for restricting our attention in this chapter to the mathematically more tractable zero potential vorticity limit. We shall, however, look at channels of more general geometry and with the addition of a net barotropic forcing.

The equations describing the interface height and layer velocity variations across the channel for an arbitrary uniform potential vorticity were given in sections 6.2 to 6.4. In this section we shall examine in more detail these equations for regions containing an attached flow in the limit of vanishingly small potential vorticity.

When $\Pi = 0$, the interface adopts a linear cross-channel profile and equation (6.11) reduces to

$$h = D (\partial^2 A + B \gamma / \xi). \quad (8.1)$$

As before, the interface height coefficient $A = A(x)$ and the interface slope coefficient $B = B(x)$ must be determined to fully specify the flow. In all realisable zero potential vorticity flows the slope coefficient will be negative, corresponding to the current being banked up on the right-hand side of the channel and a positive exchange flow rate. If $B > 0$ the exchange flow rate is negative; this violates the basic assumptions of the flow. Nevertheless we shall allow $B$ to take either sign in this analysis as it is necessary to consider such unrealisable solutions to ensure proper convergence of numerical solutions.

Integrating the potential vorticity relations, and relating the constants of integration using the geostrophic balance, gives the velocities as
where the constant of integration \( c \) is obtained from conservation of the net barotropic flow rate, viz.

\[
c = \frac{(ABD + \frac{1}{2} Q/D)}{\xi}. \tag{8.3}
\]

The exchange flow rate is given by

\[
\bar{q} = 4D \left( \frac{1}{2} B \xi^2 - \frac{1}{2} B D + A c \xi \right). \tag{8.4}
\]

If the exchange flow rate for the channel is known, then the interface slope coefficient may be calculated from (8.4) in terms of the interface height coefficient as

\[
B = \frac{1}{4D} \frac{\bar{q} - 2A Q}{\xi^2 - \frac{1}{2} B D + A^2 D}. \tag{8.5}
\]

Together these relations may be substituted into the definition of the hydraulic functional given in equation (6.30) which is most conveniently written as

\[
J(\cdot, Q; \bar{q}, \xi; A) = Q + \left( \frac{1}{2} B Q + A B^2 D^2 \right)/\xi^2 - H - D (1/2 + A). \tag{8.6}
\]

Differentiating (8.6) in terms of the interface height coefficient (recall that \( B \) is also a function of \( A \)) gives

\[
\frac{dJ}{dA} = \left\{ \frac{Q^2}{4D \xi^2} \left[ \begin{array}{c}
-\frac{Q^2}{9} + \frac{D \xi^2}{6} + A^2 D \xi^2 - \frac{D^2}{16} - \frac{3}{4} D^2 A^2 \\
+ \frac{A Q \bar{q}}{4 \xi^2} \left[ -\xi^2 + \frac{3}{4} D + A^2 D \right] \\
+ \frac{\bar{q}^2}{16 \xi^2} \left[ \frac{1}{2} B \xi^2 - \frac{1}{2} B D - 3 A^2 D \right]
\end{array} \right] \right\} \\
+ \left[ \frac{1}{4} B \xi^2 - \frac{1}{2} B D + A^2 D \right]^3 - D. \tag{8.7}
\]

The complete derivative \( d/dA \) was defined by equation (6.34). Roots of \( J(\cdot, A) = 0 \) with \( dJ/dA < 0 \) correspond to subcritical flow, \( dJ/dA = 0 \) to critical flow and \( dJ/dA > 0 \) to supercritical flow in
a manner analogous to the nonrotating limit, though this time the
the Froude number varies across the width of the channel (except
if the flow is critical - see section 6.4).

If the position of one of the control sections is known along
with the interface height coefficient at that section, then the
exchange flow rate may be calculated as the more positive root of
(8.7) (which is quadratic in \( \bar{q} \)).

Note that for the flow to be attached \( A - B < \frac{\omega}{\alpha} \) and
\( A + B > -\frac{\omega}{2} \), where \( B \) is given by (8.5). If this is not true, then
the flow is separated. We introduce the theory associated with
separated flows in the next section.

8.2 Separated flow

As in section 7.2, we shall consider only separated flows
which intersect the channel floor \( (\frac{\omega}{2} + A + B < 0) \). The interface
intersecting the channel roof may be treated in an identical
manner. Suppose the interface intersects the floor at \( y = \zeta(x) \). As
the interface height must be defined continuously, \( h(y=\zeta(x)) = 0 \),
and so

\[
\zeta = -\frac{\frac{\omega}{2} + A}{B} \zeta. \tag{8.8}
\]

The resulting equations will depend on the sign of \( B \), as was
the case in section 6.3. We have said previously that \( B \leq 0 \) for
all realisable flows; nevertheless it is important that we develop
the equations for \( B > 0 \), as stated in the previous section. When
\( B < 0 \) the interface intersects the \( y = -\zeta \) wall and

\[
h = \begin{cases} 
D \left( \frac{\omega}{2} + A + B \frac{y}{\zeta} \right) ; & -\zeta \leq y < \zeta \\
0 ; & \zeta < y \leq \zeta.
\end{cases} \tag{8.9}
\]

Integrating the potential vorticity relation for \( y > \zeta \) again gives
a linear dependence on \( y \). The constant of integration may be
eliminated by assuming the velocity in the upper layer is
continuous at \( y = \zeta \). Clearly the velocity of the lower layer is
undefined for \( y > \zeta \). Thus
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\[ u_1 = \begin{cases} 
  y - \frac{1}{2} BD/\xi + c ; & -\xi \leq y < \xi \\
  - & ; \xi < y < \xi' 
\end{cases} \]

and

\[ u_2 = y + \frac{1}{2} BD/\xi + c. \tag{8.10} \]

where the velocity constant \( c \) may be obtained from the net barotropic flow relation as

\[ c = \frac{Q/D + BD (\frac{1}{2} - A)}{2 \xi} + \frac{BD [B^2 + (\frac{1}{2} + A)^2]}{2 \xi}, \tag{8.11} \]

and the exchange flow rate is

\[ \bar{q} = \frac{2}{3} B^3 - (\frac{1}{2} + A)^3 \xi^2 - \left[ \frac{\frac{1}{2} + A + B/c}{\xi} \right] \frac{B^2 - (\frac{1}{2} + A)^2}{B} \xi^2 - 2 (\frac{1}{2} - A)c \xi - 2 c \frac{(\frac{1}{2} + A)^2}{B} \xi - BD \]. \tag{8.12} \]

If the slope coefficient is positive, the interface is banked up against the \( y = \xi \) wall and

\[ h = \begin{cases} 
  0 ; & -\xi \leq y < \xi, \\
  D (\frac{1}{2} + A + BU/\xi) ; & \xi < y < \xi', 
\end{cases} \tag{8.13} \]

giving the layer velocities as

\[ u_1 = \begin{cases} 
  - & ; -\xi \leq y < \xi \\
  y - \frac{1}{2} BD/\xi + c ; & \xi < y < \xi' 
\end{cases} \]

and

\[ u_2 = y + \frac{1}{2} BD/\xi + c. \tag{8.14} \]

The velocity constant \( c \) is again obtained from the net barotropic flow relation, \( \text{viz.} \)
Zero potential vorticity \[ \frac{c}{2} = \frac{Q/D + A D [\eta + A + B]}{2} \tag{8.15} \]

The exchange flow rate in this case is

\[
\bar{q} = D \left\{ \frac{2}{3} \left[ B + \frac{(\eta + A)^3}{B^3} \right] \xi^2 + (\eta + A) \left[ 1 - \frac{(\eta + A)^2}{B^2} \right] \right. \\
- D B \left. + \left[ B + \frac{(\eta + A)^2}{B} \right] c \xi - 2 (\eta - A) c \xi \right\}. \tag{8.16}
\]

If the exchange flow rate is known then it is possible to evaluate the slope coefficient \( B \) from (8.12) or (8.16). Unfortunately it is not possible to write down an explicit expression for \( B \), and so any evaluation must be numerical. Furthermore, there will in general be more than one value of \( B \) satisfying (8.12) or (8.16). The appropriate one is determined by recalling that the hydraulic functional must be continuous for all values of the interface height coefficient \( A \). The hydraulic functional is given in terms of \( A, B \) and \( c \) for this separated flow as

\[
J(\eta; A) = \bar{q} + D B c \xi - H - D (\eta + A). \tag{8.17}
\]

Criticality may again be discussed in terms of \( \frac{dJ}{dA} \), even though this may not be written explicitly. At the control sections \( \frac{dJ}{dA} \) must vanish whether the flow is attached or separated. While the required exchange flow rate for a given value of \( A \) could be calculated from turning points in (8.17) directly, it is more convenient to solve

\[
\left\{ \frac{2}{3} \left[ B^3 - (\eta + A)^3 \right] \xi^2 - \left[ B^2 - (\eta + A)^2 \right] B c \xi - D B^3 \right\} \frac{dB}{dA} \\
- \left\{ \left[ B^2 + (\eta + A)^2 \right] B^2 \xi^2 - 2 (\eta - A) B^3 \xi \right\} \frac{dc}{dA} \\
+ \left\{ - \left[ B^2 - (\eta + A)^2 \right] B \xi^2 + 2 \left[ B - (\eta + A) \right] B^2 c \xi \right\} = 0, \tag{8.18}
\]

for negative \( B \) (we shall not present the equivalent result for
B > 0), where

\[ \frac{dB}{dA} = \frac{\xi}{c} - \frac{B}{c} \frac{dc}{dA} \]

Here (equation (8.19)) we are solving \( dJ/dA = 0 \) for \( B \) along a trajectory maintaining constant \( A \) and \( \bar{q} \) (i.e. \( dq/dA = 0 \) where \( \bar{q} \) is given by equation (8.16)). The root obtained may then be substituted back into (3.2#5) to obtain the desired \( \bar{q} \). As is frequently the case with separated flows, there is not necessarily one unique value of \( B \) satisfying (8.14). The correct one is chosen on this occasion through the need for \( \bar{q}_{\text{crit}} \) to be continuous over some range of \( A \).

The next section explores the controlled solution of flow in a channel of constant depth. The section is effectively the same as sections 3.2 and 3.6 with the addition of rotation.

### 8.3 Channels of constant depth

In this section we are interested in the effects of net barotropic forcing \( Q \) through a channel of constant depth and varying width. Results will be presented in terms of the specific forcing \( Q/b_c \) to isolate the effects of rotation. The analysis is analogous to that given in section 3.6 with the addition of rotation. The \( Q = 0 \) limit was solved in the previous chapter for more general potential vorticity and will not be reconsidered here, except to note that both controls are at the narrowest section and the interface height coefficient vanishes for critical conditions.

From our knowledge of the \( \xi \rightarrow 0 \) limit, we are able to state that there will be, in general, two distinct control sections when \( Q \neq 0 \). At each, equation (7.15) must hold, requiring either \( d\xi/dx = 0 \) (i.e. the contraction) or \( \partial J/\partial \xi = 0 \). The primary control, \( x = x_c \), will always be at the contraction where \( d\xi/dx \) vanishes. The second condition \( \partial J/\partial \xi = 0 \) must apply at the virtual control, \( x = x_v \), whether or not it is distinct from the contraction. We are
able to solve $\partial J/\partial \xi = 0$ for $A_V$ (the interface height coefficient at the virtual control) in terms of $Q$ and $\tilde{Q}$ provided the flow is attached at this point, viz.

$$A_V = 2 \frac{Q}{\tilde{Q}} \left( \frac{\tilde{Q}}{Q} - \frac{\xi_V^2}{Q} \right).$$

Note the similarity with equation (3.20) for nonrotating channels.

The position of the virtual control $x_V$ is, as yet, unknown. By solving $J(\cdot; x; A) = 0$ simultaneously at the contraction $x_c$ and the virtual control, and insisting that the flow is critical at both locations, we may, in principle, fully determine the flow. Unfortunately an explicit solution is not possible for general values of the contraction width $\xi_c$ and net forcing $Q$, though we are able to solve the problem asymptotically for small $Q$ and $\xi_c$:

$$A_c \sim Q / (8 \xi_c),$$
$$A_V \sim \frac{Q}{\xi_c} - \frac{1}{128 \left( \frac{1}{8} \xi_c^2 - \frac{1}{4} \xi_c^2 \right)^3} \left[ \frac{Q}{\xi_c} \right]^3,$$
$$\tilde{Q} \sim \xi_c \left( 1 - \frac{4}{3} \xi_c^2 \right) + \frac{1}{16} \left[ \frac{Q}{\xi_c} \right]^2,$$
$$\xi_V \sim \left\{ 1 + \frac{9}{16 (1 - 4 \xi_c^2)} \left[ \frac{Q}{\xi_c} \right]^2 \right\} \xi_c.$$  \hspace{1cm} (8.21)

Note that for fixed specific forcing, $Q/\xi_c$, increasing $\xi_c$ (i.e. increasing the rotation rate) causes the virtual control to move back towards the contraction ($\xi_V/\xi_c$ decreases).

For more general flows we must determine the solution numerically, a task much more difficult than in the nonrotating limit through the need to consider separated flows where it is necessary to track the solution to ensure $B$ is evaluated correctly. As with the flow's nonrotating conterpart, we are able to state some limits on the values of $A_c$ and $A_V$.

Suppose that $\tilde{Q}_{\text{crit}}$ is the value of the exchange flow rate required to give critical conditions in some section where the interface height coefficient is $A$. As in section 3.6, we shall define $A_{\text{max}}$ as the value of $A$ which maximizes $\tilde{Q}_{\text{crit}}$ (i.e. gives $d^2 J/\partial A^2 = 0$ in addition to $dJ/\partial A = 0$). If the corresponding flow is attached then
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\[ A_{\text{max}} = \frac{Q}{(\pi D^{3/2} \xi)}. \]  

(8.22)

exactly as for the nonrotating limit. However, if the flow is separated, \(|A_{\text{max}}|\) will differ from that given by (8.22), its value being determined numerically. As \(A = A_{\text{max}}\) represents the vanishing of \(d^2J/dA^2\), if the flow is critical, the height coefficients at the control sections must be on opposite sides of their respective values of \(A_{\text{max}}\). In particular

\[ (A_c - A_{\text{max}}(x=x_c))(A_v - A_{\text{max}}(x=x_v)) \leq 0, \]

(8.23)
equality only occurring (in channels of constant depth) when \(Q = 0\). Moreover, as we do not expect any bifurcations in the behaviour of the flow, \(|A_c|\) and \(|A_v|\) must increase monotonically with \(|Q|\), thus \(A_c\) will be bounded between zero and \(A_{\text{max}}\).

In the nonrotating limit we were able to place an upper bound of \(\frac{\sqrt{2}}{3}\) on \(|A_v|\), representing the vanishing of one of the two layers. However when rotation is included, the interface has an additional degree of freedom and may adopt values of \(|A| > \frac{\sqrt{2}}{3}\) through the interface tilting. The corresponding limit may be given implicitly as \(|A_v| - |A_v| < \frac{\sqrt{2}}{3}\), equality being approached at the virtual control only as one of the layers is brought to rest by the forcing. The lower bound on \(|A_v|\) is again \(|A_{\text{max}}|\).

For no rotation, one of the layers is first brought to rest by the net forcing with a front (\(|A_v| = \frac{\sqrt{2}}{3}\)) forming at the virtual control when this is positioned in the reservoir (the ratio \(b_v/b_c\) becomes infinite). The value of \(|Q/b_c|\) required is \((2D/3)^{2/3}\). If, in the presence of rotation, the front were to form first at infinity, then we are able to show that, for flows attached at the contraction, the height coefficient at the front is

\[ |A_f| = \frac{\sqrt{2}}{3}, \]

and

\[ |A_c| = \frac{\sqrt{2}}{3} \left\{ \left[ 1 - \frac{5}{3} \xi_c \right] - \left[ \frac{4}{3} - \frac{5}{3} \xi_c + \frac{16}{9} \xi_c^4 \right] \right\}^2. \]
Note that $|Q_f/b_c|$ decreases as $\xi_c$ increases. However, it is possible to show that, unless $\xi_c = 0$, such a flow is subcritical at the front, and hence is submaximal. Thus we are able to conclude that stagnation of a maximal flow will first occur with the virtual control in a region of the channel of finite width. This was hinted at by equation (8.21) where rotation reduced the movement of the virtual control away from the contraction with increasing $Q/b_c$. Furthermore, the net forcing required to bring the layer to rest will be greater than $Q_f$ of equation (8.24) and the interface at the virtual control will be separated from one of the sidewalls as the limit is approached.

Because the flow is separated at the virtual control, we are unable to determine analytically the strength of forcing $Q_{stag}$ required to bring one of the layers to rest (except in the nonrotating limit). The stagnation forcing may, however, be determined numerically by insisting the flow is critical at the front. Figure 8.1 shows how the strength of the forcing required to produce stagnation ($Q_{stag}/b_c$) increases as the rotating channel width is increased. Correspondingly, figure 8.2 plots variations in the interface height coefficient at the contraction, and figure 8.3 the slope coefficient at the contraction. The interfacial slope immediately downstream of the front (equal to $B_v$ for $|Q|$ just less than that required to form the front; this is plotted on figure 8.3 as a dotted line) increases slightly more rapidly than at the contraction. The front forms closer to the contraction as the rotation rate is increased as is shown by the ratio $b_v/b_c$ plotted in figure 8.4 (the piece-wise linear nature of the curve is the result of the small number of solutions used to plot it).

For the nonrotating channels of chapter 3 and 5, once one of the layers was brought to rest, the flow behaved exactly as a single-layer flow with an overlying passive layer. Treatment of the present rotating system as such is not possible: the requirement that the potential vorticity of the layer brought to rest is zero requires motion to exist in it for all values of $Q$. Thus it may never be passive and the two-layer results do not
Figure 8.1. Strength of specific forcing required to bring one of the fluid layers to rest in a channel of constant depth.

Figure 8.2. Interface height coefficient at the contraction (solid line) and adjacent to the front (dotted line) in a constant depth channel where one of the layers has been brought to rest by the forcing.
Figure 8.3. Slope coefficient at the contraction (solid line) and adjacent to the front (dotted line) in a constant depth channel where one of the layers has been brought to rest by the forcing.

Figure 8.4. Position of the front in terms of the ratio of the channel width at the front ($b_v$) to that at the contraction ($b_c$). The discrete steps in the curve are the result of sparse data.
collapse to the single-layer results when the forcing is sufficiently strong. We note, however, that neglected processes such as friction may well alter the potential vorticity of the layer at rest so that it becomes stationary and passive. This situation is outside the scope of the present work as it requires the passive layer to have a nonuniform potential vorticity. The single-layer limit has been investigated by a number of authors (e.g., Gill, 1977).

Figures 8.5 to 8.8 show how the flow responds to varying the channel width and the strength of the net barotropic forcing. In all plots the surface over the $b_c$-$Q$ plane is present only if the combination of parameters represents a flow in which both layers are flowing (i.e., $|Q| < Q_{stag}$). Uncoloured regions represent flows attached at both control sections, blue regions flows attached at the primary ($x_c$) control but not the virtual control, and pink regions flows separated at both controls.

For a given value of $Q/b_c$ the specific exchange flow rate (figure 8.5) decreases with increased rotation, provided the flow is attached at both control section (this requires that the forcing is relatively weak). Separation at the virtual control slows the rate of decrease or even increases $Q/b_c$ with increasing $b_c$. This behaviour continues even if the flow is separated at both control sections.

Variations in the interface height coefficient at the contraction are plotted in figure 8.6. For small forcings and low rotation rates $A_c$ varies linearly in $Q/b_c$ and is approximately constant with respect to $b_c$ as predicted by the asymptotic limit of equation (8.21). Separation at the virtual control causes the value of $A_c$ to increase slightly more rapidly with the specific net forcing, though this behaviour is moderated by separation at the contraction. The corresponding interface slope is shown in figure 8.7. For flows attached at both controls, $B_c$ behaves much like the $Q = 0$ limit ($B_c = -t_c$). Separation at the virtual control increases the variations with $Q/b_c$, as does separation at the contraction.

Figure 8.8 shows the response of the interface to changing the specific forcing is more rapid at the virtual control than at the primary control (figure 8.6). Rotation has only a weak influence on $A_y$, even for separated flows. In contrast the virtual
Figure 8.5. Variations in the exchange flow rate through a contraction as a function of the channel width $b_c$ and specific forcing $Q/b_c$. Colour coding: uncoloured - attached at both controls; blue - separated at virtual control (attached at contraction); pink - separated at both controls; no surface - $|Q/b_c|$ greater than that required to form a front.

Figure 8.6. Variations in the interface height coefficient at the primary control at the contraction of a channel of constant depth as a function of channel width $b_c$ and specific forcing $Q/b_c$. Colour coding as for figure 8.5.
Figure 8.7. Variations in the interface slope coefficient at the primary control as a function of channel width $b_c$ and specific forcing $Q/b_c$. Colour coding as for figure 8.5.

Figure 8.8. Response of the interface height coefficient at the virtual control ($A_v$) to changes in channel width $b_c$ and specific forcing $Q/b_c$. Colour coding as for figure 8.5.
control slope coefficient $B_V$ is strongly influenced by rotation, but only weakly by the forcing.

Rotation has its greatest influence on the interface profile through the introduction of a cross-channel slope. Direct changes in the height coefficient are small by comparison. The effect of rotation on the specific exchange flow rate is relatively small (the greatest differences occur when $Q/b_c = 0$ with $\bar{q}/b_c$ being 33% lower for $b_c = 1$ than for $b_c = 0$). For small specific forcings rotation decreases $\bar{q}/b_c$. This trend is reversed with strong forcings. While the specific forcing required for stagnation is increased markedly at higher rotation rates, the differences so introduced will normally be of little importance as the specific exchange flow rate is very close to $|Q/b_c|$ once the latter is greater than around 0.5. Evaluation of the layer velocities at the two controls shows that the velocity in the upper layer, for $Q > 0$ (or lower layer if $Q < 0$), may change sign at some point across the channel at the primary control if the flow is separated at the virtual control. However, it will never change sign at the virtual control. The velocity in the other layer never changes sign at either control. We thus assert that, provided $b_c < 1$, all streamlines may be traced to the upstream reservoir.

In the next section we explore the asymmetry introduced by the presence of a sill in the channel. This work is essentially the same as that presented in section 3.3 with the addition of rotation.

### 8.4 Simple sills - no net flow

As with the nonrotating limit, the presence of a sill in the channel may fundamentally alter the nature of the flow. In this section we shall consider channels with the same basic geometry as section 3.3 (see figure 3.3). We shall discuss the response of the flow to the sill height and then, in the next section, proceed to look at the effects of net barotropic forcing.

When $Q = 0$, equation (2.11) requires that at the control sections either $dD/dx = 0$, or $\partial J/\partial D + (\partial J/\partial H)(\partial H/\partial D) = 0$. The first of these defines the primary control at the sill crest. For nonrotating channels we found that the virtual control was also
defined by a specific geometric feature with \( dD/dx \) and \( db/dx \) both zero. The condition \( \partial J/\partial D + (\partial J/\partial H)(\partial H/\partial D) = 0 \) fails to yield any solutions in the range of interest, and so does not correspond to the virtual control. Thus, as for the nonrotating channel, it is possible to show the virtual control is positioned where the channel width begins to expand into the dense reservoir.

For flows attached at the sill crest (we shall take \( D_c = 1 \)), we are able to obtain asymptotic solutions to the flow in the limits of \( D_w \to 1 \) and \( D_w \to \infty \) in a manner analogous to that employed in section 3.3. In particular, for small sills (i.e., \( (D_w - D_c) \ll D_c \)),

\[
A_c = -\frac{1}{4} \left[ 1 - 4 \xi^2 \right] \xi + \frac{9 - 4 \xi^2}{72 \left[ 1 - 4 \xi^2 \right] \xi} \xi^2 + \\
+ \frac{(13/24) - (37/27) \xi^2 - (2/81) \xi^4}{\left[ 1 - (4/3) \xi^2 \right] \xi^3} + O(\xi^4),
\]

\[
A_v = \frac{1}{4} \left[ 1 - 4 \xi^2 \right] \xi + \frac{9 - 4 \xi^2}{72 \left[ 1 - 4 \xi^2 \right] \xi} \xi^2 - \\
- \frac{(13/24) - (37/27) \xi^2 - (2/81) \xi^4}{\left[ 1 - (4/3) \xi^2 \right] \xi^3} + O(\xi^4),
\]

\[
\frac{3}{2} \frac{\bar{q}}{\xi} = \frac{1}{2} \left[ 1 - \frac{\xi^2}{3} \right] - \frac{3}{8} \left[ 1 - 4 \xi^2 \right]^{2/3} \xi^2 + \\
+ \frac{3}{8} \left[ 1 - \frac{\xi}{3} \right] \xi - \\
+ \frac{3}{32} - \frac{1}{3} \xi^2 - \frac{67}{54} \xi^4 - \frac{5}{81} \xi^6 + O(\xi^5),
\]

where

\[
D_v = D_w = 1 + \xi^3,
\]

(8.25)

for \( \xi \to 0 \). Note that the \( O(\xi^2) \) and \( O(\xi^3) \) terms for \( A_c \) and \( A_v \) become infinite as \( \xi \to \xi_2 \) (which gives separation in a channel of constant depth), so the expansion is formally valid only for \( \xi \ll \xi_2 \).

When the channel depth away from the sill crest becomes very large (\( D_v \to \infty \)), we pose an expansion in terms of \( 1/D_v \). It is not
possible to evaluate the coefficients of the terms explicitly, thus we introduce a further expansion in $\xi_c$, insisting that $\xi_c \ll 1$. The first important terms in the expansion (evaluated numerically) are

$$A_c \sim -0.12544 + 0.15785 \xi_c^2 + 0.5 \xi_c^4,$$

$$A_v \sim \frac{1}{2} - \left(0.35104 + 0.071196 \xi_c^2 - 0.17038 \xi_c^4\right) / D_v,$$

$$\left(\frac{\bar{q}}{\xi_c}\right)^2 \sim 0.69214 - 0.155055 \xi_c^2 + 0.15003 \xi_c^4 \quad (8.26)$$

The exact (with respect to $\xi_c$) solution of the $D_v \to \infty$ equations shows that the flow will separate at the sill crest when $\xi_c = 0.44708$.

Figure 8.9 plots how equations (8.25) vary as a function of both $D_v$ and channel width. The height coefficients at the two controls, $A_c$ and $A_v$, are shown in figure 8.9a. Agreement with the exact sill solutions (which shall be presented shortly) is good for narrow channels (as discussed in section 3.3) even for comparatively large $D_v$. However, the agreement is much more tightly confined to $D_v \to 0$ as the channel width is increased and the second and third order terms become close to singular.

Variations in the specific exchange flow rate are plotted in figure 8.9b, the apparent turning point being the result of the solution breaking down due to the binomial expansion of a square root.

Figure 8.10 plots the asymptotic solutions for $D_v \to \infty$. The height coefficients at the two controls are given in figure 8.10a & b. Note that $A_c$ increases towards zero as the width of the channel increases. This suggests that rotating channels are influenced less by the asymmetry introduced by the sill than are their nonrotating counterparts. A possible explanation of this stems from the Taylor-Proudman theorem. For a continuously stratified fluid the penetration, parallel to the axis of rotation, of any disturbance increases with the rotation rate as $f L/N$ (section 6.15, Pedlosky, 1979), where $f$ is the Coriolis parameter, $L$ the length scale of the disturbance and $N$ the Brunt-Väisälä frequency. It is reasonable for this to carry over to the two-layer case (if $N$ is replaced by $(g'/D)^{1/2}$ and $L$ by $b$ then, in our current dimensionless system, the vertical penetration should scale like...
Figure 8.9. Variations with $D_v$ in the small sill asymptotic limit. 
Top: interface height coefficients at sill crest (solid lines) and virtual (exit) control (dashed lines); bottom specific exchange flow rate. Channel widths as marked.
Figure 8.10. Variations with $b_c$ in the large sill asymptotic limit. (a) Interface height coefficients at sill crest. (b) $D_V(\frac{1}{2}-A_V)$, a measure of the interface height at the channel exit.

Continued on next page.
(c) The specific exchange flow rate. Solid lines are exact solution of the $D_v \to \infty$ equations; dashed lines are an expansion in $L_c$. Dotted lines indicate flow is separated at the sill crest (equations no longer valid).
the channel width) so that increasing the rotation increases the
communication between the two layers. This increased communication
results in a reduction in the asymmetry of the forcing associated
with the presence of the sill. The solid curve is the exact (with
respect to $\xi_c$) solution of the $D_V \to \infty$ equations, while the dashed
curves are the small $\xi_c$ approximations given by equations (8.26).
Figure 8.10c shows how the exchange flow rate per unit width is
reduced as the width of the channel is increased, the two sets of
curves having the same meaning as for the height coefficients.

While in principle asymptotic solutions could be found when
the flow is separated, we expect them to be very messy and of
comparatively little value since the exact solution may be
evaluated numerically. Figures 8.11 to 8.16 plot variations in the
controlled solution over the $b_c$-$D_V$ plane.

The height coefficient at the sill crest is plotted in figure
8.11. For channel widths less than approximately one Rossby
radius, the flow is attached (uncoloured regions) at both control
sections and $A_c$ varies with the depth at the virtual control $D_V$
($= D_W$) in a manner very much like the nonrotating limit.
Increasing the channel width within these bounds decreases the
value of $A_c$, though only slightly. Increasing the rotation rate
causes the flow to become separated first at the sill crest
(yellow regions). The interface height coefficient responds by
increasing in magnitude rapidly as $b_c$ is increased. Separation at
the virtual control (pink) for still wider channels does little to
modify this trend.

The slope coefficient $B_c$, shown in figure 8.12, shows
comparatively little variation with $D_V$ — the variations that do
exist are confined to $D_V \approx 1$. The magnitude of the coefficient
increases approximately linearly with the channel width
(uncoloured regions; in dimensional terms this means that the
slope of the interface is approximately the depth divided by the
internal Rossby radius) up until the flow separates at the sill
crest (yellow). For wider channels, the variation is still nearly
linear, though slightly more rapid.

Provided the flow is not separated at both the sill crest and
virtual control (i.e. not pink), the interface height parameter $A_V$
(plotted in figure 8.13) behaves almost exactly like the
nonrotating limit, approaching its asymptotic form for $D_V$ greater
Figure 8.11. Variations in the interface height coefficient $A_c$ at the sill crest with the channel width $b_c$ and depth away from the sill at the virtual control ($D_v$). Colour coding: uncoloured - attached at both controls; yellow - separated at sill crest but attached at virtual (exit) control; pink - separated at both controls.

Figure 8.12. Variations in the interface slope coefficient $B_c$ at the sill crest with the channel width $b_c$ and depth away from the sill at the virtual control ($D_v$). Colour coding as for figure 8.11.
Figure 8.13. Variations in the interface height coefficient $A_v$ at the virtual control with the channel width $b_c$ and depth away from the sill at the virtual control ($D_v$). Colour coding as for figure 8.11.

Figure 8.14. Surface as for figure 8.13, but colour coding based on the velocity of the upper layer: uncoloured - velocity unidirectional at both controls; yellow - $u_2$ unidirectional at virtual control but changes sign at sill crest; pink - $u_2$ changes sign at both controls.
than approximately 1.5. Separation at both controls results in the $A_V$ surface increasing in both $b_c$ and $D_V$.

Figure 8.14 also shows the $A_V$ surface, though now the colour coding indicates features of the upper layer velocity rather than separation at the controls. Uncoloured regions signify the velocity is strictly unidirectional at both control sections, yellow colouring that the $u_2$ is unidirectional at the virtual control, but undergoes a reversal at the sill crest, and pink that the velocity undergoes reversal at both the virtual control and the sill crest. In this last case it is not possible to assert that all upper layer streamlines originated in the less dense reservoir, and so the resulting solution may not be valid. Note that this only occurs for $b_c > 1$ (as was stated in section 6.6).

The interface slope coefficient at the virtual control is shown in figure 8.15, the colour code indicates whether the flow is attached or separated as was used for figures 8.11 to 8.13. This plot shows that variations in the channel depth greatly reduce the cross-channel slope at the virtual control. The value of $B_V$ varies approximately linearly with $b_c$ regardless of separation, though the constant of proportionality is a strong function of $D_V$.

As the channel width increases, the specific exchange flow rate (plotted in figure 8.16; the colour code indicates separation) decreases for all values of $D_V > 1$. The rate of decrease is similar to that for a channel of constant depth. Variations with $D_V$ follow closely the nonrotating form, the magnitude decreasing as $b_c$ increases in the manner predicted by the asymptotic limits of equations (8.25) and (8.26).

As in the nonrotating limit, channels with $D_V$ greater than approximately 1.5 behave very much as though $D_V \to \infty$. The height coefficients at the sill crest and virtual control and the exchange flow rate are all within a few percent of their asymptotic limits. Increasing the channel width leads first to separation at the sill crest, then to velocity reversal at both controls in the upper layer if the flow is still attached at the virtual control. In the next section we shall explore how the introduction of a net barotropic forcing modifies these results.
Figure 8.15. Variations in the interface slope coefficient $B_v$ at the virtual control with the channel width $b_c$ and depth away from the sill at the virtual control ($D_v$). Colour coding as for figure 8.11.

Figure 8.16. The response of the specific exchange flow rate $q/b_c$ to changes in the channel width $b_c$ and depth at the virtual control $D_v$. Colour coding as for figure 8.11.
In this section we shall present a limited number of solutions to the problem of a barotropically forced flow over an isolated sill in a rotating channel. Solutions in the nonrotating limit were presented in section 3.7 where we demonstrated that, if the depth of the channel at the expansion in width was greater than approximately 1.5 times that at the sill crest, the sill behaved much as though the depth at the exit was infinite. This feature carries over to rotating channels as was shown in the preceding section for $Q = 0$.

Due to the large parameter space and much greater computational expense for the rotating system (compared with the nonrotating limit), we shall restrict ourselves in this section to channels where the depth becomes very large away from the sill crest. For computational reasons very large can not be too large; we shall take $D_C = 1$ and $D_W = 5$.

As for all the flows over a simple sill we have considered so far, the primary control is located at the sill crest. In the absence of net barotropic forcing, the virtual control is at the exit to the dense reservoir. We showed in section 3.6 that, for rectangular channels in the nonrotating limit, if $D_W$ is greater than 1.5, the virtual control will remain at the channel exit for all negative values of $Q$ which result in two flowing layers. However, for parabolic channels (section 5.3), the flow will bifurcate from sill-like behaviour to coincident behaviour (with $x_C = x_V$ at the sill crest) as $Q$ is made more negative if $D_{0w} > 1.107$. A further increase in $Q$ will cause a subsequent bifurcation to contraction-like behaviour before the forcing is strong enough to bring the lower layer to rest. It is not clear a priori whether the flow in a rotating channel will behave like the rectangular channel in the nonrotating limit, or whether it will adopt some of the characteristics of the parabolic channel. Note that we can draw a parallel between the parabolic channel and separated flow in a rectangular channel in that for both channels changes in the interface position also cause a change in the width of the two-layer region.

Stagnation occurs for $Q < 0$ over a nonrotating rectangular sill with the front forming at the primary control at the sill
Zero potential vorticity

Section 8.5

crest (when $D_w = 5 > 3/2$). There is no other control section, even though for a smaller value of $|Q|$ the virtual control is at the channel exit. The flow effectively undergoes a bifurcation to coincident behaviour as $Q$ approaches that for stagnation. Numerical evaluation of the hydraulic problem in the rotating channel shows the solution bifurcates from sill-like behaviour to coincident behaviour as $Q/b_c$ is decreased from zero. The strength of the specific forcing required to cause this bifurcation varies depending on $b_c$: the bifurcation occurs for smaller $|Q/b_c|$ as $b_c$ increases. We expect the flow to undergo a second bifurcation to contraction-like behaviour as the specific forcing is made more negative, and subsequently that stagnation will occur with the primary control at the sill crest and the front forming somewhere towards the less dense reservoir. The strength of the specific forcings for these two changes in behaviour have not been calculated due to the associated numerical difficulties, though in the asymptotic limit $D_w \to \infty$ both will require $Q \to -\infty$. Moreover the precise values of $Q$ for these two changes are not of great importance as the exchange flow rate approaches very close to $-Q$ for much smaller values of $Q$.

When the net forcing is positive, both the rectangular and parabolic channels behave in a similar manner, thus there is little reason to think rotation should introduce any new features. Figure 8.17 shows how the specific forcing required to bring the upper layer to rest ($q_2 = 0 \text{ not } u_2 = 0$) varies as a function of the channel width. Unlike the nonrotating limit, the value of $Q/b_c$ for stagnation is a function of depth away from the sill crest due to the introduction of a cross-channel tilt of the interface near the front. Notice that the forcing varies only weakly with the channel width. In the limit $D_w \to \infty$ we expect stagnation to occur at $Q/b_c = ((2/3) D)^{3/2}$ independently of the channel width: the infinite depth would allow the slope coefficient at the virtual control to vanish while still allowing an exchange flow. Figure 8.18 shows how small the variation in $A_v$ is for the present $D_w = 5$ channel. In contrast $A_c$, also shown in figure 8.18, increases monotonically with $b_c$: the form of the curve changing when the flow separates at the sill crest. We expect $A_c$ to maintain this behaviour in the $D_w \to \infty$ limit.

As the width of the channel increases, so does the slope of
Upper layer brought to rest over a sill

![Figure 8.17. Strength of specific forcing required to bring the upper layer to rest as a function of channel width $b_c$.](image)

Upper layer brought to rest over a sill

![Figure 8.18. Variations in the interface height coefficients at the sill crest (solid line) and adjacent to the front (dotted line) when the upper layer is brought to rest by the forcing.](image)
Figure 8.19. Response of the interface slope coefficients to variations in the channel width when the upper layer is brought to rest. Curves for the sill crest (solid line) and adjacent to the front (dotted line) are shown.
the interface at the sill crest. However, at the virtual control, the interface remains essentially horizontal in line with the nearly constant \((X)\) value of \(A_V\). This behaviour is shown by figure 8.19. Increasing \(D_W\) to infinity would reduce \(B_V\) to zero for all channel widths.

The position at which the front first forms changes as the channel width is increased. For \(D_W \to \infty\), the front will be positioned in the dense reservoir for nonrotating channels \((b_C = 0)\), but at the foot of the sill for all \(b_C > 0\). Finite values of \(D_W\) allow the front to be away from the foot of the sill provided the channel is sufficiently narrow (less than approximately 0.2 for \(D_W = 5\)).

Figure 8.20 shows how the interface height coefficient at the sill crest varies in response to changing the channel width and specific forcing. Uncoloured regions of the surface represent the flow is attached at both control sections; yellow colouring means the flow is separated at the crest but not at the virtual control; blue colouring if the interface is separated only at the virtual control; pink if the flow is separated at both controls; and green if the flow has coincident behaviour \((x_C = x_V\) at the sill crest; the flow is separated under these circumstances).

Separation induced by positive forcing has very little effect on the form of \(A_C\) even when separated at both controls. In contrast separation due to negative forcing or increasing the width of the channel causes \(A_C\) to decrease rapidly, leveling off close to the bifurcation to coincident behaviour.

The interface height coefficient at the virtual control varies only slightly with the channel width and net forcing (shown in figure 8.21), at least so long as the bifurcation to coincident behaviour is not considered. In the \(D_W \to \infty\) limit we expect all variations in \(A_V\) to vanish, though for the present \(D_W = 5\) separation at virtual control causes a slight increase in \(A_V\). The bifurcation to coincident behaviour is shown as a large jump in the value of \(A_V\). Note that this jump is misleading as it is associated with the shifting of \(x_V\) to the sill crest; there is not a jump in the height of the interface at any fixed section.

The slope coefficient at the sill crest is plotted in figure 8.22, the colour coding is the same as for figure 8.20. So long as the flow is attached at both controls \(B_C\) varies only slightly with
Figure 8.20. Variations in the interface height coefficient $A_c$ at the sill crest in response to the channel width $b_c$ and specific net forcing $Q/b_c$. Colour coding: uncoloured - attached at both controls; yellow - separated at sill crest but attached at virtual control; blue - attached at sill crest but separated at virtual control; pink - separated at both controls; green - maximal behaviour ($x_c = x_v$; separated).
Figure 8.21. Response of interface height coefficient $A_v$ at the virtual control to changes in the channel width $b_c$ and specific forcing $Q/b_c$. The jump in the value plotted is due to the bifurcation from *sill-like* to *contraction-like* behaviour. Colour code as per figure 8.20.

Figure 8.22. The effect of changes in the channel width $b_c$ and specific forcing $Q/b_c$ on the interface slope coefficient $B_c$ at the sill crest. Colour coding as per figure 8.20.
the specific forcing. Separation at either control (yellow or blue) forces $|B_c|$ to increase to compensate for the reduction in the portion of the cross-section occupied by one of the layers. The transition to coincident behaviour is comparatively smooth.

As with $A_V$, the slope coefficient at the virtual control (figure 8.23) varies only slightly with the specific forcing and channel width: the bifurcation to coincident behaviour (green) causes a large jump in $B_V$. Again this jump is attributable to the change in the position of $x_V$ and not any jump in the interface height. In the limit $D_W \to \infty$ variations in $B_V$ would vanish, except those associated with the bifurcation to coincident behaviour.

Figure 8.24 plots how the specific exchange flow rate varies with the forcing and channel width. For small forcings, rotation causes a slight reduction in $\tilde{q}/b_c$, as seen for $Q = 0$ in the previous section. As the specific forcing is increased, $\tilde{q}/b_c$ approaches its nonrotating value, exceeding it if the forcing is sufficiently strong so that stagnation occurs at larger $|Q/b_c|$ for wider channels.

As with the channels of constant depth of section 8.3, the dominant effect of rotation is the introduction of a cross-channel slope to the interface. So long as the flow is attached the height coefficient changes only a little with increased rotation. Once the flow is separated from the sill crest, however, the effects of rotation on the mid-channel interface height are much more pronounced (though continuous in both $b_c$ and $Q$) with the introduction of a bifurcation to coincident behaviour. The specific exchange flow rate is decreased slightly for small forcings, but increased for larger forcings. Thus we would expect the net effect to be very small if the flow were barotropically modulated, even for comparatively wide channels. Within the portion of the parameter plane investigated in this section we find that velocity reversal may occur at the sill crest for the upper layer or the virtual control for the lower layer. However the velocity does not change sign at either control and so the hydraulic treatment remains valid (in the previous section we showed that $u_2$ will have a change of sign at both controls if the channel is a little over one Rossby radius in width when $Q = 0$).
Figure 8.23. The effect of changes in the channel width $b_c$ and specific forcing $Q/b_c$ on the interface slope coefficient $B_V$ at the virtual control. Colour coding as per figure 8.20.

Figure 8.24. Variations in the specific exchange flow rate $q/b_c$ in response to changes in the channel width $b_c$ and specific forcing $Q/b_c$. Colour coding as per figure 8.20.
8.6 Short summary of zero potential vorticity flows

In this chapter we have investigated how rotation affects the exchange flow rate through channels of rectangular cross-section. We have found that for small specific forcings ($|Q|/b_c << 1$) rotation causes a reduction in the exchange flow rate, 33% for $b_c = 1$ when compared with $b_c = 0$ with $Q = 0$. However, as the strength of the specific net flow rate is increased, the change in the specific exchange flow rate with rotation is reversed. As $b_c$ becomes order one, the strength of forcing required to bring one of the layers to rest increases with the rotation rate. In terms of the position of the interface, rotation tends to suppress asymmetry introduced by asymmetric channel geometry due to the Taylor-Proudman theorem. Provided the flow is attached at a given section, rotation does not alter the qualitative response of the interface height coefficients, at the two controls, to net forcing; however if the flow is separated then $A_c$ and $A_v$ vary much more rapidly with $Q/b_c$.

For attached flows the interface slope coefficients $B_c$ and $B_v$ tend to vary as $b/D^{1/2}$ (where $b$ and $D$ take the values appropriate for the section) and only weakly with the specific forcing. This picture changes if the flow is separated through the need for greater velocities (i.e. greater $|B|$) to compensate for the associated reduction in the height coefficient and the area occupied by a given layer.

In the next chapter we report the results of the experimental work undertaken to support the hydraulic theory we have developed in this and the preceding two chapters. Series 1 experiments, reported in sections 9.2 and 9.3, were performed in a channel of constant depth and varying width. Of particular interest is section 9.3 which reports the behaviour of the flow at high rotation rates where the analysis of this chapter breaks down. The experiments in series 2 (section 9.4) and series 3 (section 9.5) are for channels of varying depth, series 2 having a horizontal plane of symmetry (and so the analysis is that for channels of constant depth) while series 3 corresponds to simple sills. Finally the series 4 experiments (section 9.6) were for net barotropic forcing through a channel of constant depth.
Extensive experimental work has been undertaken to support the theoretical development of two-layer flows through rotating channels. The experimental set-up is shown in figure 9.1 in a typical configuration. Figure 9.1a is a photograph of the apparatus, while figure 9.1b is a sketch showing the key elements of the rig. The different channel configurations are summarised by figure 9.2.

A Perspex tank, 0.89 m in diameter, was mounted on an accurately levelled precision rotating table driven by a DC servo motor at up to $2\pi$ rad s$^{-1}$ (though typically experiments were performed at rotation rates in the range $[0,1]$ rad s$^{-1}$) and capable of maintaining the rotation rate constant to within one part in $10^4$. A Perspex barrier passing through the centre of the tank dividing it into two reservoirs of (approximately) equal capacity. These reservoirs were linked by a Perspex channel, 300 mm long, passing through the barrier. The channel could be configured so as to produce a channel of either constant depth (50 mm) and varying width ($>100$ mm; figure 9.2a), or constant width (100 mm) and varying depth ($>50$ mm); in the latter case the depth could vary in either a symmetric (ie. the channel was reflectionally symmetric about a horizontal plane; figure 9.2b, solid lines) or asymmetric (ie. a sill; figure 9.2b with false bottome) manner. In all cases the minimum depth or width occurred at the centre of the channel.

Prior to the running of an experiment, the two reservoirs would be filled with tap water (a total of approximately 160 litres), and a barrier positioned at one end of the channel to isolate the reservoirs. Salt was added to one or both reservoirs to give $g' = 0.03$ ms$^{-2}$, this value being chosen to maximize the density difference (and hence minimize the viscous effects) whilst avoiding the undesirable effects of large amplitude Kelvin-Helmholtz instabilities within the channel. In sections 2.4 and 6.4 we showed that the flow is unstable to the long wave disturbances, which produce Kelvin-Helmholtz instabilities, only
Figure 9.1. Photograph of experimental set-up. Overlay shows key elements of the rig. A: dense reservoir; B: light reservoir; C: Perspex channel (simple sill configuration); D: turntable; E: viewing tanks to reduce optical distortion; F: framework over turntable.
Caption on next page.
Figure 9.2. Alternative channel configurations. (a) Constant depth channel (series 1 and series 4). (b) Symmetric depth variations (series 2); false bottom (dashed line) converts the channel into a simple sill (series 3).
in the supercritical regions. If \( g' \) was made too large these disturbances grew to an amplitude such that they altered the density structure and/or potential vorticity of the flow.

The tank was allowed a minimum of 30 minutes to spin up before the experiment was started. This is much longer than the spin-up time; the spin-up time is greatly reduced by vortices being shed from the ends of the channel protruding into the two reservoirs. The barrier across the tank has negligible effect on the spin-up time, at least in the linear limit. During this time samples were taken from both reservoirs and their densities measured using a DMA 602 densitometer (this instrument utilises the resonance of a U-tube, containing the sample, to determine the density) to determine \( g' \) to an accuracy of better than 0.1%.

Experimental runs were started by the removal of the gate separating the two reservoirs, and finished by replacing this barrier; the time between the start and end of the experiment was recorded to the nearest five seconds (the runs ranged between five and twenty minutes in duration depending on the rotation rate). Experiments were typically run for as long as possible (so as to minimize the effects of the initial transients), though never so long that the level of the interface in the reservoirs would start to influence the control mechanism (e.g. by flooding one or both hydraulic controls). In some runs (series 4) net barotropic flow was included by pumping fluid from the lower layer in the dense reservoir to a light-weight plastic bag submerged in the light reservoir. The purpose of the plastic bag was to prevent mixing of the pumped fluid. Care was also taken in withdrawing the fluid from the dense reservoir to minimize changes in the potential vorticity of the fluid.

The experiments were recorded photographically, using either a 35 mm SLR camera or directly digitising images from a monochrome video camera. Plan, elevation or cross-sectional views were recorded for different experimental runs (only one type of view was recorded for a given run). The flow was visualised using fluorescein dye (either injected at a point or with one entire layer dyed), neutrally buoyant polystyrene particles or by shadowgraph techniques (applicable only for elevations). The merits of the different techniques will be discussed in later sections as and when appropriate.
On completion of an experimental run, the two reservoirs were thoroughly mixed and their densities again measured. The two initial and two final densities, in combination with the time the gate was open, provided two ways of calculating the exchange flow rate. These generally agreed to within a few percent - results from experiments where the agreement was not so good were discarded.

The two most serious short-comings of the apparatus were the finite size of the reservoirs and the non-slowly varying nature of the entry and exit geometry of the channel. The finite reservoir size meant that the experiments could only be run for a comparatively short period of time before the reservoirs began to influence the controlled flow (we relied on the supercritical nature of the flow to isolate the reservoir conditions from the controlled solution during the experimental runs), particularly in the flows with a net barotropic forcing. Additionally, the change in the layer thickness within the reservoir may lead to a corresponding change in the potential vorticity of the fluid within that layer as it resides there sufficiently long for viscous effects to at least partially spin it up (we note that while the spinup time for a two-layer system is much greater than a single layer reservoir, the protruding channel geometry greatly reduces this time), though the propagation of internal Kelvin waves around the reservoir may reverse this trend.

More important is the channel entrance/exit geometry. This is sharply discontinuous and may not be described within the theoretical framework of this thesis. We note that it is not a priori obvious that potential vorticity and/or Bernoulli potential will be conserved by streamlines passing through this region. Provided the control mechanism is totally within the slowly varying region of the channel, the breakdown of the assumptions will influence the flow only through the potential vorticity distribution, as only this is not set by the control region of the flow. In section 7.2 we showed that the exchange flow rate is only a weak function of the potential vorticity, at least so long as it is uniform and the same in both layers, and in any case that the exchange flow rate for an attached flow with $T = 0$ is greater than that for any other value of $T$ (see figure 7.5). As we do not know what the actual distribution of $T$ is,
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we are forced to assume that it is uniform when making comparisons between theory and experiment. Furthermore, we shall assume that \( \Pi = 0 \) and not the value associated with stagnant fluid in the source reservoir. The use of \( \Pi = 0 \) is justified both through the weak dependence of the flow on the actual value of \( \Pi \) and that experimentally the finite size of the two reservoirs allows internal Kelvin waves exiting from the channel to propagate around the reservoir many times during the course of an experiment. Whitehead (1986) suggested that this propagation of Kelvin waves leads to the rapid formation of a cyclonic gyre within the reservoir and a reduction in the potential vorticity of the fluid.

Four main series of experiments were performed to look at different aspects of the flow. In series 1 (sections 9.2 and 9.3) the channel was arranged so that it was of varying width and constant depth (figure 9.2a). For series 2 (section 9.4) the channel was reconfigured so that it was of constant depth and symmetrically varying depth (the channel was reflectionally symmetric about a horizontal plane, so we would expect antisymmetric solutions; figure 9.2b). For both these flows we had no net barotropic forcing, and so expect there to be a single control section at the narrowest/shallowest point in the channel in the zero potential vorticity limit.

The influence of sills, in the absence of a net forcing, on the flow was investigated in series 3 (section 9.5). While the ratio of the depth at the sill crest to that at the exit to the reservoir was only 1:2 (figure 9.2b with false bottom), we can consider it to have been an infinite sill both because of the analysis of sections 3.3 and 8.3, and because of the sharp channel exit geometry. Series 4 (section 9.6) looked at some of the effects of a net barotropic forcing on the flow through a channel of constant depth and varying width.

For the first three series of experiments, the primary tool for comparison is the measured exchange flow rate. Analysis of the appropriate equations giving \( \bar{q} \) for a critical flow shows such a comparison to be reasonably robust. Some comparisons of the cross-sectional form of the flow have also been made, but as shall be shown in section 9.2, such comparisons are hampered by the small viscous effects present in the experiments.

Comparison of series 4 with the theory is only possible in
9.2 Series 1 - unforced flow through a contraction

The primary data collected during this series of experiments were measurements of the exchange flow rate and how these vary as a function of the rotation rate (dimensionless channel width). In addition photographs were taken of the cross-section of the flow at various positions along the channel, and a number runs were performed with plan views of dye streaks. Fluorescein was used to dye one of the layers or as the dye streaks - its low opacity did not interfere with the viewing of the flow. Further, when illuminated by a sheet of light from a projector, it produced an intense image easily photographed.

Figure 9.3 plots the exchange flow data collected during the experiments. The plot is in terms of \( \bar{q} \) verses \( b_c \), this being the form most convenient for interpreting the features in wide channels (\( b_c > 1 \)). The scatter in the observational data may be attributed mainly to leakage from one side of the dividing barrier to the other, the presence of dissolved air in the samples taken for density measurement, plus, for some runs, the presence of Kelvin-Helmholtz instabilities (which altered the density structure of the flow in the supercritical regions) between the two layers. Experimental error bars have been plotted to give an indication of the size of the known causes.

The most striking feature of figure 9.3 is that \( \bar{q} \) is almost independent of the channel width when \( b_c \) is greater than approximately unity. As we will show shortly, the slight decrease in \( \bar{q} \) seen in the data for \( b_c > 1 \) may be attributed to viscous effects. Recall that in section 7.2 we hypothesised the existence of a stagnant zone of fluid forming in a channel of constant depth if the channel were sufficiently wide. Such a stagnant zone would lead to the exchange flow rate being independent of the channel width (shown in section 7.2) as we see here. The solid line on the plot represents the zero potential vorticity flow in such a
Figure 9.3. Exchange flow rate data for series 1 experiments: crosses - observations; curves - predicted behaviour for inviscid and viscous theory as marked.
situation - we shall provide additional experimental and theoretical support for this idea in the next section.

The 'inviscid theory' curve (calculated from equation (7.17) with \( \Pi = 0 \) or by setting equation (8.7) with \( A = 0 \), to zero) in figure 9.3 neglects viscous processes, so it comes as no surprise that the observed exchange flow rates are less than the corresponding theoretical values. In the nonrotating system, the Reynolds number describes the relative importance of the (neglected) frictional terms, these generally being negligible for a single-layer flow when \( Re > 300 \) (p. 81, Barna, 1973), where the Reynolds number is based on the mean hydraulic radius (cross-sectional area divided by the wetted perimeter). For the current two-layer experiments the mean hydraulic radius is \( D_b/(2D + 3b) \), giving a Reynolds number of

\[
Re = \frac{(D'g')^\nu}{\nu} \frac{D_b}{(2D + 3b)}, \tag{9.1}
\]

where \( \nu \) is the kinematic viscosity. For typical experimental runs \( Re \) was approximately 500. As this value of \( Re \) only just exceeds the 300 stated by Barna (1973), viscous effects may have some influence on the two-layer flow. The magnitude of this effect increases with the length of the channel as the viscous boundary layers grow in thickness further from the reservoirs.

When the channel is rotating, the importance of the viscous stresses are more aptly described in terms of the Ekman number

\[
E = \frac{2}{f} \frac{\nu}{D^2}, \tag{9.2}
\]

the ratio of the viscous to Coriolis forces (the Reynolds number is the ratio of the inertial to viscous forces). When the Ekman number is small, viscous forces are negligible. In the experiments reported in this chapter, \( E \sim 10^{-3}/f \) with \( f \) typically in the range \([0,1]\). Note that as \( f \rightarrow 0 \) the formation time of the Ekman layers becomes infinite and so they are no longer appropriate, thus the singularity in (9.2) at \( f = 0 \) is a function of the breakdown of this treatment of the boundary layers.

The boundary layer solutions in the rotating system are well known and may be found in most standard texts (eg. p. 30,
Greenspan, 1968). The leading order effect is the formation of Ekman layers with a thickness of order $D E^{1/2}$ at the channel top and bottom, and at the interface between the two fluid layers. Within the top and bottom Ekman layers, at a given position across the channel, there is a net cross-channel transport of fluid in layer $i$ of

$$Q_{\text{noslip}.i} = \frac{1}{2} E^{1/2} u_1. \quad (9.3)$$

For zero potential vorticity flows, mass conservation requires an Ekman pumping into the top and bottom boundary layers. The different nature of the boundary condition at the interface gives the corresponding expression as

$$Q_{\text{int}.i} = \frac{1}{2} E^{1/2} (u_1 - u_2) \quad (9.4)$$

for the lower layer, and negative this expression for the upper layer. We do not expect this result to be altered significantly by the interface not being sharp as there is no Ekman pumping into or out of the interface Ekman layer for zero potential vorticity flows ($u_1 - u_2$ is independent of $y$; all the flux in the interfacial Ekman layer is supplied by the associated Stewartson layers). Under most circumstances $Q_{\text{noslip}.i}$ and $Q_{\text{int}.i}$ will be in the same direction for a given layer.

Associated with the boundary layer fluxes are Stewartson shear layers in order to provide/return the Ekman layer fluxes to the interior flow. Stewartson layers consist of two levels: one of order $D E^{1/4}$ thick to satisfy the no-slip condition on the along-channel component of velocity and the flux condition on the cross-channel component; the second inner solution is of order $D E^{1/4}$ thick to satisfy the boundary conditions on the vertical component of velocity.

The $E^{1/4}$ Stewartson layers represent the most significant effect of viscosity on the exchange flow rate in a rotating channel flow. Note that the along-channel coordinate enters the solution only parametrically, and so we are able to analyse the effects at a given section solely in terms of the inviscid solution for that section. We shall ignore the vortex stretching associated with the Ekman pumping, which is a function of length of the channel, as this is of order $E^{1/2}$. 

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We can show that within the $E^\perp$ layer, the velocities satisfy

$$\left\{ \frac{\partial}{\partial \eta} - \frac{3}{2} \frac{D}{(D-h) h} \right\} \frac{\partial^2}{\partial \eta^2} u_1 = 0,$$

where

$$\frac{b^2}{\delta^2} E^\perp \eta = \xi = y. \quad (9.5)$$

Solving (9.5), subject to it matching the internal flow and wall boundary conditions, and integrating over $\eta \in [0, \infty)$ gives the effective reverse flux on each sidewall due to the $E^\perp$ Stewartson layers as

$$\bar{q}_{\text{rev}} = E^\perp \frac{b^2}{\delta^2} \left\{ h \left[ \frac{C_{11} + C_{21}}{\lambda_1} \right] - (D-h) \left[ \frac{C_{12} + C_{22}}{\lambda_1} \right] \right\},$$

where

$$\lambda_1^2 = \frac{3}{4} \frac{D}{(D-h) h} \left\{ 1 + \left[ 1 - \frac{32 (D-h) h}{9 D^2} \right]^{1/2} \right\},$$

$$\lambda_2^2 = \frac{3}{4} \frac{D}{(D-h) h} \left\{ 1 - \left[ 1 - \frac{32 (D-h) h}{9 D^2} \right]^{1/2} \right\},$$

$$C_{11} = \frac{(3 - 2 h \lambda_2^2) u_1 - u_2}{2 h (\lambda_1^2 - \lambda_2^2)},$$

$$C_{21} = - \frac{(3 - 2 h \lambda_1^2) u_1 - u_2}{2 h (\lambda_1^2 - \lambda_2^2)},$$

$$C_{12} = \frac{(3 - 2 (D-h) \lambda_2^2) u_2 - u_1}{2 h (\lambda_1^2 - \lambda_2^2)},$$

$$C_{22} = - \frac{(3 - 2 (D-h) \lambda_1^2) u_2 - u_1}{2 h (\lambda_1^2 - \lambda_2^2)}, \quad (9.6)$$

and $\delta$ is the ratio of the dimensional channel width to depth. Note that $\lambda_1, \lambda_2$ are real and positive.

If the geostrophic balance continues to hold, at least approximately, then we expect the interface slope to decay exponentially (with length scales $1/\lambda_1, 1/\lambda_2$), vanishing at the wall.
If $b \leq E^{\frac{1}{2}}$ is sufficiently small, we could evaluate $h$, $u_1$, $u_2$ at the wall provided the flow is not close to separation or indeed separated. However, in the experiments, the Stewartson layers were not negligible compared with the width of the channel, so matching onto the inviscid solution at the wall is not the correct approach. Instead, a first estimate of the length scale of the shear layer was made from $\lambda_1$ and $\lambda_2$, these having been obtained from calculating $h$, $u_1$, $u_2$ at some arbitrary position across the channel. This scale was then used to obtain an improved estimate of the interface height and layer velocities for calculation of (9.6).

This procedure becomes more important as the inviscid flow approaches separation, as the presence of even a negligible viscosity will prevent the appropriate layer from vanishing in the manner specified by the inviscid theory. This effect of viscosity will be dealt with more fully later in this section.

We define the viscous exchange flow rate as

$$\tilde{q}_{\text{visc}} = \tilde{q} - \tilde{q}_{\text{rev}}(\gamma=\xi) - \tilde{q}_{\text{rev}}(\gamma=-\xi),$$  \hspace{1cm} (9.7)$$

and the effective channel width as

$$b_{\text{eff}} = b_c (\tilde{q} - \tilde{q}_{\text{visc}}) / \tilde{q}.$$  \hspace{1cm} (9.8)$$

The curve marked 'viscous theory' on figure 9.3 plots $\tilde{q}_{\text{visc}}$ against the appropriate channel width. The observed values of $\tilde{q}$ still fall below the theoretical curve, though the remaining error is approximately of the order of the observational error (and the neglected $E^{\frac{1}{2}}$ terms).

Observations of the cross-section of the flow at the contraction showed very good agreement with the expected $A_C = 0$, $B_C = -\frac{1}{2} b_c$ provided $b_c$ was not close to (or larger than) unity. Figure 9.4 shows the digitised (binary) images of a number of such flows, the lower layer appears white due to the presence of fluorescein illuminated by a sheet of light. The views are downstream with respect to the upper layer. The Ekman and Stewartson layer thicknesses are indicated, as is the theoretical (viscous) interface profile. Notice that the interface is approximately horizontal where it intersects the channel walls; this is in accordance with the viscous theory.
Figure 9.4. Digitised cross-sections of the flow through a contraction: (a) $b_c = 0.374$; (b) $b_c = 0.520$; (c) $b_c = 0.611$; (d) $b_c = 0.820$; (e) $b_c = 1.113$. Boundary layer thicknesses are indicated by arrows and the predicted interface by a broken line (except for (a) where theory and observations coincide almost exactly). The channel boundaries have been emphasised.
When the effective channel width is greater than unity we expect (from section 7.2) the flow to behave in a fundamentally different manner at the contraction. In particular, the interface will be separated from one (or both) of the channel walls and there will be either a change in sign in the velocity profiles across the channel, or a zone of stagnant fluid will be present in the single layer region(s). Figure 9.5 shows the cross-section for a run with $b_c = 2.37$. Notice that the symmetry of figure 9.4 is gone. The interface has a significant slope over approximately half the channel width (i.e. close to one Rossby radius, as predicted in section 7.2 in the presence of a zone of stagnant fluid), and is nearly flat over the other half. If the fluid is in geostrophic balance, then this implies that the fluid velocity vanishes (or nearly vanishes) in the left-hand half of the cross-section.

The presence of upper layer fluid in the left-hand half of the channel of figure 9.5 may be explained by noting that the boundary layer flux in the upper layer Ekman layers is directed towards the left in this diagram. The inviscid solution corresponding to this situation would have the interface vanishing at about the mid-point of the section, and so no Stewartson layer would be able to form to return this flux to the interior if viscosity were turned on. Instead a tongue of fluid would grow into the single-layer region of the channel to absorb the excess flux. This tongue of fluid will continue to grow until either it is thick enough to return the fluid to the inviscid two-layer current on the right of this diagram, or there is some small along-channel velocity in the left half of the diagram capable of carrying the excess flux off downstream (note that the image was captured a long time after the flow was set-up and so should represent an equilibrium state). Thus, for separated flows, even a very small viscosity may be capable of fundamentally altering the flow. If the flow is attached, the same phenomenon may cause the interface to slope less than suggested by the viscous modification to the inviscid theory given earlier in this section.

In the next section we look in greater detail at the flow when the channel width is greater than one Rossby radius. In particular we attempt to justify the idea of stagnation zones, introduced in sections 6.6 and 7.2, and explain why the
Figure 9.5. Digitised cross-section for flow through contraction with $b_c = 2.371$. The channel boundaries have been emphasised.
cross-section shown in figure 9.5 is not symmetric with respect to the mid-point of the channel.

9.3 *Series 1 - channel-crossing*

For channels wider than one Rossby radius a number of new and interesting features are introduced. The theory introduced in sections 7.1, 7.2 and 8.3 leads us to expect that if the channel is exactly one Rossby radius wide at the contraction ($b_c = 1$) the interface will lie from corner to corner within the channel. A naive application for $b_c$ greater than unity would lead us to expect the fundamental symmetry of the flow to be maintained and the interface to intersect both the channel top and bottom away from the side walls. The introduction of viscous effects in the previous section suggests that this will not quite happen, though the extrapolation of the interface in the central inviscid region should still follow this pattern. However, as we noted in section 7.2, separation from both channel walls at the control section in a channel of constant depth requires that the velocities of the two layers have a change of sign somewhere across the width of the channel. If this were to occur, the basic assumptions used in this thesis would be broken. We have already noted (sections 6.6, 7.2 and 9.2) the introduction of a zone of stagnant fluid may allow the hydraulic theory to be applied. The purpose of this section is to explore the mechanism by which such a zone of stagnant fluid may be formed and the validity of this approach.

We shall proceed by noting that the observed exchange flow rates (figure 9.3) are consistent with the idea of the formation of a zone of stagnant fluid, but not with the potential vorticity being uniform everywhere within the channel. Further, the fluid in the channel of figure 9.5 was banked against the $y = \frac{1}{2}b$ wall of the channel and that there is no a priori reason for this in the steady state dynamics. The interface could equally well have been banked against the $y = -\frac{1}{2}b$ wall or in the centre of the channel. In the presence of a stagnant zone, the inviscid solution does not feel any difference between one section of the channel and another, and so either situation is possible. Observations of the interface profile at other sections along the channel when $b_c >> 1$...
Experimental work revealed that close to the dense reservoir the interface was banked against the $y = \frac{1}{2}b$ wall, and close to the light reservoir it was banked against the $y = -\frac{1}{2}b$ wall, exactly as expected from the hydraulic solution for narrower channels. At some point along the channel the two-layer region crossed from one side of the channel to the other.

Hydraulic theory for channels less than one Rossby radius wide shows that the presence of an hydraulic transition causes the interface to be banked against the $y = \frac{1}{2}b$ wall close to the dense reservoir, and the $y = -\frac{1}{2}b$ wall close to the light reservoir. The interface may be either attached or separated at such sections, depending on the channel geometry and rotation rate. The interface may separated from one of the channel walls at some sections to either side of the contraction and so, like the interface in channels with $b_c > 1$, the two-layer region must cross from one side of the channel to the other. This occurs through the flow becoming attached close to the contraction and passing through an hydraulic transition before separating from the other side wall. As the flow is attached to both side walls it is forced directly by the changes in the channel geometry (width); the length scale over which the current crosses the channel is therefore that of variations in the channel geometry.

In contrast, if $b_c > 1$ and a stagnant zone is present, the two-layer region will not feel changes in the width of the channel and so the channel-crossing must occur over a region the order of one Rossby radius in length, this being the only other length scale present. In a number of experimental runs at high rotation rates we observed an oscillation in time (not locked to the rotation period) between the interface being banked against first one wall and then the other at the contraction. Such an oscillation could be attributed to the point at which a two-layer current crosses the channel shifting from one side of the contraction to the other and back again. This is consistent with the flow not feeling variations in the width of the channel (such variations would fix the crossing at the contraction), though the crossing typically occurred close to the contraction.

Additional support for the $O(1)$ channel-crossing comes from the linear Rossby adjustment problem discussed in section 7.3 and shown to be equivalent to Gill's (1976) single-layer Rossby
adjustment. There Gill showed that the flow due to an initial step discontinuity in the free surface height leads to $O(1)$ cross-channel velocities in a region one Rossby radius wide centered on the position of the initial discontinuity. If the channel is sufficiently wide then nonlinear dam-break in the two-layer system will again yield this result.

Consider a very wide rotating channel of uniform width (i.e. $b = b_c \gg 1$) containing fluid initially at rest. A barrier is positioned across the channel separating fluids of two slightly different densities. When this barrier is removed at $t = 0$ the interface between the two will initially slump with the fluid moving normal to the original position of the barrier. Over an inertial period the Coriolis force will turn the fluid to the right, releasing internal Poincaré waves (we shall not consider these oscillatory modes - refer back to sections 7.4 and 7.4 for more details) and forming geostrophic jets within the two-layer region. The structure of these jets is identical to the flow at the control section in a channel of width such that the interface goes from corner to corner of the channel with $\Pi = 1$ (i.e. $b_c = 1.697$).

For an infinitely wide channel this Margules front (e.g. Csanady, 1971), $1.697$ Rossby radii in width (the width of a channel for stagnation zone to form with $\Pi = 1$ in chapter 7), would represent the steady-state solution. As the system has a rigid lid the velocity will remain zero outside the two-layer region. However for finite width channels the geostrophic jets will impinge on the channel walls, turn to the right and propagate in opposite directions along the channel. For a single-layer flow Garvine (1987) has shown that a supercritical jet impinging on a coast produces a shock-wave. We do not expect this to occur in the current two-layer situation as the flow is only just critical. The propagation of a density intrusion, such as that produced by the jet impinging on the coast, has been analysed by Nof (1987b) who (assuming that energy losses and friction are small) found the front propagates at approximately $0.8(D g')^{1/2}$. Once the fronts have reached the reservoirs the hydraulic control may be set-up by the propagation of small amplitude internal Kelvin waves and the advection of fluid of the reservoir potential vorticity along the channel. Initially the structure of the current behind the front
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will be identical to that through the control section of the channel with $b_c = 1.697 \ (i.e. \ separation \ with \ \Pi = 1)$, and so will be critical with respect to small amplitude waves (in fact, as $d^2J/dA^2 = 0$ in addition to $dJ/dA$ only standing waves are possible — see sections 2.3, 6.6, and 7.1). As fluid of the reservoir potential vorticity is advected along, the structure will change but will remain critical with respect to small amplitude waves, maintaining the (albeit weak) isolation from the reservoirs. Thus we expect the overall structure of the flow, viz. channel-crossing and boundary currents, to remain.

If the width at the channel-crossing is only slightly greater than one Rossby radius, the overall flow will be somewhat more complicated due to the highly nonlinear turning regions where the baroclinic jets impinge on the channel walls. Nevertheless we expect the behaviour to be qualitatively the same and note that treating the flow as possessing a zone of stagnant fluid away from the channel-crossing (where the geometry may still be essentially that of the contraction) allows analysis of the flow in terms of hydraulics for all channels wider than that to give separation.

In our experiments the barrier initially dividing the two layers was at one end of the channel. For this discussion we shall assume the barrier was at the dense reservoir end. While this was not always the case, the same arguments will apply. When the barrier was removed an intrusion of dense fluid would begin to propagate along the channel. Small imbalances between the two reservoirs would create some net flow oscillations, but these would soon die away. The intrusion was typically wedge-shaped and confined against the right-hand wall of the channel (looking downstream with respect to the lower layer). Near the head the flow was unstable in a manner similar to that observed for a non-rotating intrusion. The size of the head was slightly greater than the flow immediately behind it. The head occupied an approximately triangular region of the channel, though its precise form varied with the Kelvin-Helmholtz billows on it. No measurements were made of the speed of propagation of the head.

Behind the head the intrusion widened out, expanding to fill a greater portion of the channel width. The mechanism through which the intrusion grew to form the controlled flow was not obvious. For channels less than one Rossby radius wide, the flow
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at a particular cross-section appeared to grow upwards from the relatively small initial intrusion. Presumably this growth was due to the propagation of small amplitude Kelvin waves along the channel in an attempt to set-up the hydraulic control. For wide channels \((b_c > 1)\) Kelvin waves could also propagate along the initial intrusion as it is not of the critical form of the earlier dam-break discussion. The movement of the channel-crossing to close to the narrowest section is probably a response of these Kelvin waves to the channel width before the steady state is attained. Viscous forces would be necessary to reduce the velocity to zero in the single-layer dense region set-up by such an adjustment process.

Figure 9.6 shows the channel-crossing process at two different rotation rates. The images shown are of a neutrally buoyant plume of fluorescein introduced to the flow at the entry to the channel. Inviscid theory would predict that a zero potential vorticity flow should just be separating in the channel shown in figure 9.6a. The dye streak crosses from one side of the channel to the other over a length scale clearly governed by the channel geometry. The channel-crossing is somewhat more abrupt in figure 9.6b where the channel width is approximately 1.6 Rossby radii wide. Experiments at still higher rotation rates gave a correspondingly more abrupt crossing of the channel, though unfortunately plan views of these were not recorded for the series 1 experiments. Changing the position of the dye plume across the channel width showed that a significant flow velocity was present only in the two-layer region, the width of this region being approximately one Rossby radius. Regretably the channel geometry used changed over a length scale comparable with the channel width, making it more difficult to analyse the channel-crossing phenomenon in isolation from the geometric forcing.

The experimental evidence presented here suggests that the formation of a Margules-type front at high rotation rates is likely. It explains both the independence of \(\bar{q}\) on the width of the channel and the observed channel-crossing/zones of stagnant fluid. In a channel of infinite length and constant cross-section, the front would remain fixed at the position of the initial density discontinuity; in a channel of constant depth and varying width the position will be determined by a combination of the channel
Figure 9.6. Digitised plan views of dye streaks showing the channel-crossing phenomenon. Top: $b_c = 1.02$; bottom: $b_c = 1.56$. 
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9.4 Series 2 - unforced flow through symmetric depth changes

The purpose of this second series of experiments was to examine whether the exchange flow rate was again independent of the channel width in wide channels with symmetric depth variations (as found in series 1 for channels of constant depth) and to further isolate the channel-crossing mechanism by fixing it in a known position (the point of minimum depth). The channel geometry was arranged so as to produce depth variations with a horizontal plane of symmetry (and hence antisymmetric flows) in a region of channel of constant width. While the exits of the channel to the reservoirs were again discontinuous, the size of the discontinuity was less severe, at least as far as the channel depth was concerned.

The theoretical solution for this channel is precisely the same as for the constant depth channel of the previous section for either constant or zero potential vorticity, with the exception that the single control section remains at the shallowest point for all channel widths. Similarly, the viscous correction may be evaluated in the same manner.

Figure 9.7 plots the observed exchange flow rates along with both the inviscid zero potential vorticity solution and the viscously corrected solution. Again, we have assumed the flow rate is independent of the channel width for the theoretical solutions when \( D_C > 1 \); the observational data support this idea. The inviscid solution overestimates the exchange flow rate, though less so than for the constant depth channels of the previous section. With the viscous correction included, the theoretical solution slightly underestimates \( Q \), suggesting the neglected terms act differently in this flow to that of a constant depth channel (e.g. the vortex stretching caused by Ekman pumping is counteracted by the more continuous change in depth along the channel). Error bars on the experiments are included to give an idea of the relative accuracies of the experiments.

Four cross-sectional views of the flow are given in figure 9.8. When the channel is less than one Rossby radius in width at
Figure 9.7. Exchange flow rate data for series 2 experiments: crosses - observations; curves - predicted behaviour for inviscid and viscous theory as marked.
Figure 9.6. Cross-sectional views of flow through a symmetric flow path.
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the shallowest point, the viscous theory gives good agreement with the observed profile (figures 9.8 a, b and c), the quality of the match decreasing as the rotation rate increases.

Figure 9.8d is for a channel approximately 4.5 Rossby radii in width. The form of the interface in this plot is not consistent with the theory developed in earlier chapters, though it still follows a sinh-like profile. Visual inspection of the interface along the entire channel showed that the plot presented here is for a section close to the middle of region over which the two-layer current crosses the channel. The flow on either side of the channel is essentially along its length, the sloping interface corresponding to the geostrophic balance. In the central region the flow is mainly across the channel. The geostrophic balance here requires an along-channel slope to the interface. To either side of the shallowest section, the flow satisfies the relatively straight assumption used in the theoretical development and thus takes on the expected form. For the linear adjustment problem in section 7.3 we predicted a sinh profile for the interface at the position of the initial discontinuity, with the interface ranging between the heights on either side of the initial discontinuity. The sinh-like profile of the present figure (the best fit sinh curve, slightly offset from the centre of the channel, has an root mean square deviation of 0.0064 of the channel depth, using 19 evenly spaced points excluding the wall intercepts) may therefore not be surprising, though we note that the channel-crossing for hydraulic flows is a highly nonlinear feature.

The streak photographs of figure 9.9 demonstrate that the scale over which the channel-crossing occurs is given by the internal Rossby radius and not the along-channel geometry. For both photos the particles are in the lower layer and the view extends from the shallowest section towards the denser reservoir. The Rossby radius of deformation is shown beneath the photographs. The channel-crossing in figure 9.9b occurs over a length scale significantly less than for figure 9.9a, reflecting the difference in the dimensionless channel width. Notice also that over a significant portion of the channel the fluid velocity is very small compared with that in the two-layer current-crossing the channel. This observation is consistent with the idea of stagnation zones forming in the fluid, though it can be shown (by
Figure 9.9. Streak photographs (plan view) of channel-crossing phenomenon in a symmetric vertical contraction. (a) $b_c = 2.79$; (b) $b_c = 4.42$. 

(a) 

(b)
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an analysis similar to that presented in section 7.2) that for this geometry the stagnation point does not coincide with the interface intersecting the channel top/bottom except at the control section: the velocity will be nonzero over part of the single-layer region away from the control.

As with the previous section, these observations may be interpreted as the formation of a Margules-type front when the channel becomes sufficiently wide. As the flux associated with such a front is minimized by minimizing the depth, it is reasonable for the geometry to ensure such a front would drift from its initial position to the shallowest section as a steady state is approached. However, provided the channel geometry varies sufficiently slowly, and the channel width is greater than a Rossby radius, the channel-crossing will occur over a width of one Rossby radius and not the length scale associated with variations in the channel geometry. Unfortunately the length scale of variations in geometry for the experimental apparatus are of the same order as the channel width so this criteria will not be strictly correct.

9.5 Series 3 - unforced flow over sills

The purpose of the experiments of series 3 was two-fold: to confirm that asymmetric changes in the channel depth result in an interface which is depressed towards the channel bottom at the sill crest, and to discover if the exchange flow rate behaves, at higher rotation rates, in a manner similar to that found for the series 1 and series 2 experiments.

The measured exchange flow rates are plotted in figure 9.10, along with the inviscid and viscously corrected theoretical curves. The calculations for the inviscid model without stagnation zones were presented in section 8.4. For channels wider than that in which there is a velocity reversal of the upper layer at both controls ($b_c = 1.07$ for an inviscid flow, or 1.25 for the viscous model of the experiments) we have assumed the exchange flow rate becomes independent of the channel width. We shall call this model for the behaviour in a wide channel (i.e. greater than approximately one Rossby radius) the no reversal model. Recall
Figure 9.10. Exchange flow rate data for series 3 experiments. Crosses - observations; curves - predicted behaviour for inviscid and viscous theory as marked.
that if we do not introduce a stagnation zone, but instead allow velocity reversal at both controls, it is not possible to trace some of the upper layer streamlines back to the less dense reservoir, and so the potential vorticity need not be zero everywhere. We note however that the simple no reversal model introduced here can not be entirely correct. This model implies that a shear layer would form within the channel at the sill crest (and elsewhere; no shear layer was observed) - $u_2$ has a change in sign at the crest for narrower channels than produce a change in sign at the virtual control. Later in this section we shall introduce an alternative description of the behaviour in wide channels.

The nonlinear channel-crossing of series 1 and series 2 experiments is not predicted by this naive stagnant zone model. Up until this point where $u_2$ changes sign at both controls, the two controls are distinct, and their position, and hence the length scale of any crossing, is governed by the channel geometry. For wider channels, if stagnation zones form with the controls still at the crest and channel exit, then the geometry will always dictate the length scale of the channel-crossing. This is inconsistent with the limit of a vanishingly small sill.

The theoretical results of section 8.4 suggest that, so long as the potential vorticity remains zero, the exchange flow rate over a sill continues to increase with $b_c$ even after the channel is wider than that for $u_2$ to have a change in sign at both controls. When $b_c$ is approximately 1.3 this sill solution produces a value of $\bar{q}$ greater than the contraction solution with stagnant zones and a channel-crossing. It is therefore conceivable that the flow may bifurcate to this form of solution if the channel is sufficiently wide. This scenario is consistent with the naive notion that the controlled solution maximizes the exchange flow. We shall term this the bifurcating model.

The two competing models (no reversal and bifurcating) are plotted in figure 9.11 (solid lines), along with the exchange flow rate for a channel of constant depth (dotted line). Figure 9.12 compares the experimental observations with the more sophisticated bifurcating model for wide channels. The inviscid exchange flow rate is continuous over the bifurcation, but the interface configuration is not. Thus the viscous correction, which relies on
Figure 9.11. Comparison of two competing stagnation zone models for wide sills - see text.

Figure 9.12. Observed exchange flow rate data compared with the bifurcating sill model for wide channels (inviscid and viscous forms as marked).
the interface and velocity structure, will change with the bifurcation, rendering $q_{\text{visc}}$ discontinuous. In the real channel this would not occur; the neglected effects would ensure the continuous behaviour of $q_{\text{visc}}$. Unfortunately the quality of the exchange flow rate observations is not sufficient to choose between the two models.

In addition to the measurements of the exchange flow rate, photographs of the along-channel structure of the flow were taken using a shadow graph to show the intersection of the interface with the channel walls (the necessary curvature in the path integrated salinity profile was provided by interface curvature induced by the Stuartson shear layers). Subsequent analysis of the photographs enabled the determination of the height of the wall intersections to better than $\pm 5\%$ of the channel depth. The accuracy achieved depended on a combination of the thickness of the interface, the quality of the photograph, the position of the shadowgraph and the extent to which Kelvin-Helmholtz instabilities in the supercritical region produced anomalous density features. Estimates of the interface height coefficient at the sill crest were made from the mean of the two wall intersections; we shall call this the wall average. Note however that the inherent asymmetry of the sill flow will alter the structure of the Stuartson shear layers in an asymmetric manner, introducing further error to the estimate of $A_c$. No attempt was made to estimate $B_c$ from these photographs as the result would be very sensitive to the form of the Stuartson shear layers.

Figure 9.13 is a scatter plot of the estimated values of the wall average, $(h(y=\xi) + h(y=-\xi))/2$, accompanied by the calculated inviscid curve for this quantity (details of the calculation are similar to those given in section 8.4). While the scatter in the observations and their associated error bars are comparatively large, the trend is consistent with the hydraulic theory: the wall average increases towards zero as the rotation rate is increased, at least up until the upper layer has a change in sign at both controls. For wider channels ($b_c > 1.07$) the theoretical curve is that for the no reversal model, the observed interface profiles being more in line with this than the bifurcating model where we would expect a sudden jump to a zero wall average at some channel width.
Series 3 experiments

Figure 9.13. Plot of observed average wall intersect height and the predicted value, for the no $u_2$ reversal wide channel model, as a function of channel width.
Figure 9.14. Photograph of typical experimental run showing a shadowgraph of the interface wall intersection for $b_O = 0.264$. Overlay: theoretical (inviscid) wall intersection. The side nearer the observer is shown using long dashes, and that further away using short dashes. The positions of the two control sections are indicated by arrows.
Figure 9.14 shows the wall intersections for a typical experimental run. This experiment was chosen for the clarity of the photograph and not for how well it agrees with the theory (most experimental runs were performed with the shadowgraph in a different location to avoid the optical distortion present in this figure; unfortunately the resulting negatives were of a much lower quality). Photographs of experiments at other rotation rates produced a similar match with the inviscid theory, but the poor quality of most of the shadowgraph images prevented them from being printed (the analysis was undertaken by projecting the negative up to A3 size). The overlay shows the inviscid hydraulic solution. Over most of the length of the channel theory and experiment are in very close agreement. Towards the right-hand end of the channel, the theory predicts that the interface should separate from the wall further from the observer. However this was never observed for a flow with $Q = 0$ regardless of the rotation rate. In section 9.2 we explained that this is due to the cross-channel flux in the Ekman layer. The channel exit geometry for the experiments is not modelled accurately by the shallow water hydraulic solution, so the poor agreement towards the end of the channel is to be expected.

9.6 Series 4 - forced flow through a contraction

The final series of experiments was performed to observe the effects of a net forcing on the along-channel interface profile. The channel geometry was identical to that used for the series 1 experiments reported in sections 9.2 and 9.3. The net barotropic forcing was produced by pumping fluid from the bottom of the dense reservoir into a light-weight plastic bag within the light reservoir. The fluid displaced by the filling bag produced a net barotropic flow along the channel equal to the rate at which fluid was pumped into the bag. This rate was measured by a carefully calibrated flow meter. Care was taken in positioning the inlet to the pump in the dense reservoir and the plastic bag in the light reservoir to minimize the disturbance of these to the flow.

No attempt was made to measure the exchange flow rates: such measurements were not feasible due to the method of forcing the
flow. Measurements were confined to shadowgraph images of the along-channel structure of the where the interface intersects the channel wall. In particular the wall average of the interface height at the contraction \( \bar{X} \) was estimated from these images. As in the previous section, the accuracy of the measurements was limited by a combination of the position of the shadowgraph, the stability of the flow away from the control region, and the overall quality of the photograph. In addition, the agreement between theory and experiment was affected by the viscous boundary layers which were not symmetric for \( Q \neq 0 \).

Figure 9.15 plots the estimated observed values of \( \bar{A}_c \) as a function of the net forcing for a number of different channel widths. Note that the scale for \( \bar{A} \) corresponds to only \( 1/5 \) of the total depth of the channel. The corresponding inviscid curves are also plotted, the piecewise linear appearance of the curves is the result of the small number of solutions used to calculate them. At low to moderate rotation rates (figure 9.15 a to c) the agreement between theory and experiment is very good. The rate at which \( \bar{A} \) increases with increasing net forcing follows the same trend in observations and theory. The theoretical curves generally pass through the error bars of the experimental points. Measurements for \( Q/b_c > 0.5 \) were not possible as one of the layers was brought to rest and the flow became unstable.

The plots at higher rotation rates (figure 9.15 d to f) are for experiments in which the channel was subsequently found to be not horizontal. This is visible on the plots as \( \bar{A} \neq 0 \) when \( Q/b_c = 0 \). Allowing for the offset in the results due to this error in the channel levelling, figures 9.15 d and e still show a very good qualitative agreement between theory and experiment.

When the channel is wider than one Rossby radius, our present theory is unable to predict how the flow will respond to net barotropic forcing. The simple idea of a zone of stagnant fluid which proved useful in analysing the series 1 to 3 experiments may not be applied if we have a net barotropic forcing along the channel. Figure 9.15f represents the highest rotation rate \( (b_c = 1.087) \) at which we performed an experiment with a net flow. Note the different scale for \( \bar{A} \) compared with the other plots of figure 9.15. The offset from \( \bar{A} = 0 \) with \( Q/b_c = 0 \) is again attributable to the channel not being levelled adequately. As with
0.2-

bc = 0.136

bc = 0.276

Caption on next page.
Figure 9.15. Observed and predicted values of the average wall intersect as a function of the specific forcing $Q/b_c$ for a number of different channel widths. (a) $b_c = 0.136$; (b) $b_c = 0.276$; (c) $b_c = 0.452$; (d) $b_c = 0.681$; (e) $b_c = 0.882$; (f) $b_c = 1.087$. 
0.1

\[
\begin{align*}
V & , h_0 \cd L_a, + \cd a \\
S & ! I , I I I \\
bc & = 1.087
\end{align*}
\]
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the lower rotation rates, the observed wall average increases almost linearly with the specific forcing. No theoretical curve is plotted as the theory is not valid.

The question as to how a flow responds to barotropic forcing in channels wider than one Rossby radius remains open. It may be possible for zones of stagnant fluid to exist, bounded perhaps by a single layer region of nonzero velocity, though this does not seem likely. An alternative is the existence of recirculating streamlines, confined to one side of a highly nonlinear channel-crossing. This channel-crossing may be skewed by the net forcing, so that the virtual control remains upstream of the contraction (the results of section 8.3 support this idea as the virtual control still moves away from the contraction as the \( b_c = 1 \) limit is approached). On the downstream side of the channel-crossing either a zone of stagnant fluid, or a second set of recirculating streamlines may exist.

The second scenario presented in the preceding paragraph is the more likely as it would suggest a flow consistent with that observed in the \( Q = 0 \) and the \( b_c = 1 \) limits. The exchange flow rate may well become independent of the channel width for channels wider than one Rossby radius. Analysis of this highly nonlinear flow is beyond the scope of this thesis.

9.7 Summary of experimental findings

This chapter has reported how the exchange flow is modified by the effects of rotation in channels of three different along-channel geometries. The dimensionless channel widths range up to approximately 5.5 (Rossby radii). The most important conclusion which may be drawn from this work is that the exchange flow rate is independent of the channel width if the latter is greater than approximately one Rossby radius. This result may be restricted to zero net flow (\( Q = 0 \)), but insufficient experiments with a net forcing were performed to confirm this.

The independence of \( \dot{Q} \) upon \( b_c \) was first suggested in section 7.2 as an alternative to the velocity profiles changing sign across the width of the channel. In section 9.2 we showed that a model including zones of stagnant fluid produced very good
agreement with the experimental results. A viscous correction for the Stewartson boundary layers was shown to account for most of the difference between the theory and observations. In section 9.3 we developed further the ideas of zones of stagnant fluid forming, showing that they may be formed by the set-up process if this was from a dam-break. If not, the small viscous forces may act over the long time scales associated with the steady state to produce the necessary stagnant zones. Observations of streamlines in sections 9.3 and 9.4 supported the notion of a stagnation zone and channel-crossing.

The exchange flow through symmetric depth changes and over a simple sill were also found to be independent of the channel width for wide channels in sections 9.4 and 9.5. The simple stagnation zone model produced a good agreement with the observations, though it was noted that the naive application of stagnation zones for a sill would require the formation of a shear layer. This was not observed. An alternative model for wide channels was suggested, though observations of the interface profile favoured the more naive stagnation zone approach, without any velocity reversals.

The response to net barotropic forcing of the flow through a contraction was found to agree very well with the theoretical predictions in section 9.6. As the theory is valid only for channels less than one Rossby radius in width, only experiments in this range were performed. Some suggestions were made as to how the stagnation zone model, for wide channels, may be extended to allow for net barotropic forcing along the channel.
A number of previous researchers have applied two-layer hydraulics as a model of the flow through the Strait of Gibraltar. The most recent and sophisticated of these was Farmer & Armi (1986) who considered the effects of both a sill and a contraction on the flow through the Strait. Their model was a rectangular channel of the type outlined in section 3.8 with \( D_w = \omega \), i.e. a sill within a region of constant width joined to a contraction in a region of infinite depth. They included the effects of barotropic modulation to model the tidal flow in and out of the Mediterranean Sea. This section introduces a model geometry which more closely follows the geometry of the Strait. In section 10.2 we present complete solutions for rectangular and parabolic nonrotating channels, and in section 10.3 the effects of rotation are added to the rectangular model.

Farmer & Armi (1986) noted four main geometrical features in the Strait of Gibraltar. From west to east the Strait consists of: the Spartel Sill (width \( \sim 18 \) km, depth \( \sim 300 \) m); the Tangier Basin (width \( 12 \sim 18 \) km, depth \( \sim 500 \) m); the Camarinal Sill (width \( \sim 12 \) km, depth \( \sim 280 \) m); and the Tarifa Narrows (width \( \sim 9 \) km, depth \( \sim 900 \) m). Farmer & Armi included only the Caraminal Sill and Tarifa Narrows (where they assumed infinite depth) in their model.

A salinity difference of approximately 2% exists between the Mediterranean and Atlantic waters, this being the result of the much higher evaporation rate in the Mediterranean basin. The salinity difference drives the exchange flow through the Strait (the salinity difference is much more significant than the temperature difference between the two waters of 1.7°C).

Figure 10.1 is a simplified map of the area around the Strait of Gibraltar showing the coast lines of both Spain and North Africa, plus the 200 m depth contour. The positions of the two sills are marked by dot-dash lines, and the Tarifa Narrows by open arrows. The width of the model channels used is also indicated at these features, the parabolic channel being a factor of 1.5 wider than that used in the rectangular model.

*Bryden (pers. comm.) suggests a depth of 360 m is more appropriate for Spartel Sill. This greater depth makes little difference to the overall results.
Figure 10.1. Simplified chart of the Strait of Gibraltar showing the Spanish and North African coasts (solid lines) and the 200 m depth contour (dotted line). The position of the two sills is indicated by dot-dash lines and the contraction by arrows. The width of the model geometry is also marked for the rectangular (I) and parabolic (+) cross-sections.
Figure 10.2. Idealised geometry for the Strait of Gibraltar. Nondimensional depths and widths are shown. Dimensionally one unit of depth is 280 m and one unit of width is 12 km for the rectangular model, or 18 km for the parabolic model. The denser Mediterranean water is on the left. Note that the Strait is shown looking south, the opposite way around to the map of figure 10.1.
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The model geometry we shall use is shown in figure 10.2, the channel widths being those for the nonrotating rectangular model (the widths for the parabolic model are $3/2$ times these values and for the rotating rectangular model $b_0$ times the values here). Unlike Farmer & Armi (1986), we shall include both sills and note that the channel is of finite depth everywhere. To be consistent with the earlier analyses and plots, we have placed the denser Mediterranean to the left in figure 10.2, thus we are effectively looking south towards the Strait. As with all earlier channels, the along-channel length scale is not of importance so long as the channel remains slowly varying. This restriction is satisfied over most of the Strait, the exception being the exit into the Alboran Sea. A number of researchers (eg. Henderson, 1966) have shown that hydraulic flows are not very sensitive to departures from the slowly varying assumption, so the exit to the Alboran Sea should not affect the flow too greatly, particularly as much of the time it will be isolated from the central control region by supercritical flow. We note, however, that Whitehead & Miller (1979) has suggested the sharp change in direction of the North African coast is the cause of the gyre observed in the Alboran Sea.

Farmer & Armi (1986) noted that it takes around 1.5 hours for a long internal wave to travel between the Camarinal Sill and Tarifa Narrows (longer between the Spartel Sill and the Tarifa Narrows), compared with the 12.5 hour period of the semi-diurnal tide. Thus a quasi-steady solution offers a reasonable approximation to the flow, at least qualitatively, provided the internal reservoir formed by the Tangier Basin does not affect the dynamics (Armi & Farmer, 1988, showed that this internal reservoir is important and so the quasi-steady approximation is only of limited value; this feature is discussed more fully in section 10.4).

10.2 Nonrotating models

The Strait of Gibraltar are located at approximately $36^\circ$ North which, in combination with the channel geometry and density difference, gives the internal Rossby radius at Camarinal Sill as
approximately four times the width of the Strait at this point. From the analysis of chapter 8, we do not expect rotation to have a significant effect on the overall form of the hydraulic solution, nor on the actual exchange flow rate, though rotation will introduce a significant cross-channel tilt to the interface. In this section we look at how the geometry of the channel cross-section alters the hydraulic solution if we assume no rotation. Complete results are presented for both rectangular and parabolic cross-sections.

If we assume there is no barotropic modulation of the flow through the Strait, the model geometry behaves much like the isolated contraction and sill model of section 3.8. One control is at the (Camarinal) sill crest and the other at the contraction (Tarifa Narrows). The flow rate, contraction width, contraction depth relationship will be identical to that for the isolated sill and contraction, only the interface profile towards the lighter reservoir will be different. For both the rectangular and parabolic models the exchange flow rate will be very close to its asymptotic value (with $D_w \to \infty$; see figure 3.21 for the nonrotating rectangular model), justifying Farmer & Armi's use of this limit. The nonrotating rectangular model produces an exchange flow rate of 0.349 (multiply by the dimensional $b_c D (D g')^{1/2} = 7.88 \times 10^6 \text{m}^3\text{s}^{-1}$ to obtain the dimensional flow rates), 4% higher than the 0.336 for the parabolic model and 19% higher than the observed 0.305 ($1.2 \times 10^6 \text{m}^3\text{s}^{-1}$, Whitehead et al., 1974).

Farmer and Armi's (1986) nonrotating rectangular model of Gibraltar ignored the presence of the second sill at Spartel and of the finite depth of the channel in the Tarifa Narrows. In the absence of net forcing the Spartel Sill has no influence on the exchange flow rate (it is isolated by a region of supercritical flow), though it will modify the along-channel interface configuration. The depth through the Tarifa Narrows will only have a small effect on the flow (the depth there gives a behaviour very close to its asymptotic limit - see chapters 3 and 5). However when the forcing is strong (close to that bringing one of the layers to rest), these two features may become important.

For example, we may calculate the strength of forcing required to bring the lower layer to rest and where the controls will be in such a situation. For both the rectangular and
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parabolic models the lower layer is brought to rest with the primary control at the contraction and the front forming at the Spartel Sill (in Farmer and Armi’s model the front formed at the Camarinal Sill). This occurs with a forcing of \( Q = -0.481 \times 10^6 \text{ m}^3\text{s}^{-1} \) into the Mediterranean) for the rectangular channel or \( Q = -0.729 \times 10^6 \text{ m}^3\text{s}^{-1} \) for the parabolic channel. As when \( Q = 0 \) the controls are at the contraction and the Camarinal Sill, there must be a bifurcation in the behaviour of the channel for some negative value of \( Q \).

When \( Q > 0 \), the upper layer is first brought to rest with the primary control at the Caraminal Sill crest and the front forming in the Alboran Sea. This result is independent of the depth away from the Caraminal Sill and occurs when \( Q = 0.544 \times 10^6 \text{ m}^3\text{s}^{-1} \) out of the Mediterranean) or \( Q = 0.459 \times 10^6 \text{ m}^3\text{s}^{-1} \) for the rectangular and parabolic models respectively.

Details of the calculations we have performed are summarised by figures 10.3 and 10.4. The first of these figures shows how the interface profiles for both models vary in response to the strength of the net barotropic forcing. The left-hand column shows the rectangular channel, and the right-hand column the parabolic channel. The position of each control is indicated by a solid triangle, and the strength and direction of the forcing by chevrons (an open chevron indicating one of the layers is at rest while a solid chevron that both layers are mobile). If the forcing is sufficiently strong the interface will intersect the channel top or bottom at one or more places; such places are shown by open triangles.

Consider figure 10.3f in which no barotropic forcing is present. To the east (left) of the Caraminal Sill the interface profile is much like that through an isolated sill and contraction. However to the right, the lower layer of the supercritical flow would have to form a thin, fast moving layer through the Tangier Basin and then rise (still supercritical but decreasing Froude number) over the Spartel Sill before draining into the Atlantic Ocean. Such a flow is likely to be unstable and would not be realised in practice. Instead an hydraulic jump would form within the Tangier Basin, returning the flow to subcritical conditions. As the lower layer passes over the Spartel Sill the flow would again go supercritical before draining into the
Figure 10.3. Predicted interface profiles for the Strait of Gibraltar. The left-hand column is for a rectangular cross-section, the right for a parabolic section. The position of the controls is indicated by \( \uparrow \); the intersection of the interface with the channel top or bottom by \( \uparrow \). The strength and direction of the net barotropic forcing is shown by the chevron arrows. An open chevron (\( \uparrow \)) indicates only one layer is flowing; a solid chevron (\( \downarrow \)) that both layers are moving. Hydraulic jumps are shown by wavy lines; the supercritical solution in the absence of the jump by dotted lines.

Continued on next page.
In terms of the rectangular channel the strength of the net barotropic forcing \((Q)\) is as follows (negative values indicate tidal flow into the Mediterranean Sea):

- \((a)\) -1.0 \((-7.88 \times 10^6 \text{m}^3\text{s}^{-1})\);
- \((b)\) -0.8 \((-6.30 \times 10^6 \text{m}^3\text{s}^{-1})\);
- \((c)\) -0.6 \((-4.73 \times 10^6 \text{m}^3\text{s}^{-1})\);
- \((d)\) -0.4 \((-3.15 \times 10^6 \text{m}^3\text{s}^{-1})\);
- \((e)\) -0.2 \((-1.58 \times 10^6 \text{m}^3\text{s}^{-1})\);
- \((f)\) 0 \((0 \text{m}^3\text{s}^{-1})\);
- \((g)\) 0.2 \((1.58 \times 10^6 \text{m}^3\text{s}^{-1})\);
- \((h)\) 0.4 \((3.15 \times 10^6 \text{m}^3\text{s}^{-1})\);
- \((i)\) 0.6 \((4.73 \times 10^6 \text{m}^3\text{s}^{-1})\);
- \((j)\) 0.8 \((6.30 \times 10^6 \text{m}^3\text{s}^{-1})\);
- \((k)\) 1.0 \((7.88 \times 10^6 \text{m}^3\text{s}^{-1})\).
Atlantic where a further hydraulic jump may occur (not sketched). We are able to calculate the profile to the west (right) of the hydraulic jump as the flow must be critical over the Spartel Sill; tracing the interface from such a flow back to the Caraminal Sill produces a lower elevation than that for controlled flow there so that maximal exchange (controlled by the Tarifa Narrows and Caraminal Sill) will still occur. Such a composite flow situation is shown — the hydraulic jump is indicated by a wavy line. The interface height after the jump is shown correctly, but no attempt has been made to calculate the position of the jump. Note that any disturbances in the subcritical region within the Tangier Basin will not affect the overall flow as they are isolated from the Tarifa-Caraminal control mechanism.

Figures 10.3a to e show how the situation changes with the strength of the incoming tide. When the tidal flow is small (figure 10.3e and f) the flow has the same qualitative features as in the absence of forcing. While the flow is still controlled by the Tarifa Narrows and Camarinal Sill, the interface over both these is lower. The interface after the hydraulic jump in the Tangier Basin is also lower, but proportionally less so, making it closer to the height of the control at the crest of the Caraminal Sill.

As the net forcing is decreased from the $Q = -0.4$ of figure 10.3d, the level of the Tangier Basin interface continues to rise, reaching the level of that over the sill for $Q = -0.479$ for the rectangular channel (slightly less than the $Q = -0.481$ required to bring the lower layer to rest) and $Q = -0.520$ for the parabolic channel. In doing so it floods the control at the Caraminal Sill; the flow remains critical at both the Tarifa Narrows and Spartel Sill, though now without an hydraulic jump. This flow configuration is shown in figure 10.3b for the parabolic channel; it occurs only for a much smaller range of $Q$ in the rectangular channel and thus did not manifest itself in this series of plots.

Decreasing the net forcing past $Q = -0.481$ or $-0.729$, for the rectangular and parabolic channels respectively, brings the lower layer to rest. The flow remains critical through the Tarifa Narrows, but the interface intersects the channel bottom to form a front somewhere further west. When $Q$ is only just strong enough to bring the layer to rest, the front forms at the crest of the
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Spartel Sill, moving down towards the Tangier Basin as Q is decreased further. This is shown in figure 10.3b for the parabolic model, though also occurs (over a smaller range of Q) in the rectangular model. The interface height over the Caraminal Sill continues to drop and will eventually intersect the sill crest to form a second front, trapping Mediterranean water in the Tangier Basin to the east of the Caraminal Sill (figure 10.3a and b for the rectangular channel; it also occurs for the parabolic channel though requires a forcing stronger than Q = -1). In a real flow this trapped water would not be present once a steady state had been achieved as Kelvin-Helmholtz instabilities are likely to cause it to be mixed with the Atlantic water and be carried into the Mediterranean. However the time scale for such a process may be much longer than the time scales associated with the barotropic modulation.

Figures 10.3g to k trace the evolution of an outflowing tide. For small values of Q the controls remain at the Tarifa Narrows and Caraminal Sill with an hydraulic jump in the Tangier Basin. As the forcing becomes closer to bringing the upper layer to rest, the control at the contraction will start to move out into the Alboran Sea, increasing the size of the subcritical region (in practice the control is unlikely to move past the point where the North African coast turns sharply southwards). The front which forms when the layer is brought to rest starts off in the Alboran Sea, slowly being brought closer to the contraction as the forcing is increased.

The height of the interface over the Spartel Sill, where the flow goes critical after the hydraulic jump in the Tangier Basin increases with Q, bringing the interface in the subcritical Tangier Basin region closer to the sea surface. Once Q is sufficiently strong it will intersect the sea surface, removing the second layer over the Caraminal Sill and transferring the control to the Spartel Sill (figure 10.3k). Increasing the forcing past Q = 1.660 for the rectangular channel or Q = 1.355 for the parabolic channel will remove all form of control from this region of the Strait and the front will move out into the Atlantic Ocean (the flow being supercritical to the west side of the front).

For many applications it is not the instantaneous exchange flow rate through the Strait of Gibraltar that is of interest, but
the time averaged value. To calculate this we shall assume the flow is quasi-steady and that the barotropic forcing varies as

\[ Q = Q_{\text{max}} \cos \omega t, \]  

(10.1)

where \( Q_{\text{max}} \) is the amplitude of the modulation, \( \omega = 2\pi / \Omega \) the period and \( t \) time.

Figure 10.4 shows how the time-averaged exchange flow rate is increased by increasing the amplitude of the barotropic modulation. Curves are shown for the nonrotating rectangular model (solid line), the parabolic model (long dashes) and the rotating rectangular model with \( b_c = \frac{1}{4} \) Rossby radii (details of the solution associated with this last curve is given in the next section). These three curves are fairly close over the whole range of \( Q_{\text{max}} \). The line shown by short dashes is \( 
\bar{Q}_{\text{max}} = 2 \frac{Q_{\text{max}}}{\Omega}, \) the effective averaged exchange flow rate if baroclinic flow is not important except as a passive tracer. This is the asymptotic limit as \( Q_{\text{max}} \to \infty \).

The observed time averaged flow rate (Whitehead et al., 1974) is shown by the dot-dash line. The calculated exchange flow rate, averaged over a tidal cycle, is greater than the observed value for all values of \( Q_{\text{max}} \). Some of this discrepancy may be due to the model geometry introduced in the previous section giving a poor estimate of the real channel (in particular the quantity \( \frac{3}{2} g \beta D \) may be over-estimated). However, as we shall show in section 10.4, a large portion of the difference may be attributable to departures from the quasi-steady assumption introduced by the internal reservoir (Tangier Basin).

Calculations performed with the Spartel Sill removed from the model geometry produce a response to tidal modulation almost indistinguishable from that plotted in figure 10.4 (which includes Spartel Sill). The parabolic model with Spartel Sill removed shows only small differences. Similarly, doubling the width of the channel at the Spartel Sill or changing the depth of the Tarifa Narrows both have little effect on the exchange flow. Changing the depth of the Tangier Basin does not alter the flow, at least so long as it remains deeper than the Spartel Sill. The flow is most sensitive to the ratio of the width at the Tarifa Narrows to that at the Camarinal Sill. The ratio of the depths at these two locations has relatively little effect, at least provided that the
Figure 10.4. Time averaged, tidally (barotropically) modulated exchange flow rate through the Strait of Gibraltar as a function of the amplitude of the modulation ($Q_{\text{max}}$). The solid line is for the rectangular model, the long dashes for the parabolic model and the dotted line for the rotating rectangular model with $\beta_c = 0.25$. The effective exchange flow rate due to the barotropic modulation in the absence of any baroclinic flow is shown by short dashes. The observed time-averaged exchange flow rate is shown by the dot-dashed line.
Narrows are more than twice the depth of the sill (see section 3.8).

A more detailed discussion of these results in light of the recent "Gibraltar Experiment" (Armi & Farmer, 1988) is delayed until section 10.4 after the effects of rotation have been investigated in the next section.

10.3 Rotating rectangular model

The effects of rotation on the exchange flow rate through the Strait of Gibraltar are comparatively small. While the ratio of the channel width to the internal Rossby radius at the Camarinal Sill is approximately 0.25, we shall investigate a broader range of Gibraltar-like channels, with the channel width ranging from zero (i.e. no rotation) to unit width (i.e. one Rossby radius wide), in order to gain a clearer understanding of the role of rotation in such exchange flows. We shall identify the various geometric features of the model by their names in the Strait of Gibraltar, viz. Spartel Sill, Tangier Basin, Camarinal Sill and Tarifa Narrows.

As with contractions and sills, rotation decreases the specific exchange flow rate for small forcings but increases it if the forcing is sufficiently strong. When \( Q = 0 \) the specific exchange flow rate is \( 0.349 \ (2.75 \times 10^6 \text{ m}^3\text{s}^{-1}) \) when \( b_C = 0 \), reducing to \( 0.272 \ (2.14 \times 10^6 \text{ m}^3\text{s}^{-1}) \) for \( b_C = 1 \); with \( b_C = 0.25 \) the reduction is only 1.4% of the \( b_C = 0 \) value. The strength of the forcing required to bring the upper layer to rest when the flow is into the Atlantic is increased only slightly by rotation from \( Q/b_C = 0.544 \) when \( b_C = 0 \) to \( Q/b_C = 0.622 \) when \( b_C = 0.25 \); increasing \( b_C \) past around 0.75 reduces the specific forcing for stagnation slightly. In contrast, when the tidal flow is into the Mediterranean, rotation causes an increase in the forcing required for stagnation from \( Q/b_C = -0.481 \) in the nonrotating limit to \( -1.57 \) if the width of the channel were one Rossby radius wide at Camarinal Sill.

Figure 10.5 plots the dimensionless specific exchange flow rate as a function of the rotating channel width (in terms of Rossby radii) at the Camarinal Sill and the strength of the
Figure 10.5. The response of the instantaneous specific exchange flow rate $Q/b_c$ to variations in the channel width $b_c$ and specific forcing $Q/b_c$. Colour coding: uncoloured - controls at crest of Camarinal Sill and in Tarifa Narrows; yellow - controls at crest of Spartel Sill and in Tarifa Narrows; pink - controls at crest of Camarinal Sill and virtual control moving out into Alboran Sea.
specific net forcing. Only that portion of the $b_c - Q/b_c$ parameter plane which results in two flowing layers has been plotted; when only one layer is flowing the surface would be given by $\tilde{q}/b_c = |Q|/b_c$. The colour coding indicates the position of the two control sections for different combinations of channel width and forcing. Uncoloured regions indicate that the two controls are at the crest of the Camarinal Sill and in the Tarifa Narrows. Yellow regions denote the flooding of Camarinal Sill and the control shifting to the Spartel Sill, while pink regions represent the primary control at the Camarinal Sill with the virtual control moving out into the Alboran Sea.

The surface plotted in figure 10.6 is precisely the same as for figure 10.5 except that the axes are now labelled with their dimensional quantities and the colour coding of the surface represents separation rather than control position. For small forcings the flow is attached at both controls (uncoloured areas). As $Q/b_c$ is made more negative, the flow separates at the sill crest (yellow when separated at the Camarinal sill and green after the bifurcation with separation at Spartel Sill), though the flow always remains attached at Tarifa provided there are two flowing layers. If sufficiently strong, positive forcing introduces separation at the virtual control, whether this be at the Tarifa Narrows or in the Alboran Sea. If the channel width is less than approximately 0.55 Rossby radii then the flow will be attached at the sill crest regardless of forcing (blue regions), though wider channels may cause separation at this location (pink regions).

Figure 10.7 shows how the specific exchange flow rate, averaged over a tidal cycle, varies in response to the amplitude of the barotropic modulation. The curves are plotted at intervals of 0.1 Rossby radii, the three for the narrowest channels being practically indistinguishable. The dotted curve represents the effective exchange flow rate if baroclinic effects were not present; this is also the limit as $Q_{max} \to \infty$. The effect of rotation is to decrease the averaged specific exchange flow rate. In the absence of barotropic modulation the $b_c = 1$ value is 22% less than that for no rotation (though in Gibraltar the reduction is only around 1.4% as $b_c = 0.25$). As the amplitude of the modulation is increased, the rotating and nonrotating solutions converge rapidly, as the specific exchange flow rate for a
Figure 10.6. Surface plotted as for figure 10.5, but axes labeled with dimensional quantities. Colour coding: uncoloured - flow attached at both controls; yellow - separated at Camarinal control but attached at Tarifa control; green - separated at Spartel control but attached at Tarifa control; blue - attached at sill Camarinal control but separated at virtual control (Tarifa Narrows or in Alboran Sea); pink - separated at both controls.
Figure 10.7. Variations in the specific exchange flow rate averaged over a tidal cycle as a function of the modulation amplitude $Q_{max}$ for a number of different channel widths.
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separated flow increases above its nonrotating counterpart. This suggests that even if Gibraltar were situated at a somewhat higher latitude or that the density difference were less (both lead to a smaller Rossby radius), a nonrotating model would provide a very good estimate of the exchange flow rate through the Strait. Only when the width of the channel approaches that required for the channel-crossing phenomenon to occur (wider than approximately one Rossby radius) will the effects of rotation on \( \frac{q}{b_c} \) become profound.

The averaged (specific) exchange flow rate for Gibraltar with \( b_c = 0.25 \) was plotted accompanied by the nonrotating rectangular and parabolic models in figure 10.4. Over much of the range of modulation amplitudes the rotating rectangular model produces an averaged exchange flow rate intermediate to that of the other two models. For all values of \( Q_{\text{max}} \) the difference between the three models is less than 13% (the maximum occurring with \( Q_{\text{max}} = 0 \), between the nonrotating rectangular and parabolic channels, and decreasing as \( Q_{\text{max}} \) is increased), suggesting that neither geometry nor rotation influence the flow significantly. While it is probable that rotating parabolic geometry will lead to a further reduction in the exchange flow rate at the lower values of \( Q_{\text{max}} \), it is unlikely that the effect will be very great, especially when \( Q_{\text{max}} \) is increased to moderate values.

Figure 10.8 details how the along-channel form of the density interface varies with rotation rate in the absence of a net barotropic forcing. For these conditions the solution is controlled at both the Camarinal Sill and the Tarifa Narrows. The supercritical flow down the west (right) face of the Camarinal Sill goes through an hydraulic jump into the Tangier Basin. Flowing out of the basin the flow again becomes critical over the Spartel Sill. The height of the jump is correct (this being determined by critical conditions over Spartel Sill), but no attempt has been made to determine its position. When \( b_c > 0 \) the interface adopts a cross-channel tilt.

The dotted line indicates the intersection of the interface with the northern (right or \( y = -h / 2 \)b) bank of the Strait, and the solid line the southern bank. The difference in height between these two lines is given by \( 2BD \) provided the flow remains attached. Even for comparatively low rotation rates (e.g. \( b_c = 0.1 \) page : 181
Figure 10.8. The response of the along-channel interface profile to the rotation rate (characterised by $b_c$) when there is no net flow through the channel. Dotted lines indicate the intersection with the northern bank and solid lines with the southern bank. Hydraulic jumps are shown by near-vertical lines. Channel widths:

- a) $b_c = 0.0$
- b) $b_c = 0.1$
- c) $b_c = 0.2$
- d) $b_c = 0.3$
- e) $b_c = 0.4$
- f) $b_c = 0.5$
- g) $b_c = 0.6$
- h) $b_c = 0.7$
- i) $b_c = 0.8$
- j) $b_c = 0.9$
- k) $b_c = 1.0$. 
in figure 10.8b) the flow may become separated from one of the walls both on the Atlantic side of the Spartel Sill and in the approaches to the Alboran Sea. As the effects of rotation are increased, the point at which the flow separates on the Spartel Sill moves back towards its crest, and a similar point of separation forms on the Camarinal Sill. The point of separation in the Alboran Sea also moves back towards the Tarifa Narrows. For \( b_c \) greater than around 0.75 the flow is separated at the crest of the Camarinal Sill, though remains attached at Tarifa for all channel widths in this series. Notice that as \( b_c \) is increased the height of the northern wall intersect (dotted line) starts to adopt an increase in its height as it approaches both the sill crests. This is due to the interface slope increasing more rapidly than the height coefficient is decreasing and should not be thought of as a potential instability.

The evolution of the interface profile through a tidal cycle is plotted in figure 10.9 for both \( b_c = 0 \) and \( b_c = 0.25 \). The specific net forcing is restricted to the range -0.5 to 0.5 (in steps of 0.1) as we are not able to determine the interface configuration for forcings stronger than those giving stagnation as the potential vorticity of layer in which there is no flow is not known, except in the \( b_c = 0 \) limit. As with figure 10.8 a dotted line represents the intersection of the interface with the northern bank of the channel, and the solid line with the southern bank.

The profile when \( Q = 0 \) is plotted in figure 10.9f. The flow is controlled at the Camarinal Sill and Tarifa narrows with an hydraulic jump in the Tangier Basin adjusting the flow to critical over the Spartel Sill. As the net forcing is made more negative (figures 10.9e back to b), the interface is lowered over the whole length of the channel, with the flow separated at the two sill crests for figure 10.9b and c. Decreasing \( Q/b_c \) to -0.5 (figure 10.9a) brings the lower layer to rest in the \( b_c = 0 \) channel (though after the bifurcation to control by Spartel and Tarifa), but not for \( b_c = 0.25 \) where the Camarinal Sill has been flooded to produce critical conditions at Tarifa and Spartel. Decreasing \( Q/b_c \) past -0.539 would cause a front to form at the Spartel Sill for the \( b_c = 0.25 \) channel.

Increasing the specific forcing from zero causes the
Figure 10.9. The evolution of the quasi-steady interface profile through a tidal cycle. Left-hand column is for $b_{c} = 0.0$ and right for $b_{c} = 0.25$. Hydraulic jumps are shown in Tangier Basin when appropriate. The position of the controls is denoted by closed triangles. Dotted lines indicate the intersection with the northern bank and solid lines with the southern bank. The strength and direction of the forcing is indicated by chevrons as per figure 10.2$\#1$.

Continued on next page.
The strength of the specific forcings are:

(a) $-0.5 \cdot (-3.94 \times 10^6 \text{ m}^3\text{s}^{-1})$;  
(b) $-0.4 \cdot (-3.15 \times 10^6 \text{ m}^3\text{s}^{-1})$;  
(c) $-0.3 \cdot (-2.36 \times 10^6 \text{ m}^3\text{s}^{-1})$;  
(d) $-0.2 \cdot (-1.58 \times 10^6 \text{ m}^3\text{s}^{-1})$;  
(e) $-0.1 \cdot (-0.79 \times 10^6 \text{ m}^3\text{s}^{-1})$;  
(f) $0 \cdot (0 \text{ m}^3\text{s}^{-1})$;  
(g) $0.1 \cdot (0.79 \times 10^6 \text{ m}^3\text{s}^{-1})$;  
(h) $0.2 \cdot (1.58 \times 10^6 \text{ m}^3\text{s}^{-1})$;  
(i) $0.3 \cdot (2.36 \times 10^6 \text{ m}^3\text{s}^{-1})$;  
(j) $0.4 \cdot (3.15 \times 10^6 \text{ m}^3\text{s}^{-1})$;  
(k) $0.5 \cdot (3.94 \times 10^6 \text{ m}^3\text{s}^{-1})$.  

The specific forcings are shown in the diagrams as follows:

- (f) Narrowed
- (g) Cameral
- (h) T rough Be a
- (i) Sp eti S
- (j) N narro w
- (k) Cameral S
- (l) Tr ough B
- (m) Sp eti S

Each diagram represents the specific forcing with the corresponding value.
interface profile to rise over the length of the channel. No new features are introduced until \( Q/b_c \) is greater than approximately 0.45 when the virtual control moves out from the Tarifa Narrows and into the Alboran Sea. The interface separates at the virtual control at a similar strength of forcing. Increasing the specific forcing past 0.544 for the \( b_c = 0 \) channel or 0.577 for the rotating channel will cause a front to form somewhere in the Alboran Sea. When \( b_c = 0 \) the width at the front is infinite, whereas \( b_c = 0.25 \) gives a width of approximately 0.3 Rossby radii (hence not very far from Tarifa).

Figure 10.10 shows four transects at various locations across the Strait of Gibraltar. The plots are based on data collected during the "Gibraltar experiment" (see next section for more details) and presented in Armi & Farmer (1988). Superimposed on these plots are approximate (linear) interface positions based on the temperature profiles (for the slope only) and along-channel velocity contours. The keys on the right-hand side of each of the four transects show the tidal height, position of the transect and temperature and salinity scales. Direct quantitative comparison between these observations and the current hydraulic model is hampered due to the unknown phase relationship between the tidal height and the strength of the net forcing. Moreover there are comparatively large errors introduced by the fitting by eye of the approximate interface slopes. The fitting was done before the predictions of the model were studied in detail. Nevertheless we are able to make a qualitative comparison by utilising the observed interface height coefficient to estimate the net forcing, and then compare the observed and predicted slope coefficients. Table [1] details the comparison of the observations and model results. Tabulated are the observed values of \( A/k_B \) along with the theoretical values associated with a range of net forcings giving \( A \) close to that observed.
Caption on next page.
Figure 10.10. Transects across the Strait of Gibraltar (taken from Armi & Farmer, 1988). Solid lines are temperature profiles, dotted lines density profiles (where available), dashed lines contours of constant along-channel velocity. The heavy dot-dash line is the estimated position of a linear interface. The state of the tide, position of the transect and temperature (and density) scale are shown by the cartoons on the right-hand side.

a) Crest of Camarinal Sill.
b) Foot of Camarinal Sill on the Mediterranean side.
c) Tarifa Narrows.
d) Exit to Alboran Sea.
Overall, the observed interface profiles at these four sections are consistent with the predicted profiles when $b_c = 0.25$. The one exception to this is that the observed slope on the interface at the Tarifa-Cires transect is a factor of two greater than predicted. We note, however, that the zero along-channel velocity contour undergoes large excursions across the width of the channel, as do the temperature profiles close to the northern shore. Moreover, it is reassuring that the ranges of $Q/b_c$ predicted from the observed height coefficient are essentially the same for the Camarinal and Tarifa-Cires transects as these were made at the same stage of the tidal cycle.

10.4 Limitations of the quasi-steady model

Armi & Farmer (1988) present results of the "Gibraltar experiment", during which detailed observations were made of a wide variety of features associated with the flow through the Strait of Gibraltar in October 1985 and April 1986. They suggest a number of features which effect the hydraulics of the flow through the Strait which are not covered by the current model. Data was collected by a large number of techniques including profile measurements (using both expendable profilers and acoustic methods) for both ship traverses and time series observations at key locations, plus moored instruments (current meters,
thermistors, conductivity sensors and pressure sensors) at four locations within the Strait.

Figure 10.11 is taken from Armi & Farmer (1988) and schematically summarises the observed time dependent hydraulic response of the flow through the Strait at different stages in the tidal cycle. The key down the left-hand side of this diagram represents the stage of the tidal cycle the diagram is meant to represent. Note that the strength of the barotropic forcing will be out of phase with the tidal height, but not by simply 90° (Armi & Farmer do not present sufficient information for the phase relation to be determined accurately, though it is obvious from the velocities given by the arrows in figure 10.11 that it is not 90°).

The top diagram represents the interface configuration on an incoming tide, the net forcing \( (Q/b_c) \) appears close to its maximum value. The flow is controlled by Camarinal Sill and Tarifa Narrows: an hydraulic jump forms just to the west of Camarinal Sill, raising the interface in the Tangier Basin so that it may again go critical out over Spartel Sill. Comparing this picture with the quasi-steady models of the previous two sections shows that the observed interface in the Tangier Basin is lower than expected (figures 10.3g, h and 10.9g to i). This is consistent with the lower layer flow rate being less (and the upper layer flow rate more) over Spartel Sill than Camarinal Sill, such as would occur if the internal reservoir of the Tangier Basin were filling.

As high tide is approached (second diagram in figure 10.11) the Tangier Basin will continue to fill with Mediterranean water. The net forcing will decrease at the same time, causing a drop in the height of the interface over Camarinal Sill and through Tarifa Narrows. If \( Q/b_c \) decreases sufficiently rapidly the Tangier Basin will not be able to adjust through its flow out over Spartel Sill and the interface height within the basin may well become higher than that over the sill. This would cause a weak internal bore (reported in Armi & Farmer, 1988) to form, propagating eastwards to flood the control at Camarinal Sill. Overall the flow is still controlled as the hydraulic transitions at Spartel and Tarifa remain.

The net flow into the Mediterranean \( (Q/b_c < 0) \) reaches its
Figure 10.1. Cartoon summarising the time dependent hydraulic response observed in the Strait of Gibraltar at different states of the tide. The symbol on the left-hand side indicates the state of the tide.
peak in the third diagram of the series. The forcing requires a further decrease in interface height through Tarifa to accommodate the inflow of water to the Mediterranean. Tracing the solution back from Tarifa to Camarinal continues to give an interface height lower than in the Tangier basin, allowing outflow in both directions from the basin, hydraulic jumps forming to the east of Camarinal and the west of Spartel.

The approach of low tide sees $Q/b_C$ again increase towards zero and become positive, increasing the interface height through Tarifa, and subsequently returning forward flow and control to Camarinal Sill. The Tangier Basin begins to refill with Mediterranean water, the inflow over Camarinal Sill being somewhat greater than the outflow over Spartel Sill. The hydraulic jump to the west of Spartel Sill is maintained and the one to the west of Camarinal is restored.

Armi & Farmer (1988) noted that although the observed flow does not satisfy the quasi-steady assumption, it is nevertheless controlled and maximal. At all stages through the tidal cycle the flow through the Tarifa-Spartel region is isolated by supercritical regions to the east and west. The flows through Tarifa and over Spartel remain critical even when the Camarinal control is lost. The observation that maximal flow is maintained is of great importance as it validates our treatment of the flow as an hydraulics problem and allows us to ignore details of what is happening within the Mediterranean and Atlantic.

The fundamental feature of this control mechanism which has not been taken into account by the quasi-steady model of the previous two sections is the volumetric capacity of the control region. The quasi-steady model requires the interface height to adjust instantaneously and does not allow time for water to drain out of the Tangier Basin. The decrease/increase of Mediterranean or Atlantic water between Camarinal and Tarifa is a less restricting problem due to the much greater overall depth.

The present hydraulic treatment could be adapted to take this internal reservoir into account by relaxing the restriction that $\bar{q}$ is conserved over the entire length of the channel. As a first approximation we shall assume that the change in the overall water level within the Strait is small compared with changes in the height of the interface; this allows us to assert that $Q$ is
conserved throughout the channel. We shall continue to specify $Q$ as in the earlier models. As we noted above, the finite capacity of the Tarifa-Camarinal region is not important due to its great depth, thus we are free to insist that $\bar{Q}$ is conserved everywhere to the east of the Tangier Basin. Likewise we shall assert that $\bar{Q}$ is conserved to the west of Tangier Basin, though the value of $\bar{Q}$ here will not in general be equal to that through Tarifa. Rather than equating $\bar{Q}$ to the east with $\bar{Q}$ to the west we could relate them through the rate of change of height of the interface in the Tangier basin by

$$\frac{\Delta h_{\text{Tangier}}}{\Delta t} = \frac{\bar{Q}_{\text{west}} - \bar{Q}_{\text{east}}}{2 \text{ Area}} \quad (10.2)$$

where $h_{\text{Tangier}}$ is some characteristic interface height within the Basin and Area some measure of the area the Basin covers such that the product Area $\Delta h_{\text{Tangier}}/\Delta t$ is a good measure of the rate of change of volume of Mediterranean water in the Basin.

Solution would proceed by time stepping through the tidal cycle from some initial state until steady periodic behaviour is achieved. For a given state of the tide $Q$ and interface height in Tangier $h_{\text{Tangier}}$:

- Calculate the controlled solution for controls at Camarinal and Tarifa to give the exchange flow rate $\bar{Q}_{\text{CTarifa}}$ and interface height over Camarinal $h_{\text{C; Tarifa}}$.
- Using $h_{\text{Tangier}}$, calculate the height of the interface over Spartel Sill $h_g$ to give critical conditions and the associated exchange flow rate $\bar{Q}_{\text{west}}$.
- Using $h_{\text{Tangier}}$ and ignoring the flow to the east of Camarinal, calculate the height of the interface over Camarinal Sill $h_{\text{C; Tangier}}$ and the associated exchange flow rate $\bar{Q}_{\text{CTangier}}$.
- If $h_{\text{C; Tarifa}} > h_{\text{C; Tangier}}$ then the Tarifa-Camarinal control exists and the flow to the east of the Tangier Basin is specified by this control with $\bar{Q}_{\text{east}} = \bar{Q}_{\text{CTarifa}}$.
- If $h_{\text{C; Tarifa}} < h_{\text{C; Tangier}}$ then the Tarifa-Camarinal control has been flooded. The flow over Camarinal is given by $h_{\text{C; Tangier}}$ and $\bar{Q}_{\text{east}} = \bar{Q}_{\text{CTangier}}$. To the east of the hydraulic jump which will form between Camarinal and Tarifa, the flow is calculated by assuming the exchange flow rate is
and critical conditions exist at Tarifa.

- The flow to the west of Tangier Basin is specified by the conditions in the Tangier Basin, characterised by $\bar{q}_{west}$, regardless of what is happening at Camarinal.

- Calculate the new interface height using equation (10.2). This calculation should be relatively straightforward but is beyond the scope of the present work.

Further, but less important shortcomings of the present model are related to the channel geometry and neglected processes such as friction and mixing. These are likely to change the fine-scale features of the realised flow, but not the overall hydraulic picture. For example, Armi & Farmer (1988) noted that Camarinal Sill is "topographically complex" and that the "interfacial displacements are closely coupled through the hydraulic response to the local depth of the sea floor which is highly variable over small scales" (see the transect in figure 10.10). Nevertheless they concluded that the flow could be modelled successfully by a two-dimensional obstacle.

Pratt (1986) demonstrated how the presence of frictional effects could shift the control section in a single-layer flow downstream from its inviscid position. Moreover he demonstrated that the friction affected subcritical and supercritical flows in a different manner (friction causes the thickness of a supercritical layer to increase and a subcritical layer to decrease downstream). Much of these results apply also to two-layer nonrotating hydraulics, though in a somewhat modified form. However, we do not include such an analysis here as the effect is likely to be very small compared with the complex time dependent response of the flow to the modulation. Further, the change in the structure of the boundary layers with the presence of rotation prevent the simple drag coefficient approach of Pratt (1986).
Conclusions

11 Conclusions

11.1 General remarks

The structure of this chapter is as follows: in this section we highlight the scope of the work covered by the dissertation. The formulation of the two-layer hydraulic problem is reviewed in section 11.2, and the flow through channels of constant depth is summarised in section 11.3. Section 11.4 recalls the asymmetry introduced by the flow over a simple sill, while section 11.5 reconsiders the flow through more complex geometries and the Strait of Gibraltar. Finally, section 11.6 recapitulates the two most important findings of this work.

In this thesis we have investigated some of the phenomena associated with two-layer buoyancy-driven inviscid flows. Our main concern has been the steady state limit of exchange flow along a channel connecting two reservoirs. We have seen that over a large range of reservoir conditions such flows may be hydraulically controlled in the sense that details of processes and disturbances within the reservoirs are not able to communicate any information to the control region and alter the overall flow. Controlled flows are found to represent the maximal exchange flow. If conditions within the reservoirs are outside the range for which hydraulic control may be achieved (this range is discussed in sections 3.5 and 6.5), one of the control sections bounding the control region is flooded. The other, however, will remain intact (unless the two reservoir interface heights lie on the same subcritical solution), providing partial control of the flow. Disturbances in only one of the two reservoirs are able to affect the flow. This situation is similar to single-layer controlled flows except that both layers are active.

The main focus has been on fully controlled flows as these are frequently realised in practice, even when some of the basic assumptions are violated (Armi & Farmer, 1987). Further, the isolation from the reservoir conditions allows the overall flow to be analysed without the need for a detailed knowledge of the conditions within the reservoirs. We have shown that hydraulic control is a meaningful feature of flows in nonrectangular and
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rotating channels in addition to the nonrotating rectangular channels it has been associated with traditionally. The mechanism creating the isolation of the control region, and the associated hydraulic transitions, is the same regardless of the channel geometry and presence of system rotation.

11.2 Hydraulic functional

In chapter 2 we derived a functional formulation of two-layer hydraulic flows based on the formalism presented by Gill (1977). We demonstrated that if the functional is related to the difference in the Bernoulli potentials between the two layers, a simple relationship exists between the functional (at a given section) and the composite Froude number traditionally used to describe such flows. The functional was derived in a form which did not set restrictions on the cross-sectional geometry of the channel (other than an implicit restriction that the interface is continuous across the channel), allowing for the possibility of the cross-section varying along the channel. Further, the relationship with the Froude number is independent of the cross-section, showing that the flow is always critical (unity Froude number) at a solution branch point.

The hydraulic functional was further developed in section 6.4 to allow for the effects of rotation. While discussion was confined to rotating channels of rectangular cross-section with uniform layer potential vorticities, the formulation of the functional remains valid for any cross-section (so long as there are no discontinuities in the interface across the channel) and any potential vorticity distribution. We were able to demonstrate that solution branch points (hydraulic transitions) correspond with critical flow. The notions of subcritical and supercritical flow in terms of information propagation may be applied even though, by the classical definition, the Froude number varies across the width of the channel. This is possible as the phase velocities of long, small amplitude internal gravity (Kelvin) waves are constant over the width of the channel.

The main advantage of the functional formulation over the approach of Arm (1986) and subsequent authors is clear: it is not
necessary to determine the relationship between a given interface configuration and the layer Froude number. For nonrotating channels there is no real difficulty in utilising a Froude number plane solution for any arbitrary cross-section, though the present functional formulation may prove more attractive computationally. However, when the channel is rotating, the velocity of fluid particles varies across the width of the channel rendering the Froude number a function of the channel width. Nevertheless the relationship between subcritical, critical and supercritical flow continues to hold.

The algorithm presented in section 3.4 was utilised essentially unchanged to produce all the numerically evaluated solutions in this thesis. It was, however, necessary to add further levels for rotating flows where separation prevented an explicit expression from being given for the hydraulic functional. Unfortunately this added greatly to the computational expense: the nonrotating channel computations were performed on a small microcomputer (BBC, based on the 8-bit 6502), while the rotating computations took (on average) slightly longer, per situation, on a Sun 3/60 workstation. The programs on the BBC were in an interpreted Basic, while those on the Sun were in highly optimised, compiled Pascal.

In principle it should be possible to extend the formulation to model multi-layered systems in a similar manner. The definition of subcritical and supercritical flows would again be in terms of information propagation: if all the phase velocities were in the same direction, then the flow would be supercritical (or, if one or more of them vanishes and the others are in the same direction, the flow would be critical). Similarly, if the phase velocities are in both directions, the flow will be subcritical. Any description of an n-layered system would require n-l interface height coefficient-like variables $A_i$ to describe it, in addition to n initially unknown parameters (for the two-layer system they were $\bar{q}$ and $\zeta$). The functional would be defined in a space with n-l dimensions representing possible interface configurations as well as the geometric, flow parameter ($\bar{q}$, $\zeta$, etc.) and J dimensions. The criteria relating the functional to information propagation would also have to be determined. However, this extension is beyond the scope of the present work.
11.3 Channels of constant depth

For a rectangular cross-section, the flow through a channel of constant depth is the most easily analysed and understood. In the absence of rotation and net barotropic forcing, the controlled flow is antisymmetric, with respect to the geometry, and has a single unique control section at the contraction. If the channel is rotating, there will still be one unique control section at the contraction provided the channel is less than approximately one Rossby radius in width. Channels containing nonzero potential vorticity fluid and with a contraction width close to one Rossby radius may have two distinct control sections. These would be positioned symmetrically to either side of the contraction. At these control sections the interface will pass from corner to corner of the channel. The specific exchange flow rate in a rotating channel (with $Q = 0$) is always less than that in the nonrotating limit, regardless of the potential vorticity.

For rotating channels, the value of the potential vorticity is found to make very little difference to the exchange flow rate and appearance of the flow (sections 7.1 and 7.2), provided the channel is less than approximately one Rossby radius wide. In wider channels, the assumption that all streamlines may be traced to the upstream reservoir is broken, though the experiments of chapter 9 suggest that the region of flow reversal may be replaced by a zone of stagnant fluid, at least so long as there is no net barotropic forcing. The presence of a zone of stagnant fluid makes the exchange flow rate independent of the channel width for channels wider than approximately one Rossby radius. The associated channel-crossing may be understood in terms of set-up from dam-break. The assumption that the flow is relatively straight is no longer satisfied within the region where the flow crosses the channel, although it remains valid to either side and so may be used to infer conditions elsewhere in the channel.

If the cross-sectional geometry of the channel is not symmetric about the mid-depth, then the $Q = 0$ flow will also be asymmetric. This was found in section 5.2 for a channel of parabolic cross-section. The essential difference between such a
channel and its rectangular counterpart is the vertical distribution of channel width. The flow adopts a compromise between the fluid velocities in the two layers being equal at the control section (thus minimizing the associated kinetic energy component) and maintaining the interface at half the channel depth. This is achieved by the virtual control being positioned on the dense reservoir side of the contraction such that \( u_1 = -u_2 \) at this point.

The inclusion of net barotropic forcing breaks the symmetry of the flow through a rectangular cross-section. The virtual control separates from the primary control at the contraction and moves upstream, with respect to the net flow. If the channel is not rotating, increasing the net flow will eventually bring one of the layers to rest with a front forming at the virtual control in the source reservoir. Rotation retards the outward movement of the virtual control, causing the layer to be brought to rest with a front forming closer to the contraction, but requiring a stronger (specific) net forcing. For weak barotropic forcings, rotation decreases the exchange flow rate. However as \( Q/b_c \) is increased, rotation may bring about an increase in the exchange flow rate.

Nonrectangular cross-section geometry does little to alter the qualitative response to net forcing. As with the \( Q = 0 \) limit, the interface is always displaced towards the top of the channel (relative to the rectangular cross-section channels), and requires a weak negative forcing to cause both controls to coincide. The overall response to forcing is asymmetric (with respect to the sign of \( Q \)) due again to the asymmetry of the cross-section width distribution.

As pointed out by Armi (1986), the flow over a sill differs fundamentally from that through a contraction. In the latter, variations in the channel width affect both layers equally, whereas the presence of a sill has a direct effect only on the lower layer. The upper layer feels the presence of the sill only through the weak coupling between the two layers.

Earlier investigators (Armi, 1986; Farmer & Armi, 1986)
considered only the flow over a sill (nonrotating, rectangular cross-section) where the depth away from the crest went to infinity. The analysis of sections 3.3 and 3.7 filled in the details of the flow for finite sill heights.

The asymmetry of the flow introduced by a simple sill is a robust feature. It occurs in a very similar form for parabolic channels (section 5.3) and rotating channels (sections 8.4 and 8.5), though in the latter the Taylor-Proudman theorem reduces the asymmetry somewhat by allowing a greater vertical penetration of the topography. If the width of a rotating rectangular channel is greater than one Rossby radius, naive application of the hydraulic theory predicts that the velocity in the upper layer will change sign at both controls somewhere over the width of the channel. This violates the assumption that all streamlines may be traced back to the upstream reservoir. The simple inclusion of a zone of stagnant fluid instead of any velocity reversal is inappropriate, although it predicts the experimental results of section 9.5 with a high degree of accuracy. A more accurate picture of the flow in wide channels is beyond the scope of this thesis.

In the absence of net forcing we found that if the depth away from the crest is greater than approximately 1.5 times the depth at the crest, the flow was very close to that over an infinite sill. This finding holds regardless of the form of the cross-section of the channel and whether or not the channel is rotating. Asymptotic expansions were derived to show the behaviour at small and large sill heights for rectangular cross-sections (with and without rotation). Agreement between the exact solution and the asymptotic expansions was very good, even well outside the range of sill geometries for which they were formally valid, provided the channel was much less than one Rossby radius in width.

The inclusion of a net barotropic flow along the channel was found to introduce a number of new features to the flow over a simple sill of finite height: under some circumstances changing the net flow rate can lead to a sudden change in the behaviour and appearance of the flow. At low net flow rates the virtual control remains fixed at the channel exit; the flow appears little different from \( Q = 0 \) other than a slight displacement of the interface. However, at high net flow rates, the effects of the
geometrically introduced asymmetry may be swamped by the expanding of the channel on either side of the sill. This allows the flow to adopt a contraction-like behaviour. This bifurcation is most pronounced when the net forcing is from the light reservoir: the virtual control jumps from one side of the sill to the other. Associated with this jump there may be a sudden change in the along-channel interface profile.

For nonrotating rectangular channels, if the depth away from the sill is less than 1.5 times that at the crest, the lower layer is brought to rest with a front forming at the virtual control in the light reservoir. For some smaller value of $|Q|$ a bifurcation occurred from sill-like to contraction-like behaviour. The response of the interface height profile and the position of the virtual control is a discontinuous function of the net forcing at this bifurcation.

If the depth away from the sill crest is greater than 1.5 times that at the crest, the lower layer is brought to rest with a front forming at the sill crest before the bifurcation to contraction-like behaviour. A careful analysis of this limit shows that there is a bifurcation to coincident behaviour (both controls at the sill crest), though this does not introduce any new features for the nonrotating rectangular channel.

Forcing from the dense reservoir allows the flow to vary continuously. While no discontinuities in the appearance of the flow occur, there is still a distinction between sill-like and contraction-like behaviour based on the position of the virtual control.

Similar bifurcations exist in parabolic and rotating channels when the forcing is from the dense reservoir, though in a modified form when the net flow is from the light reservoir. If the channel is not rotating, then the strength of forcing required to bring the upper layer to rest is independent of the depth of the channel away from the sill crest. However, the greater vertical penetration produced by rotation means that this is no longer true for rotating channels: increasing depth away from the sill crest decreases the forcing required towards that for a nonrotating channel of the same geometry. In the limit of an infinite sill the forcing required is independent of the rotation rate.

For all channels, if the sill is very small, the flow
bifurcates directly from sill-like to contraction-like behaviour as \( Q \) is decreased from zero. The interface profile and position of the virtual control are discontinuous functions of \( Q \). However, if the sill height is not small and the channel is either rotating or nonrectangular \( (D_w > 1.107 \text{ for the parabolic channel; similar calculations have not been performed for rotating channels}) \), the flow undergoes an intermediate bifurcation to coincident behaviour with both controls at the sill crest. Decreasing \( Q \) further leads to a subsequent bifurcation to contraction-like behaviour at a value of \(|Q|\) less than that required to bring the lower layer to rest. Unlike the single bifurcation from sill-like to contraction-like behaviour, the interface profile is a continuous function of \( Q \) for the sill-like to coincident and coincident to contraction-like bifurcations; the only jump is in the position of the virtual control.

11.5 More complex along-channel geometries and Gibraltar

More complex along-channel geometries were analysed in detail only for nonrotating rectangular channels. The bifurcation structure of a channel containing both a sill and a contraction, isolated so that only one of \( dD/dx \) and \( db/dx \) was nonzero at a given location, was analysed in detail (section 3.8) as a precursor to considering the exchange flow through the Strait of Gibraltar. It was found that the order of the two geometric features was important in determining the response of the flow, both to net barotropic forcing and changes in the key geometric ratios. However, if the depth at the contraction was sufficiently large (the definition of large is a function of both the ratio of the channel widths at the sill crest and contraction, and the order of the two geometric features), the flow responded as though the depth were infinite away from the sill crest.

Simultaneous variations in the channel width and depth were found to produce a flow whose character responds continuously with the strength of the net forcing. The simple geometry introduced in section 3.9 was used in chapter 4 to predict the buoyancy-driven air flow through a doorway. While the quantitative agreement between the experiments and naive hydraulic model is not good,
there is a great deal of qualitative similarity. This is somewhat surprising as a number of the basic assumptions of the hydraulic flow are broken. The results suggest that the observation that the location of the neutral plane away from the mid-door level may be due to the hydraulic-like response of the flow and not directly the nonhydrostatic pressure distribution in the warm room adjacent to the doorway.

The along-channel geometry of the Strait of Gibraltar is fundamentally similar to that of section 3.8. Under the quasi-steady approximation, the presence of a second sill (Spartel) is felt but only when the net forcing is strongly into the Mediterranean. Under these circumstances Camarinal Sill is flooded with the hydraulic control being transferred to Spartel Sill.

The fundamental response of the flow to quasi-steady net barotropic forcing is similar for all three channel types looked at. The exchange flow rates, averaged over a tidal cycle, agree to within a few percent, regardless of the amplitude of the tidal modulation. The response of the interface is qualitatively similar, though details of the forcing required for certain aspects to manifest themselves differs between the channels. The parabolic model always produces an interface position higher than the rectangular cross-sections; again this is a result of the different vertical distribution of the channel width.

Rotation induces a cross-channel tilt to the interface and causes it to separate from the Spanish coast when flowing into the Alboran Sea. For the rectangular cross-section used the interface will also separate from the North African coast flowing over the two sills and into the Atlantic, though this feature will be modified by the nonrectangular geometry of the real channel.

The close agreement between the three models for the Strait shows that the most important feature not modelled is the unsteady behaviour associated with the filling and draining of the Tangier Basin, and not any fundamental hydraulic features. In section 10.4 we suggested a simple method by which this unsteadiness may be modelled within the quasi-steady framework of an hydraulic model.
11.6 Final remarks

The two most important findings of the work presented in this thesis are the power of the hydraulic functional formulation to model flows in complex situations, and the relative insensitivity of the controlled solution to the exact form of the cross-channel geometry and the presence of rotation.

The crucial step in analysing hydraulic flows is determining how information is communicated. For two-layer flows, this is by long, small-amplitude internal gravity waves. The usefulness of the hydraulic functional stems from its simple relationship with the phase velocities of such waves, regardless of the cross-section, system rotation and potential vorticity. Moreover, the functional may, in principle, be evaluated for any two-layer hydraulic flow (though it may not be possible to give an explicit representation of it in terms of a single coefficient).

The apparent insensitivity of hydraulic flows to details of the flow other than the along-channel geometry is reassuring. This insensitivity was most apparent in the analysis of the flow through the Strait of Gibraltar where the three models used were nearly indistinguishable. It suggests (as has been found for single-layer flows) that maximal exchange flow is likely to occur in a much wider variety of situations than the shallow water assumptions would suggest. An example of this is the qualitative agreement between hydraulic theory and buoyancy-driven airflow through doorways: the latter is definitely not two-dimensional or slowly varying as required by the shallow water equations.
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