Buoyant Mixing of Unstably Stratified Fluids in a Vertical Square Tube

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Abstract

Quantitative time dependent measurements of irreversible mixing caused by the development of Rayleigh-Taylor instability on an initially unstable two-layer stratification of miscible fluids are taken from a series of laboratory experiments. The experiments are conducted by overturning a stable stratification in a high-aspect ratio tank with square cross-section, and are observed until the flow is quiescent.

Using a light attenuation technique we obtain detailed time-dependent measures of the irreversible mixing, and its efficiency during flow evolution. We find that the flow passes through a number of distinct phases, beginning with classical $t^2$ growth of the instability before a rapid (but gradually slowing) growth of the instability along the entire length of the tube. The still statically unstable mean stratification continues to drive turbulent mixing along the tube, ultimately leading to a weakly stably stratified final state with a gradual exponential decay of the remaining disturbances.

1. Introduction

Rayleigh-Taylor instability, which can develop whenever the density and pressure gradients have components in the opposite direction, plays an important role in many density stratified flows. Consequently, a substantial literature has developed on the instability in its simplest form, developing from quiescent initial conditions with fluid of a higher density overlying fluid of a lower density. Recent studies have extended this to consider how the development of the instability may be changed if embedded in a more complex flow. For example, Holford et al. (2002) have looked at the development from tilted interfaces, whilst Jacobs & Dalziel (2005) have considered the instability in more complex stratifications. While most of these studies have concentrated on gross features such as the ‘growth rate’ (the coefficient in front of the $t^2$ term governing the width of the mixing region, e.g. Dimonte et al. 2005), some authors have addressed the question of how efficient the mixing process is. Redondo & Linden (1993) not only made measurements of the final state, finding a weakly stable stratification, but also used a pH indicator to illustrate the mixing occurring during the growth of the instability. Holford et al. (2002) used an LIF technique to measure the instantaneous density field in conjunction with nominally identical particle tracking experiments for the velocity field to get a handle on the instantaneous mixing efficiency of the developing instability, comparing their results with Implicit Large Eddy Simulations of the same thing. Most recently, Lawrie & Dalziel (2006) have used a pH sensitive fluorescent indicator to look directly at the fine structure of the mixing that develops.
Most of these studies, however, have been performed in geometries with an order one aspect ratio. While this may be of interest in many practical situations, there are other cases where mixing in high aspect ratio domains is important. The simplest example of this is the mixing along a pipe.

The growth of a mixing region between two miscible fluids in a long circular tube has been studied in a recent series of papers (Debacq et al. 2001; Séon et al. 2004; Séon et al. 2005). This work has illuminated many of the key features of the flow, but has concentrated on the initial development as a function of the inclination of the tube. In contrast, in this paper we quantify the mixing, adopting the Available Potential Energy framework of Winters et al. (1995) to divide the instantaneous potential energy of the system into Background Potential Energy (which represents the mixedness of the system) and Available Potential Energy (which can drive motion within the flow). We apply this formulation to the current flow in a manner similar to that employed by Patterson et al. (2006) in a recent study of Kelvin-Helmholtz billows.

2. Experimental setup

The experimental setup consists of a 2m long tube with a 50mm square cross-section mounted on a rotating pivot at its mid-point. The tube is initially filled to its mid-level with a layer of fresh water then a layer of dyed (‘red fiesta’) salt water is carefully added from below to form a stable two-layer stratification. At $t = 0$ the tank is rapidly rotated about the pivot to position the dense fluid above the light fluid. As viscous and baroclinic effects are not significant during this inversion process, the fluid remains essentially irrotational which means the interface is displaced from horizontal by approximately $\tan^{-1}2\pi \approx 81^\circ$. The tube is illuminated from behind by a 2.4m fluorescent tube and acrylic diffuser to give a nearly uniform illumination. A 12-bit, 8MPixel digital video camera operating at around 2 frames per second is used to capture the evolution of the flow with approximately 70 pixels across the width of the tube and 3300 along its length. The presence of the dye in the salty layer attenuates the light passing through the tank, thus allowing the line-of-sight averaged concentration/density field to be determined. The background illumination is recorded from an image of the tank containing fresh water, and small spatio-temporal variations in the illumination are corrected for using information of the illumination during the experiment from just outside of the tank. The subsequent processing of the images and calibration of the dye follows that outlined by Hacker et al. (1994?) and used by Patterson et al. (2006).

3. Observations

Figure 1a shows snapshots the initial development of the instability for an Atwood number

$$ A = (\rho_1 - \rho_2)/(\rho_1 + \rho_2) \quad (1) $$

of 0.005, where $\rho_1$ and $\rho_2$ are the initial densities of the upper and lower layers, respectively. If the interface between the two layers were initially horizontal, then at very early times the instability might be expected to grow as $\alpha Ag t^2$ (where $g$ is gravity and $t$ time, with $\alpha \approx 0.07$ dependent on the initial conditions). This would continue until the vertical length scales were comparable with the width of the tube. However, as noted above, there is a significant initial tilt on the interface, effectively removing the initial period of $t^2$ growth. As the width of the tube $W$ will then be the dominant length scale, we therefore expect (on dimensional grounds)
the natural time scale of the flow to be $T = (W/Ag)^{1/2}$ and the height of the mixing zone to initially scale as $(AgW)^{1/2} t = W t/T$.

However, as can be seen in figure 1a, the growth rate of the mixing zone is not constant but instead reduces gradually. This is well modelled by Inogamov et al.'s (2001) prediction of

$$\frac{h}{W} = \beta \left( \frac{t}{T} \right)^{7/5},$$

where $\beta$ is a dimensionless constant of order unity. This relationship (with $\beta \approx 1.5$) is superimposed on the snapshots in figure 1a. The vertical density profiles are self-similar during this time with a vertical structure reminiscent of an error function, suggesting a process akin to turbulent-diffusion is governing the vertical transport. Indeed, the velocity field within the tube (figure 1b) is characterised by turbulent structures on the scale of the width of the tube, and it is these structures that are responsible for the vertical transport of density. Simple turbulent diffusion with constant diffusivity proportional to $Wu'$ (where $u'$ is the turbulent intensity) would suggest a $t^{1/2}$ time-dependence, but here, however, the strength of the turbulence diminishes with time as the height of the mixing zone increases, decreasing the strength of the unstable density gradient driving the turbulence.
Once the mixing zone reaches the top and bottom of the tube the structure of the density field evolves towards a linear state with the differences over the height of the tank gradually diminishing. This evolution is clearly visible in figure 2a, which shows the vertical profiles of density (dye concentration) at 100s intervals over 6,000 s for \( A = 0.005 \). The average density gradient, obtained by fitting a line to the density profile over the central 50% of the tube, is shown in figure 2b, along with the corresponding third derivative (\( d^3 c/dz^3 \)) of the concentration field. The density gradient remains positive (unstable stratification) until about 3,700s after the beginning of the experiment, beyond which time a weak stable stratification gradually develops.

![Figure 2](image_url)

**Figure 2:** (a) Vertical profiles of the density field at 100s intervals. (b) Evolution of \( dc/dz \) (bold line) and \( d^3 c/dz^3 \) (light line) at mid-depth.

### 3. Available Potential Energy

We can gain insight into the mixing within the flow by considering the evolution of its potential energy. Following Winters et al. (1995), we divide this into two components: the Background Potential Energy (BPE), and the Available Potential Energy (APE). The BPE is obtained by adiabatically rearranging the instantaneous density field into the state with the lowest potential energy. This is the state that would be achieved it mixing were turned off and the fluid were allowed to come to rest through dissipation alone. In contrast, the APE at a given time is simply the difference between the total potential energy and the BPE.

From our experimental data it is straight forward to calculate the evolution of total potential energy within the tube throughout an experiment. However, as we only have line-of-sight averaged density measurements we cannot determine exactly the BPE: we can only determine the mean density at a given pixel in the video image, not whether this mean is achieved due to molecular mixing or simply interleaving of fluids of different densities. Reversible stirring (i.e., no molecular diffusion) in our unresolved direction gives the same signal as irreversible mixing (with molecular diffusion) in that direction. The net result of this is that we are only able to determine an upper bound on the BPE, i.e. the true density distribution might be less well mixed than it appears on our video images. Nevertheless, the bound we obtain by performing the calculation of BPE offers us considerable insight into the flow.
Figure 3 shows the evolution of the total potential energy and BPE against time. Figure 3a plots the potential energies (normalised by the initial potential energy) against time over a range of experimental Atwood numbers. Clearly, the higher Atwood number flows develop faster. However, each of the curves appears to have the same – or at least a similar – functional form.

Comparing the evolution of the mean density gradient in figure 2b with the BPE in figure 3a demonstrates that although the mean profiles are stable at late time ($t \approx 3,700$ s for $A = 0.005$), sufficient horizontal variations in the density remain for the BPE to continue to increase and the stratification to continue to evolve towards a lower total energy state.

It is tempting to replot these results with time normalised by $T$ (effectively rescaling by $A^{-1/2}$), the natural timescale for the instability developing in the tube. If we do so, however, we find that while at early times the curves for different Atwood numbers coincide, their behaviours differ late time. We find instead that rescaling time by $A^{-1/3}$ leads to a better collapse at late time. This observation is emphasised in figure 3b where we have rescaled time by $(A_0/A)^{1/3}$, with $A_0$ set to 0.005, so that the time axis corresponds to that of our lowest Atwood number flow seen in the other figures.

4. Conclusions

Rayleigh-Taylor instability in this high aspect ratio domain is clearly an efficient mixing process. Only around 7% of the initial unstable density difference between the top and bottom of the tank remains in the stable approximately linear final state, indicating an overall mixing efficiency of around 47.6%. This is close to the theoretical 50% maximum for Rayleigh-Taylor instability between layers of equal depth and significantly higher than the efficiencies found in experiments in domains with close to a unit aspect ratio (e.g. Linden & Redondo 1993; Holford et al. 2002). It might be anticipated, therefore, that the mixing efficiency will continue to increase with aspect ratio domains, asymptoting towards 50% in an infinitely long tube where the fluid must pass through an ever-increasing number of vortical structures in its passage along the tube.
The late time scaling of the development with $A^{-1/3}$ instead of $A^{-1/2}$ suggests that the flow loses much of its memory of the initial conditions and takes on a character more akin to high Rayleigh number convection rather than Rayleigh-Taylor instability. Close inspection of the late time behaviour of the potential energy supports this and suggests an approximately exponential decay towards the final state as there is no longer a significant source of energy from the available potential energy. The change-over to this exponential decay appears to take place after the stratification has established a linear profile throughout the depth of the tube (which occurs after the mixing zone reaches the limits of the tube) but while the stratification remains weakly unstable.

Although our measurements only provide an upper bound on the BPE during the development of the flow, they become exact in the final quiescent state, and illustrate that the mixing in the developing flow passes through two self-similar states, with the turbulence driven by the unstable density stratification continuing to fuel the mixing even once the mean stratification becomes stable.

References


