On the Solvability Complexity Index, the n-Pseudospectrum and Approximations of Spectra of Operators

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Abstract

We show that it is possible to compute spectra and pseudospectra of linear operators on separable Hilbert spaces given their matrix elements. The core in the theory is pseudospectral analysis and in particular the n-pseudospectrum and the residual pseudospecrum. We also introduce a new classification tool for spectral problems, namely, the Solvability Complexity Index. This index is an indicator of the "difficultness" of different computational spectral problems.

1 Introduction

This article follows up on the ideas introduced in [3, 4, 2, 1, 33] and addresses the long standing open question: Can one compute the spectrum and the pseudospectra of a linear operator on a separable Hilbert space? We show that the answer to the question is affirmative.

The importance of determining spectra of linear operators does not need much explanation as such spectra are essential in quantum mechanics, both relativistic and non-relativistic, and in general in mathematical physics. However, we would like to emphasize the importance of non-self-adjoint operators and their spectra. This is certainly not a new field, as the non-self-adjoint spectral theory of Toeplitz and Wiener-Hopf operators has been studied for about a century by many mathematicians and physicists, however, it is an expanding area of mathematics. The growing interest in non-Hermitian quantum mechanics [8, 7, 36, 35], non-self-adjoint differential operators [23, 27] and in general non-normal phenomena [22, 49, 50, 48] has made non-self-adjoint operators and pseudospectral theory indispensable.

Now returning to the main question, namely, can one compute spectra of arbitrary operators, we need to be more precise regarding the mathematical meaning of this. Given a closed operator T on a separable Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T)$, we suppose that $\{e_j\}_{j\in\mathbb{N}}$ is a basis for $\mathcal{H}$ such that $\mathrm{span}\{e_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(T)$, and thus we can form the matrix elements $x_{ij} = \langle Te_j, e_i\rangle$. Is it possible to recover the spectrum of T through a construction only using arithmetic operations and radicals of the matrix elements? (Much more precise definitions of this will be discussed in Section 3.) This obviously has to be a construction that involves some limit operation, but in finite dimensions this is certainly possible. For a finite dimensional matrix one may deduce that all its spectral information may be revealed through a construction using only arithmetic operations and radicals of the matrix elements. More precisely, for a matrix $\{a_{ij}\}_{ij\leq N}$ one can construct $\{\Omega_n\}_{n\in\mathbb{N}}$, where $\Omega_n \subset \mathbb{C}$ can be constructed using only finitely many arithmetic operations and radicals of the matrix elements $\{a_{ij}\}_{ij\leq N}$, and $\Omega_n \to \sigma(\{a_{ij}\}_{ij\leq N})$ in the Hausdorff metric as $n \to \infty$. For a compact operator C we may let P_m be the projection onto $\mathrm{span}\{e_j\}_{j\leq m}$ and observe that $\sigma(P_m C\lceil_{P_m\mathcal{H}}) \to \sigma(C)$ in the Hausdorff metric as $m \to \infty$. Thus, as we now are faced with a finite dimensional problem that we can solve (at least as sketched above), we may deduce that, yes indeed, we can compute the spectrum of a compact operator using only its matrix elements. The question is: can this be done in general?

As of today it is not known how to recover the spectrum of an arbitrary operator using its matrix elements nor is it known how to approximate the spectrum using other methods. As pointed out by Arveson in [3]: "Unfortunately, there is a dearth of literature on this basic problem, and so far as we have been able to tell, there are no proven techniques." This certainly is an obstacle to our possible understanding of non-Hermitian quantum mechanics (we emphasize non-Hermitian because much more is known in the Hermitian case). However, knowledge of spectra of linear operators is indispensable in other areas of mathematical physics, and the lack of computational theory leads to serious restrictions to our possible understanding of some physical systems. The purpose of this paper is to show that there are ways of computing spectra of arbitrary linear operators and hence filling the long standing gap in computational spectral theory. The literature is vast on spectral approximation and we can only refer to a subset here. For selected papers and books we consider to be important and related to the topic of this paper we refer to [9, 11, 10, 26, 32, 15, 14, 41, 19, 18, 28, 42, 47] for a functional analysis exposition and [50, 48, 25, 39] for a more applied mathematical treatment.

The computational spectral problem in infinite dimensions is much more delicate than the finite-dimensional case. One reason is the possibly discontinuous behavior of the spectrum as the following well known example shows. Let $A_\epsilon : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be defined by

$$(A_\epsilon f)(n) = \begin{cases} \epsilon f(n+1) & n = 0 \\ f(n+1) & n \neq 0. \end{cases}$$

Now for $\epsilon \neq 0$ we have $\sigma(A_\epsilon) = \{z : |z| = 1\}$ but for $\epsilon = 0$ then $\sigma(A_0) = \{z : |z| \leq 1\}$.

This fact may cause serious concern to a numerical analyst. One can argue that if one should do a computation of the spectrum on a computer, the fact that the arithmetic operations carried out are not exact may lead to the result that one gets the true solution to a slightly perturbed problem. As suggested in the previous example this could be disastrous. The problem above does not occur (in the bounded case) if we are considering the pseudospectrum.

Definition 1.1. *Let T be a closed operator on a Hilbert space $\mathcal{H}$ such that $\sigma(T) \neq \mathbb{C}$, and let $\epsilon > 0$. The ϵ-pseudospectrum of T is defined as the set*

$$\sigma_\epsilon(T) = \sigma(T) \cup \{z \notin \sigma(T) : \|(z - T)^{-1}\| > \epsilon^{-1}\}.$$

The reason is that the pseudospectrum varies continuously with the operator T if T is bounded (we will be more specific regarding the continuity below.) One may argue that the pseudospectrum may give a lot of information about the operator and one should therefore estimate that instead, however, we are interested in getting a complete spectral understanding of the operator and will therefore estimate both the spectrum and the pseudospectrum. We would therefore like to introduce a set which has the continuity property of the pseudospectrum but approximates the spectrum. For this we introduce the n-pseudospectrum. (This set was actually first introduced in [33].)

Definition 1.2. *Let T be a closed operator on a Hilbert space $\mathcal{H}$ such that $\sigma(T) \neq \mathbb{C}$, and let $n \in \mathbb{Z}_+$ and $\epsilon > 0$. The (n, ϵ)-pseudospectrum of T is defined as the set*

$$\sigma_{n,\epsilon}(T) = \sigma(T) \cup \{z \notin \sigma(T) : \|(z - T)^{-2^n}\|^{1/2^n} > \epsilon^{-1}\}.$$

As we will see in Section 4, the n-pseudospectrum has all the nice continuity properties that the pseudospectrum has, but it also approximates the spectrum arbitrarily well for large n.

Another ingredient that may give spectral information about the operator is the spectral distribution. More precisely, let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C^*-algebra with a unique tracial state. Then a self-adjoint operator $A \in \mathcal{A}$ determines a natural probability measure μ_A on $\mathbb{R}$ by

$$\int_{\mathbb{R}} f(x)\, d\mu_A(x) = \tau(f(A)), \quad f \in C_0(\mathbb{R}).$$

Also, if τ is faithful then $\mathrm{supp}(\mu_A) = \sigma(A)$ and one refers to μ_A as the spectral distribution. Thus knowing μ_A gives information about the density of the spectrum and hence is desirable to know. How to approximate μ_A (in the spirit of Szegö) using only the matrix elements of the operator has been investigated in [3, 6, 5, 33]. We will consider a generalisation of μ_A that is suitable for some non-self-adjoint operators, namely, the Brown measure and show how one can get Szegö-type theorems as in [3, 6, 5] for this measure.

We conclude the article with some examples suggesting that our rather abstract framework will indeed be useful in applications.

2 Background and Notation

We will in this section review some basic definition and introduce the notation used in the article. Throughout the paper $\mathcal{H}$ will always denote a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators, $\mathcal{C}(\mathcal{H})$

the set of densely defined closed linear operators and $\mathcal{SA}(\mathcal{H})$ the set of self-adjoint operators on $\mathcal{H}$. For $T \in \mathcal{C}(\mathcal{H})$ the domain of T will be denoted by $\mathcal{D}(T)$ and the spectrum by $\sigma(T)$. Also, if $T - z$ is invertible, for $z \in \mathbb{C}$, we use the notation $R(z, T) = (T - z)^{-1}$. We will denote orthonormal basis elements of $\mathcal{H}$ by e_j, and if $\{e_j\}_{j\in\mathbb{N}}$ is a basis and $\xi \in \mathcal{H}$ then $\xi_j = \langle \xi, e_j \rangle$. The word basis will always refer to an orthonormal basis. If $\mathcal{H}$ is a finite dimensional Hilbert space with a basis $\{e_j\}$ then $LT_{\mathrm{pos}}(\mathcal{H})$ will denote the set of lower triangular matrices (with respect to $\{e_j\}$) with positive elements on the diagonal. The closure of a sets $\Omega \in \mathbb{C}$ will be denoted by $\overline{\Omega}$ or $\mathrm{cl}(\Omega)$, and the interior will be denoted by Ω^o.

Convergence of sets in the complex plane will be quite crucial in our analysis and hence we need the Hausdorff metric as defined by the following.

Definition 2.1. *(i) For a set $\Sigma \subset \mathbb{C}$ and $\delta > 0$ we will let $\omega_\delta(\Sigma)$ denote the δ-neighborhood of Σ (i.e. the union of all δ-balls centered at points of Σ).*

(ii) Given two sets $\Sigma, \Lambda \subset \mathbb{C}$ we say that Σ is δ-contained in Λ if $\Sigma \subset \omega_\delta(\Lambda)$.

(iii) Given two compact sets $\Sigma, \Lambda \subset \mathbb{C}$ their Hausdorff distance is

$$d_H(\Sigma, \Lambda) = \max\left\{\sup_{\lambda\in\Sigma} d(\lambda, \Lambda), \sup_{\lambda\in\Lambda} d(\lambda, \Sigma)\right\}$$

where $d(\lambda, \Lambda) = \inf_{\rho\in\Lambda} |\rho - \lambda|$.

If $\{\Lambda_n\}_{n\in\mathbb{N}}$ is a sequence of compact subsets of $\mathbb{C}$ and $\Lambda \subset \mathbb{C}$ is compact such that $d_H(\Lambda_n, \Lambda) \to 0$ as $n \to \infty$ we may use the notation $\Lambda_n \longrightarrow \Lambda$.

As for the convergence of operators we follow the notation in [38]. Let $E \subset \mathcal{B}$ and $F \subset \mathcal{B}$ be closed subspaces of a Banach space $\mathcal{B}$. Define

$$\delta(E, F) = \sup_{\substack{x\in E\\ \|x\|=1}} \inf_{y\in F} \|x - y\|$$

and

$$\hat{\delta}(E, F) = \max[\delta(E, F), \delta(F, E)].$$

If A and B are two closed operators, with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$, their graphs

$$G(A) = \{(\xi, \eta) \in \mathcal{H} \times \mathcal{H} : \xi \in \mathcal{D}(A), \eta = A\xi\} \tag{2.1}$$

and $G(B)$ are closed subspaces of $\mathcal{H} \times \mathcal{H}$. We can therefore define (with a slight abuse of notation) the distance between A and B by

$$\hat{\delta}(A, B) = \hat{\delta}(G(A), G(B)).$$

If $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of closed operators converging in the distance suggested above to a closed operator T then we may sometimes use the notation

$$T_n \xrightarrow{\hat{\delta}} T, \qquad n \to \infty.$$

Note that $\hat{\delta}$ is not a metric. To define a metric on $\mathcal{C}(\mathcal{H})$ there are several possibilities. We will discuss two approaches here that will be useful later on in the paper. For closed operators A and B define

$$d(A, B) = \max\left[\sup_{\xi\in G(A),\|\xi\|=1} \mathrm{dist}(\xi, G(B)), \sup_{\xi\in G(B),\|\xi\|=1} \mathrm{dist}(\xi, G(A))\right],$$

where

$$\mathrm{dist}(\xi, G(A)) = \inf_{\eta\in G(A)} \|\xi - \eta\|.$$

As shown in [38] d is a metric on $\mathcal{C}(\mathcal{H})$ with the property that

$$\hat{\delta}(A, B) \leq d(A, B) \leq 2\hat{\delta}(A.B).$$

A more practical metric for our purpose is the one suggested in [20]. The definition is as follows

$$p(A, B) = [\|R_A - R_B\|^2 + \|R_{A^*} - R_{B^*}^2\| + 2\|AR_A - BR_B\|^2]^{1/2},$$

where $R_A = (1 + A^*A)^{-1}$. For our purposes it is important to link p to $\hat{\delta}$ and that follows from the fact, as proved in [20], that p and d are equivalent as metrics on $\mathcal{C}(\mathcal{H})$. In particular we have

$$d(A, B) \leq \sqrt{2}p(A, B) \leq 2d(A, B).$$

The following fact will be useful in the later developments.

Theorem 2.2. *([38] p.204) Let $T, S \in \mathcal{C}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\hat{\delta}(S + A, T + A) \leq 2(1 + \|A\|^2)\hat{\delta}(S, T).$$

Recall also the definition of a weighted shift.

Definition 2.3. *Let $\{e_n\}_{n\in\mathbb{N}}$ be a basis for the Hilbert space $\mathcal{H}$. By a weighted shift on $\mathcal{H}$ we mean an operator $W \in \mathcal{C}(\mathcal{H})$ with $\mathcal{D}(W) \supset \mathrm{span}\{e_n\}_{n\in\mathbb{N}}$ with the property that there is a sequence of complex numbers $\{\alpha_j\}_{j\in\mathbb{N}}$ and an integer k such that for $\xi \in \mathcal{D}(W)$ we have $(W\xi)_j = \alpha_j\xi_{k+j}$. The set of weighted shifts on $\mathcal{H}$ (with respect to $\{e_n\}_{n\in\mathbb{N}}$) will be denoted by $WS(\mathcal{H})$.*

Remark 2.1 Throughout the article we will only consider operators T such that $\sigma(T) \neq \mathbb{C}$ and $\sigma(T) \neq \emptyset$, hence this assumption will not be specified in any of the upcoming theorems.

3 The Main Results

3.1 The Solvability Complexity Index

Before we discuss the slightly technical main results we would like to give the reader an overview of the ideas and also justify the motivation for the essential definition of our main tool, namely, the Solvability Complexity Index (SCI). As mentioned in the introduction, it is well known that for a matrix $\{a_{ij}\}_{ij\leq N}$ one can construct $\{\Omega_n\}_{n\in\mathbb{N}}$, where $\Omega_n \subset \mathbb{C}$ can be constructed using only finitely many arithmetic operations and radicals of the matrix elements $\{a_{ij}\}_{ij\leq N}$, and $\Omega_n \to \sigma(\{a_{ij}\}_{ij\leq N})$ as $n \to \infty$. Also, for a compact operator C, if P_m is the projection onto $\mathrm{span}\{e_j\}_{j\leq m}$, then $\sigma(P_mC\lceil_{P_m\mathcal{H}}) \to \sigma(C)$ as $m \to \infty$. Thus, if we were to compute the spectrum of C we must let $\Omega_{m,n}$ denote the n-th output of the approximate computation (now a finite-dimensional problem) of the spectrum of the matrix $P_mC\lceil_{P_m\mathcal{H}}$, and then deduce that

$$\sigma(C) = \lim_{m\to\infty} \lim_{n\to\infty} \Omega_{m,n}.$$

What is crucial here is that we have to take two limits. Thus, if we are concerned with the accuracy of the approximation $\Omega_{m,n}$ to $\sigma(C)$ we must control two limiting procedures as opposed to only one in the finite-dimensional case. If there in fact had been more than two limits involved, this would have complicated the issue of accuracy even further. Hence, when faced with the computational spectral problem in infinite dimensions it is of great interest to know how many limits one must use in order to compute the spectrum (or the pseudospectrum), and, of course, the fewer the better. In particular, it is of great importance to know the number of the least amount of limits needed. This is the motivation for the Solvability Complexity Index. The Solvability Complexity Index of a spectral problem is simply the least amount of limits required to compute the desired set (spectrum, pseudospectrum, n-pseudospectrum, etc.). Hence, the Solvability Complexity Index is an indicator of optimality.

Note that the Solvability Complexity Index of spectra of compact operators may be different than the Solvability Complexity Index of spectra of bounded operators (since compact operators is a subset of bounded operators). In particular, when defining the Solvability Complexity Index of a spectral problem it is crucial to specify the class of operators considered. The ideas above are summarized in the following definitions.

Definition 3.1. *Let $\mathcal{H}$ be a Hilbert space spanned by $\{e_j\}_{j\in\mathbb{N}}$ and let*

$$\Upsilon = \{T \in \mathcal{C}(\mathcal{H}) : \mathrm{span}\{e_j\}_{n\in\mathbb{N}} \subset \mathcal{D}(T)\}. \tag{3.1}$$

Let $\Delta \subset \Upsilon$ and $\Xi : \Delta \to \Omega$, where Ω denotes the collection of closed subsets of $\mathbb{C}$. Let

$$\Pi_\Delta = \{\{x_{ij}\}_{i,j\in\mathbb{N}} : \exists\, T \in \Delta,\, x_{ij} = \langle Te_j, e_i\rangle\}.$$

A set of estimating functions of order k for Ξ is a family of functions

$$\Gamma_{n_1} : \Pi_\Delta \to \Omega, \Gamma_{n_1,n_2} : \Pi_\Delta \to \Omega, \ldots, \Gamma_{n_1,\ldots,n_{k-1}} : \Pi_\Delta \to \Omega,$$

$$\Gamma_{n_1,\ldots,n_k} : \big\{\{x_{ij}\}_{i,j\leq N(n_1,\ldots,n_k)} : \{x_{ij}\}_{i,j\in\mathbb{N}} \in \Pi_\Delta\big\} \to \Omega,$$

where $N(n_1,\ldots,n_k) < \infty$ depends on $n_1,\ldots,n_k$, with the following properties:

- *(i) The evaluation of $\Gamma_{n_1,\ldots,n_k}(\{x_{ij}\})$ requires only finitely many arithmetic operations and radicals of the elements $\{x_{ij}\}_{i,j\leq N(n_1,\ldots,n_k)}$.*
- *(ii) Also, we have the following relation between the limits:*

$$\begin{aligned}
\Xi(T) &= \lim_{n_1\to\infty} \Gamma_{n_1}(\{x_{ij}\}),\\
\Gamma_{n_1}(\{x_{ij}\}) &= \lim_{n_2\to\infty} \Gamma_{n_1,n_2}(\{x_{ij}\}),\\
&\vdots\\
\Gamma_{n_1,\ldots,n_{k-1}}(\{x_{ij}\}) &= \lim_{n_k\to\infty} \Gamma_{n_1,\ldots,n_k}(\{x_{ij}\}).
\end{aligned}$$

The limit is defined as follows: For $\omega \in \Omega$ and $\{\omega_n\} \subset \Omega$, then $\omega = \lim_{n\to\infty} \omega_n$ means that for any compact ball K such that $\omega \cap K^o \neq \emptyset$, we have $d_H(\omega \cap K, \omega_n \cap K) \to 0$, when $n \to \infty$.

Definition 3.2. *Let $\mathcal{H}$ be a Hilbert space spanned by $\{e_j\}_{j\in\mathbb{N}}$, define Υ as in (3.1), and let $\Delta \subset \Upsilon$. A set valued function*

$$\Xi : \Delta \subset \mathcal{C}(\mathcal{H}) \to \Omega$$

is said to have Solvability Complexity Index k if k is the smallest integer for which there exists a set of estimating functions of order k for Ξ. Also, Ξ is said to have infinite Solvability Complexity Index if no set of estimating functions exists. If there is a function

$$\Gamma : \{\{x_{ij}\} : \exists\, T \in \Delta,\, x_{ij} = \langle Te_j, e_i\rangle\} \to \Omega$$

such that $\Gamma(\{x_{ij}\}) = \Xi(T)$, and the evaluation of $\Gamma(\{x_{ij}\})$ requires only finitely many arithmetic operations and radicals of a finite subset of $\{x_{ij}\}$, then Ξ is said to have Solvability Complexity Index zero. The Solvability Complexity Index of a function Ξ will be denoted by $SC_{\mathrm{ind}}(\Xi)$.

Example 3.3. Let $\mathcal{H}$ be a Hilbert space with basis $\{e_j\}$, $\Delta = \mathcal{B}(\mathcal{H})$ and $\Xi(T) = \sigma(T)$ for $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\dim(\mathcal{H}) \leq 4$. Then Ξ must have Solvability Complexity Index zero, since one can obviously express the eigenvalues of T using finitely many arithmetic operations and radicals of the matrix elements $x_{ij} = \langle Te_j, e_i\rangle$.

For $\dim(\mathcal{H}) \geq 5$ then obviously $SC_{\mathrm{ind}}(\Xi) > 0$, by the much celebrated theory of Abel on the unsolvability of the quintic using radicals.

Now, what about compact operators? Suppose for a moment that we can show that $SC_{\mathrm{ind}}(\Xi) = 1$ if $\dim(\mathcal{H}) < \infty$. (It is straightforward to show this, but we consider this a problem in matrix analysis and shall not discuss it any further, nor will any of the upcoming theorems rely on such a result.) A standard way of determining the spectrum of a compact operator T is to let P_n be the projection onto $\mathrm{span}\{e_j\}_{j\leq n}$ and compute the spectrum of $P_nA\lceil_{P_n\mathcal{H}}$. This approach is justified since $\sigma(P_nA\lceil_{P_n\mathcal{H}}) \to \sigma(T)$ as $n \to \infty$. By the assumption on the Solvability Complexity Index in finite dimensions, it follows that if Δ denotes the set of compact operators then $SC_{\mathrm{ind}}(\Xi) \leq 2$.

The reasoning in the example does not say anything about what the Solvability Complexity Index of spectra of compact operators is, it only suggest that the standard way of approximating spectra of such operators will normally make use of a construction requiring two limits. We will in this article discuss only upper bounds on the Solvability Complexity Index, as we consider that the most important question to solve first, since as of today there is no general approach to estimate the spectrum of an arbitrary bounded operator. Now, after having established upper bounds, an important problem to solve would be to actually determine the Solvability Complexity Index of spectra of subclasses of operators. These questions are left for future work.

Remark 3.1 Note that there is a close connection between the Solvability Complexity Index and the question of computing zeros of polynomials. In particular, if one was to show that $SC_{\text{ind}}(\Xi) = 1$, where $\Xi(T) = \sigma(T)$ for $T \in \Delta = \mathbb{C}^{n\times n}$, $n \geq 5$, then one first has to show the insolvability of the quintic using radicals, and then show the existence of an algorithm to compute $\Xi(T)$ using only arithmetic operations and radicals and taking one limit. Note also that although the ideas behind the Solvability Complexity Index deviates slightly from the complexity theory a la Smale [44, 45, 46], they are very much in the same spirit and indeed inspired by the concepts in [44, 45, 46].

3.2 The Main Theorems

The main theorems in this chapter state that indeed it is possible to compute spectra and pseudospectra of all bounded operators given the matrix elements. For the unbounded case this is also possible if one also has access to the matrix elements of the adjoint. In this case the choice of bases is not arbitrary. We would like to emphasize that even though determining spectra and pseudospectra is the mathematical goal, another set that may be of practical interest is $\omega_\delta(\sigma(T))$ (the δ-neighborhood) for $T \in \mathcal{C}(\mathcal{H})$ and $\delta > 0$. The reason is that $\sigma(T)$ may contain parts that have Lebesgue measure zero, and therefore may be quite hard to detect. An easier alternative may then be $\omega_\delta(\sigma(T))$, although mathematically this set reveals less information about the operator.

Theorem 3.4. *Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for the Hilbert space $\mathcal{H}$ and let $\Delta = \mathcal{B}(\mathcal{H})$. Define, for $n \in \mathbb{Z}_+, \epsilon > 0$, the set valued functions*

$$\Xi_1, \Xi_2, \Xi_3 : \Delta \to \Omega, \quad \Xi_1(T) = \overline{\sigma_{n,\epsilon}(T)}, \quad \Xi_2(T) = \overline{\omega_\epsilon(\sigma(T))}, \quad \Xi_3(T) = \sigma(T).$$

Then

$$SC_{\text{ind}}(\Xi_1) \leq 2, \qquad SC_{\text{ind}}(\Xi_2) \leq 3, \qquad SC_{\text{ind}}(\Xi_3) \leq 3.$$

Theorem 3.5. *Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for the Hilbert space $\mathcal{H}$ and let*

$$\begin{aligned} \Delta = \{T \in \mathcal{C}(\mathcal{H}) : T = W + A,\, W \in WS(\mathcal{H}),\, A \in \mathcal{B}(\mathcal{H}) \\ \cap \{T \in \mathcal{C}(\mathcal{H}) : \|R(T,\cdot)^{2^n}\|^{1/2^n} \text{ is never constant for any } n\}. \end{aligned} \tag{3.2}$$

(Recall Definition 2.3). Define, for $n \in \mathbb{Z}_+, \epsilon > 0$, the set valued functions

$$\Xi_1, \Xi_2, \Xi_3 : \Delta \to \Omega, \quad \Xi_1(T) = \overline{\sigma_{n,\epsilon}(T)}, \quad \Xi_2(T) = \overline{\omega_\epsilon(\sigma(T))}, \quad \Xi_3(T) = \sigma(T).$$

Then

$$SC_{\text{ind}}(\Xi_1) \leq 3, \qquad SC_{\text{ind}}(\Xi_2) \leq 4, \qquad SC_{\text{ind}}(\Xi_3) \leq 4.$$

Remark 3.2 The assumption that $\|R(T,\cdot)^{2^n}\|^{1/2^n}$ is never constant for any n will be satisfied e.g. if $\mathbb{C} \setminus \sigma(T)$ is connected and the numerical range of T is contained in a sector of the complex plane.

Theorem 3.6. *Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for the Hilbert space $\mathcal{H}$, P_m be the projection onto $\operatorname{span}\{e_j\}_{j=1}^m$ and d be some positive integer. Let $\Delta \subset \mathcal{C}(\mathcal{H})$ have the following properties: For $T \in \Delta$ we have*

(i) $\bigcup_m P_m\mathcal{H} \subset \mathcal{D}(T)$, $\bigcup_m P_m\mathcal{H} \subset \mathcal{D}(T^*)$.

(ii) $\langle Te_{j+l}, e_j\rangle = \langle Te_j, e_{j+l}\rangle = 0$, *for* $l > d$.

(iii) $TP_m\xi \to T\xi$, $T^*P_m\eta \to T^*\eta$, *as* $m \to \infty$ *for* $\xi \in \mathcal{D}(T)$ *and* $\eta \in \mathcal{D}(T^*)$.

Let $\epsilon > 0$ and $n \in \mathbb{Z}_+$ and $\Xi_1, \Xi_2, \Xi_3 : \Delta \to \Omega$ be defined by

$$\Xi_1(T) = \overline{\sigma_{n,\epsilon}(T)}, \quad \Xi_2(T) = \overline{\omega_\epsilon(\sigma(T))}, \quad \Xi_3(T) = \sigma(T).$$

Then

$$SC_{\text{ind}}(\Xi_1) = 1, \qquad SC_{\text{ind}}(\Xi_2) \leq 2, \qquad SC_{\text{ind}}(\Xi_3) \leq 2.$$

The following corollary is immediate.

Corollary 3.7. *Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for the Hilbert space $\mathcal{H}$ and let d be a positive integer. Define*

$$\Delta = \{T \in \mathcal{B}(\mathcal{H}) : \langle Te_{j+l}, e_j\rangle = \langle Te_j, e_{j+l}\rangle = 0, \quad l > d\}.$$

Let $\epsilon > 0$ and $n \in \mathbb{Z}_+$ and $\Xi_1, \Xi_2, \Xi_3 : \Delta \to \Omega$ be defined by $\Xi_1(T) = \overline{\sigma_{n,\epsilon}(T)}$, $\Xi_2(T) = \overline{\omega_\epsilon(\sigma(T))}$ and $\Xi_3(T) = \sigma(T)$. Then

$$SC_{\mathrm{ind}}(\Xi_1) = 1, \qquad SC_{\mathrm{ind}}(\Xi_2) \leq 2, \qquad SC_{\mathrm{ind}}(\Xi_3) \leq 2.$$

Theorem 3.8. *Let $\{e_j\}_{j\in\mathbb{N}}$ and $\{\tilde{e}_j\}_{j\in\mathbb{N}}$ be bases for the Hilbert space $\mathcal{H}$ and let*

$$\begin{aligned}\tilde{\Delta} &= \{T \in \mathcal{C}(\mathcal{H} \oplus \mathcal{H}) : T = T_1 \oplus T_2, T_1, T_2 \in \mathcal{C}(\mathcal{H}), T_1^* = T_2\}\\ \Delta &= \{T \in \tilde{\Delta} : \mathrm{span}\{e_j\}_{j\in\mathbb{N}} \textit{ is a core for } T_1,\ \mathrm{span}\{\tilde{e}_j\} \textit{ is a core for } T_2\}.\end{aligned}$$

Let $\epsilon > 0$, $\Xi_1 : \Delta \to \Omega$ and $\Xi_2 : \Delta \to \Omega$ be defined by $\Xi_1(T) = \overline{\sigma_\epsilon(T_1)}$ and $\Xi_2(T) = \sigma(T_1)$. Then

$$SC_{\mathrm{ind}}(\Xi_1) \leq 2, \qquad SC_{\mathrm{ind}}(\Xi_2) \leq 3.$$

Corollary 3.9. *Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for the Hilbert space $\mathcal{H}$ and let*

$$\Delta = \{A \in \mathcal{SA}(\mathcal{H}) : \mathrm{span}\{e_j\}_{j\in\mathbb{N}} \textit{ is a core for } A\}.$$

Let $\epsilon > 0$ and $\Xi_1, \Xi_2 : \Delta \to \Omega$ be defined by $\Xi_1(T) = \sigma(T)$ and $\Xi_2(T) = \overline{\omega_\epsilon(\sigma(T))}$. Then

$$SC_{\mathrm{ind}}(\Xi_1) \leq 3, \qquad SC_{\mathrm{ind}}(\Xi_2) \leq 2.$$

Remark 3.3 What Theorem 3.8 essentially says is that given the matrix elements of the operator and its adjoint, where the matrix elements come from a reasonable choice of bases, one can compute the pseudospectra and the spectrum. Also, computing the pseudospectrum of an unbounded operator is on the same level of difficulty as computing the spectrum of a compact operator.

Remark 3.4 Note that all of our proofs regarding estimates on the Solvability Complexity Index are constructive and thus the display of a set of estimating functions yields actual algorithms for use in computations. Although the implementation of algorithms is not the focus of this paper, we demonstrate the feasibility of our approach with examples in Section 10.

4 Properties of the n-pseudospectra of Bounded Operators

We will prove some of the properties of the n-pseudospectrum, but before doing that we need a couple of propositions and theorems that will come in handy.

Proposition 4.1. *Let $\gamma : \mathbb{C} \to [0,\infty)$ be continuous and let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a sequence of functions such that $\gamma_k : \mathbb{C} \to [0,\infty)$ and $\gamma_k \to \gamma$ locally uniformly. Suppose that one of the two following properties are satisfied.*

(i) $\gamma_k \to \gamma$ monotonically from above.

(ii) For $\epsilon > 0$, then $\mathrm{cl}(\{z : \gamma(z) < \epsilon\}) = \{z : \gamma(z) \leq \epsilon\}$.

Then for any compact ball K such that $\{z : \gamma(z) < \epsilon\} \cap K^o \neq \emptyset$ it follows that

$$\mathrm{cl}(\{z : \gamma_k(z) < \epsilon\}) \cap K \longrightarrow \mathrm{cl}(\{z : \gamma(z) < \epsilon\}) \cap K, \qquad k \to \infty.$$

Proof. Let $\epsilon > 0$. We first claim that, in each case, for any $\nu > 0$ there exists an $\alpha > 0$ such that

$$\omega_\nu(\mathrm{cl}(\{z : \gamma(z) < \epsilon - \alpha\}) \cap K) \supset \mathrm{cl}(\{z : \gamma(z) < \epsilon\}) \cap K. \tag{4.1}$$

Arguing by contradiction and supposing the latter statement is false we deduce that there must be a sequence $\{\zeta_\alpha\} \subset \mathrm{cl}(\{z : \gamma(z) < \epsilon\}) \cap K$ such that $\zeta_\alpha \notin \omega_\nu(\mathrm{cl}(\{z : \gamma(z) < \epsilon - \alpha\}) \cap K)$. By compactness, we may assume without loss of generality that $\zeta_\alpha \to \zeta$ as $\alpha \to 0$. By continuity we have that $\gamma(\zeta_\alpha) \to \gamma(\zeta)$ and since $\zeta_\alpha \notin \omega_\nu(\mathrm{cl}(\{z : \gamma(z) < \epsilon - \alpha\}) \cap K)$ it follows that $\gamma(\zeta) = \epsilon$. Note that we must have

$$\zeta \in \bigcap_{\alpha > 0} \mathbb{C} \setminus \omega_\nu(\mathrm{cl}(\{z : \gamma(z) < \epsilon - \alpha\}) \cap K). \tag{4.2}$$

But there is a $\xi \in \{z : \gamma(z) < \epsilon\} \cap K$ such that $|\xi - \zeta| < \nu$. Now let $\alpha_1 = \gamma(\zeta) - \gamma(\xi)$. Then $\gamma(\xi) = \epsilon - \alpha_1$ and hence $\zeta \in \omega_\nu(\{z : \gamma(z) < \epsilon - \alpha_2\})$, for some $\alpha_2 < \alpha_1$ contradicting (4.2). We are now ready to prove the proposition, which will follow if we can show that for any $\nu > 0$ we have

$$\mathrm{cl}(\{z : \gamma(z) < \epsilon\}) \cap K \subset \omega_\nu(\mathrm{cl}(\{z : \gamma_k(z) < \epsilon\}) \cap K)$$

and $\omega_\nu(\mathrm{cl}(\{z : \gamma(z) < \epsilon\}) \cap K) \supset \mathrm{cl}(\{z : \gamma_k(z) < \epsilon\}) \cap K$, for all sufficiently large k.

Note that the first inclusion follows by using the claim in the first part of the proof and the locally uniform convergence of γ_k. Indeed, by the locally uniform convergence it follows that, for any $\alpha > 0$, we have

$$\mathrm{cl}(\{z : \gamma_k(z) < \epsilon\}) \cap K \supset \mathrm{cl}(\{z : \gamma(z) < \epsilon - \alpha\}) \cap K$$

for large k, thus by appealing to (4.1), we obtain the desired inclusion. To see the second inclusion, we first assume (i). Then $\{z : \gamma_k(z) < \epsilon\} \subset \{z : \gamma(z) < \epsilon\}$ and hence the inclusion follows. As for the second case we assume (ii). By arguing by contradiction, we suppose the statement is false and deduce that there is a sequence $\{z_k\}$ such that

$$z_k \in \mathrm{cl}(\{z : \gamma_k(z) < \epsilon\}) \cap K$$

and $z_k \notin \omega_\nu(\mathrm{cl}(\{z : \gamma(z) < \epsilon\}) \cap K)$. By compactness we may assume that $z_k \to z$ and then (by (ii)) $\gamma(z) > \epsilon$ which contradicts the fact that $\gamma_k(z_k) \to \gamma(z)$ which follows by continuity of γ and the local uniform convergence of $\{\gamma_k\}$. □

Theorem 4.2 (Shargorodsky [43]). *Let Ω be an open subset of $\mathbb{C}$, X be a Banach space and Y be a uniformly convex Banach space. Suppose $A : \Omega \to \mathcal{B}(X, Y)$ is an analytic operator valued function such that $A'(z)$ is invertible for all $z \in \Omega$. If $\|A(z)\| \leq M$ for all $z \in \Omega$ then $\|A(z)\| < M$ for all $z \in \Omega$.*

Before we continue let us define some functions that will be crucial throughout the paper.

Definition 4.3. *Let $\{P_m\}$ be an increasing sequence of projections converging strongly to the identity. Define, for $n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, the function $\Phi_{n,m} : \mathcal{B}(\mathcal{H}) \times \mathbb{C} \to \mathbb{R}$ by*

$$\Phi_{n,m}(S, z) = \min\left\{\lambda^{1/2^{n+1}} : \lambda \in \sigma\left(P_m((S - z)^*)^{2^n}(S - z)^{2^n}\lceil_{P_m\mathcal{H}}\right)\right\}.$$

Define also

$$\Phi_n(S, z) = \lim_{m \to \infty} \Phi_{n,m}(S, z),$$

and for $T \in \mathcal{B}(\mathcal{H})$

$$\gamma_n(z) = \min[\Phi_n(T, z), \Phi_n(T^*, \bar{z})]. \tag{4.3}$$

Theorem 4.4. *Let $T \in \mathcal{B}(\mathcal{H})$, γ_n be defined as in (4.3) and $\epsilon > 0$. Then the following is true:*

(i) $\sigma_{n+1,\epsilon}(T) \subset \sigma_{n,\epsilon}(T)$.

(ii) $\sigma_{n,\epsilon}(T) = \{z \in \mathbb{C} : \gamma_n(z) < \epsilon\}$.

(iii) $\mathrm{cl}(\{z : \gamma_n(z) < \epsilon\}) = \{z : \gamma_n(z) \leq \epsilon\}$.

(iv) Let $\omega_\epsilon(\sigma(T))$ denote the ϵ-neighborhood around $\sigma(T)$. Then

$$d_H\left(\overline{\sigma_{n,\epsilon}(T)}, \overline{\omega_\epsilon(\sigma(T))}\right) \longrightarrow 0, \qquad n \to \infty.$$

(v) If $\{T_k\} \subset \mathcal{B}(\mathcal{H})$ and $T_k \to T$ in norm, it follows that

$$d_H\left(\overline{\sigma_{n,\epsilon}(T_k)}, \overline{\sigma_{n,\epsilon}(T)}\right) \longrightarrow 0, \qquad k \to \infty.$$

Proof. Now (i) follows by the definition of $\sigma_{n,\epsilon}(T)$ and the fact that

$$\begin{aligned} 1/\|R(z,T)^{2^{n+1}}\|^{1/2^{n+1}} &\geq 1/\left(\|R(z,T)^{2^n}\|^{1/2^{n+1}}\|R(z,T)^{2^n}\|^{1/2^{n+1}}\right) \\ &= 1/\|R(z,T)^{2^n}\|^{1/2^n}. \end{aligned} \tag{4.4}$$

To prove (ii) we have to show that $\gamma_n(z) = 1/\|R(z,T)^{2^n}\|^{1/2^n}$ when $z \notin \sigma(T)$ and that $\gamma_n(z) = 0$ when $z \in \sigma(T)$. The former is clear, so to see the latter we need to show that when $z \in \sigma(T)$ then either $|(T-z)^{2^n}|$ or $|((T-z)^{2^n})^*|$ is not invertible. To see that, we need to consider three cases: (1) $(T-z)^{2^n}$ is not one to one, (2) $(T-z)^{2^n}$ is not onto, but the range of $(T-z)^{2^n}$ is dense in $\mathcal{H}$ or (3) $(T-z)^{2^n}$ is not onto and $\overline{\mathrm{ran}((T-z)^{2^n}} \neq \mathcal{H}$.

Case (1): Now, by the polar decomposition, we have $(T-z)^{2^n} = U|(T-z)^{2^n}|$ where U is a partial isometry, and it is easy to see that $|(T-z)^{2^n}|$ is not invertible when $(T-z)^{2^n}$ is not one to one.

Case (2): Recall that U is unitary if and only if $((T-z)^{2^n})^*$ is one to one. Thus, since $\overline{\mathrm{ran}((T-z)^{2^n})} = \mathcal{H}$ and $\ker(((T-z)^{2^n})^*) = \mathrm{ran}((T-z)^{2^n})^{\perp}$, we have that U must be unitary. But that implies that $|(T-z)^{2^n}|$ cannot be invertible since $(T-z)^{2^n}$ is not invertible.

Case (3): If $\overline{\mathrm{ran}((T-z)^{2^n}} \neq \mathcal{H}$ it follows that $\ker(((T-z)^{2^n})^*)$ is nonzero, and since

$$((T-z)^{2^n})^* = U^*\left|((T-z)^{2^n})^*\right|$$

we may argue as in Case (1) to deduce that $|((T-z)^{2^n})^*|$ is not invertible and this proves the claim.

To see (iii) we argue by contradiction and assume that $\mathrm{cl}(\{z : \gamma_n(z) < \epsilon\}) = \{z : \gamma_n(z) \leq \epsilon\}$. is false. Then there exists a $\tilde{z} \in \sigma(T)^c$ such that $\gamma_n(\tilde{z}) = \epsilon$ and also a neighborhood θ around $\tilde{z}$ such that $\gamma_n(z) \geq \epsilon$ for $z \in \theta$. Now, for $z \in \theta$, it follows that $1/\gamma_n(z) = \|R(z,T)^{2^n}\|^{1/2^n}$. Thus, $\|R(\tilde{z},T)^{2^n}\| = 1/\epsilon^{2^n}$ and $\|R(\tilde{z},T)^{2^n}\| \leq 1/\epsilon^{2^n}$ for $z \in \theta$. But $z \mapsto R(z,T)^{2^n}$ is obviously holomorphic and $\frac{d}{dz}R(z,T)^{2^n}$ is easily seen to be invertible for all $z \in \theta$. Thus, by Theorem 4.2, it follows that $\|R(\tilde{z},T)^{2^n}\| < 1/\epsilon^{2^n}$ for all $z \in \theta$, contradicting $\|R(\tilde{z},T)^{2^n}\| = 1/\epsilon^{2^n}$.

It is easy to see that to prove (iv) it suffices to show that $\gamma_n \to \gamma$ locally uniformly, where

$$\gamma(z) = \mathrm{dist}(z, \sigma(T)).$$

To see the latter, let $\delta > 0$ and let ω_δ denote the open δ-neighborhood around $\sigma(T)$. Let also Ω be a compact set such $\sigma(T) \subset \Omega^o$ and $\Omega_\delta = \Omega \setminus \omega_\delta$. Note that for $z \in \Omega \setminus \sigma(T)$ we have

$$\gamma(z) = 1/\rho(R(z,T)),$$

where $\rho(R(z,T))$ denotes the spectral radius of $R(z,T)$, and also by (4.4) it follows that $\gamma_{n+1}(z) \geq \gamma_n(z)$. Thus, by the continuity of γ and γ_n together with the spectral radius formula we may appeal to Dini's Theorem to deduce that $\gamma_n \to \gamma$ locally uniformly on Ω_δ. By choosing n large enough we can guarantee that $|\gamma_n(z) - \gamma(z)| \leq \delta$ when $z \in \Omega_\delta$. Also, since $\gamma_n(z) \leq \gamma(z)$ for $z \in \Omega \setminus \sigma(T)$ and $\gamma(z) = \mathrm{dist}(z, \sigma(T)) \leq \delta$ for $z \in \omega_\delta$ we have that $|\gamma_n(z) - \mathrm{dist}(z, \sigma(T))| \leq \delta$ when $z \in \Omega \setminus \sigma(T)$ Since, by (ii), it is true that $\gamma_n(z) = \mathrm{dist}(z, \sigma(T)) = \gamma(z) = 0$ when $z \in \sigma(T)$ we are done with (iv).

To see that (v) is true let $\gamma_{n,k}(z) = \min[\Phi_n(T_k, z), \Phi_n(T_k^*, \bar{z})]$. Then, by (ii), $\sigma_{n,\epsilon}(T_k) = \{z \in \mathbb{C} : \gamma_{n,k}(z) < \epsilon\}$. Also, since T is bounded and $T_k \to T$ in norm, there is a compact set $K \subset \mathbb{C}$ containing both $\sigma_{n,\epsilon}(T)$ and $\sigma_{n,\epsilon}(T_k)$. Thus, by appealing to (iii) and Proposition 4.1 we conclude that to prove (v) we only need to show that $\gamma_{n,k} \to \gamma_n$ locally uniformly as $k \to \infty$. It suffices to show that $\gamma_{n,k}^{2^{n+1}} \to \gamma_n^{2^{n+1}}$ locally uniformly. Now

$$\begin{aligned} &|\Phi_n(T_k, z)^{2^{n+1}} - \Phi_n(T, z)^{2^{n+1}}| \\ &\qquad \leq d_H\left(\sigma\left(((T_k - z)^*)^{2^n}(T_k - z)^{2^n}\right), \sigma\left(((T - z)^*)^{2^n}(T - z)^{2^n}\right)\right) \\ &\qquad \leq \left\|((T_k - z)^*)^{2^n}(T_k - z)^{2^n} - ((T - z)^*)^{2^n}(T - z)^{2^n}\right\| \longrightarrow 0, \end{aligned} \tag{4.5}$$

locally uniformly as $k \to \infty$. Similar estimate holds for

$$\left|\Phi_n(T_k^*, \bar{z})^{2^{n+1}} - \Phi_n(T^*, \bar{z})^{2^{n+1}}\right|,$$

and this yields the assertion. $\square$

Remark 4.1 The advantage of the (n,ϵ)-pseudospectrum is that in addition to the continuity property stated above, we now have two parameters n and ϵ to tweak in order to estimate the spectrum. It is quite easy to construct examples (even 2-by-2 matrices) of operators $\{T_n\}$ for which $\sigma_{1,\epsilon}(T_n) \subset \sigma_{\epsilon/10^n}(T_n)$. And of course, in the self-adjoint case it would not make sense to take $n > 0$ as $\sigma_{n,\epsilon}(A) = \sigma_\epsilon(A)$ for self-adjoint A.

5 Properties of the n-pseudospectra of Unbounded Operators

The theory of n-pseudospectra for unbounded operators has a lot in common with the theory of n-pseudospectra for bounded operators, however, there is a major difference; the n-pseudospectrum of an unbounded operator can "jump". We will be more specific about this below.

Theorem 5.1. *Let $T \in \mathcal{C}(\mathcal{H})$, $n \in \mathbb{Z}_+$, $\epsilon > 0$ and let $K \subset \mathbb{C}$ be a compact ball such that $\sigma_\epsilon(T) \cap K^o \neq \emptyset$. Then the following is true*

(i) $\sigma_{n+1,\epsilon}(T) \subset \sigma_{n,\epsilon}(T)$.

(ii) Let $\omega_\epsilon(\sigma(T))$ denote the ϵ neighborhood around $\sigma(T)$. Then

$$d_H\left(\overline{\sigma_{n,\epsilon}(T)} \cap K, \overline{\omega_\epsilon(\sigma(T))} \cap K\right) \longrightarrow 0, \quad n \to \infty.$$

Proof. Follows by almost identical arguments as in the proof of Theorem 4.4. □

The difference between the bounded and the unbounded case is that if $T \in \mathcal{C}(\mathcal{H})$, $z \in \mathbb{C}$ and we define

$$\gamma_n(z) = \begin{cases} 0 & z \in \sigma(T) \\ \frac{1}{\|R(z,T)^{2^n}\|^{1/2^n}} & z \in \sigma(T)^c, \end{cases} \tag{5.1}$$

then we might have that $\mathrm{cl}(\{z : \gamma_n(z) < \epsilon\}) \neq \{z : \gamma_n(z) \leq \epsilon\}$. The reason is that there exists unbounded operators for where the norm of the resolvent is constant on an open set in $\mathbb{C}$. This was proven by Shargorodsky in [43]. However, we have the following.

Theorem 5.2. *Let $T \in \mathcal{C}(\mathcal{H})$ and let γ_n be defined as in (5.1). Suppose that $\|R(\cdot,T)^{2^n}\|$ can never be constant on an open set, then $\mathrm{cl}(\{z : \gamma_n(z) < \epsilon\}) = \{z : \gamma_n(z) \leq \epsilon\}$.*

Proof. Follows by arguing similar to the argument in the proof of Theorem 4.4 (iii). □

Theorem 5.3. *Let $T \in \mathcal{C}(\mathcal{H})$ with domain $\mathcal{D}(T)$ and let $\{T_k\} \subset \mathcal{C}(\mathcal{H})$ be a sequence such that $T_k \xrightarrow{\hat{\delta}} T$. Define, for $z \in \mathbb{C}$*

$$\zeta(z) = \begin{cases} 0 & z \in \sigma(T) \\ \frac{1}{\|R(z,T)\|} & z \in \sigma(T)^c, \end{cases} \quad \zeta_k(z) = \begin{cases} 0 & z \in \sigma(T_k) \\ \frac{1}{\|R(z,T_k)\|} & z \in \sigma(T_k)^c, \end{cases} \quad k \in \mathbb{N}.$$

(i) If $z \in K$, where K is a compact ball, it follows that there is a $C_K > 0$ depending on K such that

$$|\zeta(z)^2 - \zeta_k(z)^2| \leq C_K(1 + |z|^2)\hat{\delta}(T_k, T)$$

for sufficiently large k.

(ii) Suppose that $\|R(\cdot,T)^{2^n}\|$ can never be constant on an open set. Then if $K \subset \mathbb{C}$ is a compact ball such that $K^o \cap \sigma_{n,\epsilon}(T) \neq \emptyset$, then

$$d_H\left(\overline{\sigma_{n,\epsilon}(T_k)} \cap K, \overline{\sigma_{n,\epsilon}(T)} \cap K\right) \longrightarrow 0, \quad k \to \infty, \quad \epsilon > 0.$$

Proof. To show (i) we first claim that

$$\begin{aligned} \zeta(z) = \min\Big[&\inf\{\sqrt{\lambda} : \lambda \in \sigma((T-z)^*(T-z))\}, \\ &\inf\{\sqrt{\lambda} : \lambda \in \sigma((T-z)(T-z)^*)\}\Big] \\ \zeta_k(z) = \min\Big[&\inf\{\sqrt{\lambda} : \lambda \in \sigma((T_k-z)^*(T_k-z))\}, \\ &\inf\{\sqrt{\lambda} : \lambda \in \sigma((T_k-z)(T_k-z)^*)\}\Big]. \end{aligned} \tag{5.2}$$

We will show this for ζ, and the argument is identical for ζ_k. Indeed, for $z \notin \sigma(T)$ this is quite straightforward and hence we are left to show that either $|T-z|$ or $|(T-z)^*|$ is not invertible for $z \in \sigma(T)$. This is essentially the same argument as in Theorem 4.4, but we include it for completeness and to make sure that the same conclusions can be drawn using the polar decomposition of unbounded operators. We need to consider three cases. (1), $(T-z)$ is not one to one, (2), $(T-z)$ is not onto, but the range of $(T-z)$ is dense in $\mathcal{H}$ or (3), $(T-z)$ is not onto and $\overline{\mathrm{ran}((T-z))} \neq \mathcal{H}$.

Case (1): Now, by the polar decomposition, we have $(T-z) = U|(T-z)|$ where U is a partial isometry. Note that $\ker(T-z) = \ker(|T-z|)$ and $|T-z|$ is not invertible.

Case (2): Note that $(T-z)^*$ is one to one if and only if U is unitary and so U must be unitary since $\overline{\mathrm{ran}((T-z)} = \mathcal{H}$ and $\ker((T-z)^*) = \mathrm{ran}(T-z)^{\perp}$. But that implies that $|(T-z)|$ cannot be invertible since $(T-z)$ is not invertible.

Case (3): If $\overline{\mathrm{ran}((T-z)} \neq \mathcal{H}$ it follows that $\ker((T-z)^*)$ is nonzero, and since $(T-z)^* = U^*|(T-z)^*|$ we may argue as in Case (1) to deduce that $|(T-z)^*|$ is not invertible, and thus we have shown (5.2).

Note that by the spectral mapping theorem we have that

$$\sigma((T-z)^*(T-z)) = \psi(\sigma(R_{(T-z)}), \quad \sigma((T-z)(T-z)^*) = \psi(\sigma(R_{(T-z)^*}))$$

where $\psi(x) = 1/x - 1$ (recall that $R_{(T-z)}$ is short for $(1 + (T-z)^*(T-z))^{-1}$). Now let $\zeta^2(z) = \zeta(z)^2$ and $\zeta_k^2(z) = \zeta_k(z)^2$. Then it follows that

$$\begin{aligned}\zeta^2(z) &= \min\left(\inf\{\psi(\lambda) : \lambda \in \sigma(R_{(T-z)})\}, \inf\{\psi(\lambda) : \lambda \in \sigma(R_{(T-z)^*})\}\right) \\ &= \min\left(\psi(\|R_{(T-z)}\|), \psi(\|R_{(T-z)^*}\|)\right),\end{aligned}$$

by self-adjointness of $(T-z)^*(T-z)$ and $(T-z)(T-z)^*$. Similarly,

$$\zeta_k^2(z) = \min\left(\{\psi(\|R_{(T_k-z)}\|), \psi(\|R_{(T_k-z)^*}\|)\right).$$

Recall from the definition of p and Theorem 2.2 that for $z \in \mathbb{C}$ we have

$$\begin{aligned}\|R_{T_k-z} - R_{T-z}\|^2 + \|R_{(T_k-z)^*} - R_{(T-z)^*}\|^2 &\leq p(T_k - z, T-z)^2 \\ &\leq 8\hat{\delta}(T_k - z, T-z)^2 \\ &\leq 24(1+|z|^2)^2\hat{\delta}(T_k, T)^2.\end{aligned} \tag{5.3}$$

Also, since K is compact, there is a $\delta > 0$ such that

$$0 \notin \Omega = \omega_\delta(\{\psi^{-1} \circ \zeta^2(z) : z \in K\}),$$

where $\omega_\delta(\{\psi^{-1} \circ \zeta^2(z) : z \in K\})$ denotes the δ-neighborhood around $\{\psi^{-1} \circ \zeta^2(z) : z \in K\}$, and by (5.3) it follows that

$$\{\psi^{-1} \circ \zeta_k^2(z) : z \in K\} \subset \omega_\delta(\{\psi^{-1} \circ \zeta^2(z) : z \in K\})$$

for sufficiently large k. Let C be the Lipschitz constant of $\psi\lceil_{\Omega}$. Then if $z \in \mathbb{C} \setminus \sigma(T)$ we have that $\psi(\|R_{(T-z)}\|) = \psi(\|R_{(T-z)^*}\|)$ so by (5.3)

$$|\zeta(z)^2 - \tilde{\zeta}_k^2(z)| \leq C\sqrt{24}(1+|z|^2)\hat{\delta}(T_k, T). \tag{5.4}$$

If $z \in \sigma(T)$ then at least one of $\|R_{(T-z)}\|$ and $\|R_{(T-z)^*}\|$ is equal to one. Now, suppose that $\|R_{(T-z)}\| = 1$. If $\zeta_k^2(z) = \psi(\|R_{(T_k-z)}\|)$ then (5.4) follows, so suppose that $\zeta_k^2(z) = \psi(\|R_{(T_k-z)^*}\|)$ then $\|R_{(T_k-z)^*}\| > \|R_{(T_k-z)}\|$ so

$$|\zeta(z)^2 - \zeta_k^2(z)| \leq C(1 - \|R_{(T_k-z)^*}\|) \leq C(\|R_{(T-z)}\| - \|R_{(T_k-z)}\|),$$

and hence (5.4) follows by (5.3). Similar reasoning gives the same result for

$$\zeta_k^2(z) = \psi(\|R_{(T_k-z)^*}\|)$$

and $\|R_{(T-z)^*}\| = 1$ and we deduce that (5.4) holds for all $z \in K$.

To show that

$$d_H\left(\overline{\sigma_{n,\epsilon}(T_k)} \cap K, \overline{\sigma_{n,\epsilon}(T)} \cap K\right) \longrightarrow 0, \quad k \to \infty, \quad \epsilon > 0$$

in order to deduce (ii), we will deviate substantially from the techniques used in the proof of Theorem 4.4 (v). Before getting to the argument note that, since for any $z_0 \in \mathcal{C}$ we have

$$\sigma_{n,\epsilon}(T + z_0) = \{z + z_0 : z \in \sigma_{n,\epsilon}(T)\},$$

we may assume that T is invertible. For $m \in \mathbb{N}$ consider the operator T^m defined inductively on

$$\mathcal{D}(T^m) = \{\xi : \xi \in \mathcal{D}(T^{m-1}), T^{m-1}\xi \in \mathcal{D}(T)\},$$

by $T^m\xi = T(T^{m-1}\xi)$. Then T^m is a closed operator [29]. Also, since T is invertible and T is densely defined, T^{-1} has dense range and so has T^{-m} which yields that T^m is densely defined. Note also that since $\mathcal{D}(T^m) \subset \mathcal{D}(T^{m-1})$ it follows that $p(T)$ is closed and densely defined for any polynomial p and $\mathcal{D}(p(T)) = \mathcal{D}(T^d)$ where d is the degree of the polynomial p. Thus for any $z \in \mathbb{C}$ we can define the adjoint $((T - z)^m)^*$. We can now continue with the argument. The reasoning above allows us to define

$$\begin{aligned}\gamma_{n,k}(z) = \min \Big[&\inf\{\lambda^{1/2^n} : \lambda \in \sigma(|(T_k - z)^{2^n}|)\}, \\ &\inf\{\lambda^{1/2^n} : \lambda \in \sigma(|((T_k - z)^*)^{2^n}|)\}\Big].\end{aligned}$$

Appealing to Proposition 4.1 and Theorem 5.2 (and recalling the assumption in (ii)), it suffices to show that $\gamma_{n,k} \to \gamma_n$ locally uniformly, where

$$\gamma_n(z) = \begin{cases} 0 & z \in \sigma(T) \\ \frac{1}{\|R(z,T)^{2^n}\|^{1/2^n}} & z \in \sigma(T)^c. \end{cases}$$

Claim I: We claim that $\gamma_{n,k} \to \gamma_n$ locally uniformly on $\sigma(T)$. To see that, note that for $z \in \sigma(T)$ then, by the spectral mapping theorem for polynomials of unbounded operators [29], $(T - z)^{2^n}$ is not invertible. Hence, by reasoning similar to what we did in the proof of (i), either

$$\begin{aligned} &\inf_{\|\xi\|=1,\, \xi\in\mathcal{D}(T^{2^n})} \|(T - z)^{2^n}\xi\|^{1/2^n} = 0, \\ \text{or} \quad &\inf_{\|\xi\|=1,\, \xi\in\mathcal{D}((T^{2^n})^*)} \|((T - z)^{2^n})^*\xi\|^{1/2^n} = 0, \end{aligned} \tag{5.5}$$

(or both are equal to zero). Suppose that the first part of (5.5) is true. Then, for $\delta > 0$, we can find for any $z_0 \in \sigma(T) \cap K$ a vector $\xi_{z_0} \in \mathcal{D}(T^{2^n})$ such that $\|(T - z_0)^{2^n}\xi_{z_0}\|^{1/2^n} \leq \delta/3$. Recall that, for any $m \in \mathbb{N}$ we have $\hat{\delta}(T_k^m, T^m) = \hat{\delta}(T_k^{-m}, T^{-m})$ and that $R(T_k^m) \to R(T^m)$ if and only if $\hat{\delta}(T_k^m, T^m) \to 0$, and since $R(T_k) \to R(T)$ so $R(T_k)^m \to R(T)^m$ we get that $\hat{\delta}(T_k^m, T^m) \to 0$. Hence, by the definition of $\hat{\delta}$, it follows that

$$\sup_{\substack{\xi\in\mathcal{D}(T^m)\\ \|\xi\|+\|T^m\xi\|=1}} \inf_{\eta\in\mathcal{D}(T_k^m)} \|\xi - \eta\| + \|T^m\xi - T_k^m\eta\| \longrightarrow 0, \quad k \to \infty.$$

Thus, there exists a sequence of unit vectors $\{\eta_{z_0,k}\}$ in $\mathcal{D}(T_k^m)$ such that $\eta_{z_0,k} \to \xi_{z_0}$ and $T_k^m\eta_{z_0,k} \to T^m\xi_{z_0}$ as $k \to \infty$. Now, since for any integer r we have $T_k^{-r} \to T^{-r}$ in norm, it follows that

$$T_k^l\eta_{z_0,k} = T_k^{-(m-l)}T_k^m\eta_{z_0,k} \longrightarrow T^{-(m-l)}T^m\xi_{z_0} = T^l\xi_{z_0}, \qquad k \to \infty.$$

for all $l \leq m$. In particular, it is true that $z \mapsto (T_k - z)^{2^n}\eta_{z_0,k} \to z \mapsto (T - z)^{2^n}\xi_{z_0}$ locally uniformly as $k \to \infty$. Note that $z \mapsto \|(T - z)^{2^n}\xi_{z_0}\|$ is continuous. Thus, there is a neighborhood Θ_{z_0} around z_0 such that $\|(T - z)^{2^n}\xi_{z_0}\| \leq \frac{2}{3}\delta$ for $z \in \Theta_{z_0}$ and hence $\|(T_k - z)^{2^n}\eta_{z_0,k}\| \leq \delta$ for $z \in \Theta_{z_0}$ and sufficiently large k. Covering $\sigma(T) \cap K$ with finitely many neighborhoods $\{\Theta_{z_j}\}_{j=1}^M$, of the type just described, for some $\{z_j\}_{j=1}^M \subset \sigma(T) \cap K$ and some $M \in \mathbb{N}$, we deduce that there are sequences $\{\eta_{z_j,k}\}$ and an integer k_0 such that

$$\max_{j\leq M} \sup_{z\in\Theta_{z_j}} \left\|(T_k - z)^{2^n}\eta_{z_j,k}\right\| \leq \delta^{2^n}, \qquad k \geq k_0.$$

And hence it follows that that for $z \in \bigcup_{z_j} \Theta_{z_j}$

$$\inf\{\lambda^{1/2^n} : \lambda \in \sigma(|(T_k - z)^{2^n}|)\} = (\inf_{\|\xi\|=1,\, \xi\in\mathcal{H}} \|(T_k - z)^{2^n}\xi\|)^{1/2^n} \leq \delta, \quad k \geq k_0.$$

Similar reasoning holds for the second part of (5.5) and hence we deduce that $\gamma_{n,k} \to \gamma_n$ locally uniformly on $\sigma(T)$.

Note that we have actually proved more than what we claimed, namely that if $\delta > 0$, $z_0 \in \partial\sigma(T)$ and ω is a neighborhood around z_0 such that $\gamma_n(z) \leq \delta/2$ for $z \in \omega$, then

$$\gamma_{n,k}(z) \leq \delta, \qquad z \in \omega, \qquad k \geq K, \tag{5.6}$$

for some K.

Claim II: We claim that $\gamma_{n,k} \to \gamma_n$ locally uniformly on $\mathbb{C} \setminus \sigma(T)$. Note that $z \mapsto R(z,T)$ is analytic on $\mathbb{C} \setminus \sigma(T)$ and also, since

$$T_k \xrightarrow{\hat{\delta}} T$$

and $\sigma(T)^c \neq \emptyset$, it follows that if $B_r(a)$ is an open disc with center $a \in \mathbb{C}$, radius r and $B_r(a) \subset \mathbb{C}\backslash\omega_\nu(\sigma(T))$ for some $\nu > 0$ (recall that $\omega_\nu(\Omega)$ denotes the ν-neighborhood around $\Omega \subset \mathbb{C}$), then $R(z,T_k)$ exist and is bounded on a neighborhood of $\overline{B_r(a)}$ for sufficiently large k and hence $z \mapsto R(z,T_k)$ is analytic there. Now,

$$R(z,T_k) \longrightarrow R(z,T), \qquad k \to \infty, \quad z \in B_r(a) \tag{5.7}$$

pointwise. Let $f_k(z) = R(z,T_k)$ then, by Cauchy's formula, we have for $z \in B_r(a)$

$$\begin{aligned} \|f_k(a) - f_k(z)\| &\leq \frac{1}{2\pi}\left\| \int_{\partial B_r(a)} \frac{f_k(\omega)(a-z)}{(\omega-a)(\omega-z)}\, d\omega \right\| \\ &\leq \frac{4M}{R}|a-z|, \end{aligned}$$

where M is the bound on f_k on $\overline{B_r(a)}$. Hence, $\{f_k\}$ is locally uniformly Lipschitz and therefore the convergence in (5.7) must be locally uniform. Using the reasoning above, the fact that we have $\gamma_{n,k}(z) = 1/\|R(z,T_k)^{2^n}\|^{1/2^n}$ for $z \in \mathbb{C} \setminus \omega_\nu(\sigma(T))$ and sufficiently large k, and the reasoning leading to (5.6), then Claim II easily follows.

By adding Claim I and Claim II we deduce (ii). $\square$

6 Proofs of the Main Theorems

We are now ready to prove the main theorems, but before we do that we need a couple of preliminary results.

Proposition 6.1. *$T \in \mathcal{B}(\mathcal{H})$ and $\{P_m\}$ is an increasing sequence of finite rank projections converging strongly to the identity. Let $\Phi_{n,m}$ be as in Definition 4.3. Define, for $k \in \mathbb{N}$, the functions $\gamma_{n,m}, \gamma_{n,m,k} : \mathbb{C} \to \mathbb{R}$ by*

$$\begin{aligned} \gamma_{n,m}(z) &= \min[\Phi_{n,m}(T,z), \Phi_{n,m}(T^*,\bar{z})], \\ \gamma_{n,m,k}(z) &= \min[\Phi_{n,m}(P_kTP_k,z), \Phi_{n,m}(P_kT^*P_k,\bar{z})], \end{aligned} \tag{6.1}$$

and let γ_n be defined as in (4.3). Then $\gamma_{n,m} \to \gamma_n$ as $m \to \infty$ and $\gamma_{n,m,k} \to \gamma_{n,m}$ as $k \to \infty$ locally uniformly. The convergence $\gamma_{n,m} \to \gamma_n$ is monotonically from above.

Proof. To see that $\gamma_{n,m} \to \gamma_n$ monotonically from above and locally uniformly as $m \to \infty$, define $\gamma_n^1(z) = \Phi_n(T,z)$, $\gamma_n^2(z) = \Phi_n(T^*,\bar{z})$, $\gamma_{n,m}^1(z) = \Phi_{n,m}(T,z)$ and $\gamma_{n,m}^2(z) = \Phi_{n,m}(T^*,\bar{z})$, where Φ_n and $\Phi_{n,m}$ are defined as in Definition 4.3. It follows, by the definition of $\gamma_{n,m}$, that to prove the claim it suffices to show that $\gamma_{n,m}^1 \to \gamma_n^1$ and $\gamma_{n,m}^2 \to \gamma_n^2$ monotonically from above and locally uniformly as $m \to \infty$. Now, $\gamma_{n,m}^j$ is obviously continuous as well as γ_n^j and also, since $P_{n+1} \geq P_n$ and $P_n \to I$, we have that $\gamma_{n,m+1}^j(z) \leq \gamma_{n,m}^j(z)$ and $\lim_{m\to\infty} \gamma_{n,m}^j(z) = \gamma_n^j(z)$ for $z \in \mathbb{C}$. Thus, by appealing to Dini's Theorem, we deduce that $\gamma_{n,m}^j \to \gamma_n^j$ locally uniformly.

To see that $\gamma_{n,m,k} \to \gamma_{n,m}$ as $k \to \infty$, locally uniformly we argue as follows. Using self-adjointness of

$$\begin{aligned} T_m(z) &= P_m((T-z)^*)^{2^n}(T-z)^{2^n}\lceil_{P_m\mathcal{H}} \\ T_{m,k}(z) &= P_m((P_k(T-z)P_k)^*)^{2^n}(P_k(T-z)P_k)^{2^n}\lceil_{P_m\mathcal{H}}, \\ \widetilde{T}_m(z) &= P_m(T-z)^{2^n}((T-z)^*)^{2^n}\lceil_{P_m\mathcal{H}} \\ \widetilde{T}_{m,k}(z) &= P_m(P_k(T-z)P_k)^{2^n}((P_k(T-z)P_k)^*)^{2^n}\lceil_{P_m\mathcal{H}} \end{aligned} \tag{6.2}$$

and the fact that for self-adjoint $A, B \in \mathcal{B}(\mathcal{H})$ we have $d_H(\sigma(A), \sigma(B)) \leq \|A - B\|$ it suffices to show that $T_{m,k}(z) \to T_m(z)$ and $\widetilde{T}_{m,k}(z) \to \widetilde{T}_m(z)$, as $k \to \infty$, uniformly for all z in a compact set. To see that we observe that

$$\underset{k\to\infty}{\text{SOT-lim}} P_k(T-z)P_k = T - z, \qquad \underset{k\to\infty}{\text{SOT-lim}} (P_k(T-z)P_k)^* = (T-z)^*,$$

so since multiplication is strongly continuous on bounded sets and the fact P_m has finite rank it follows that the strong convergence implies norm convergence and we deduce that $T_{m,k} \to T_m$ and $\widetilde{T}_{m,k} \to \widetilde{T}_m$ pointwise as $k \to \infty$.

A closer examination shows that the operator valued functions $z \mapsto T_{m,k}(z)$ and $z \mapsto \widetilde{T}_{m,k}(z)$ are Lipschitz continuous on compact sets with a uniformly bounded Lipschitz constant, thus the convergence asserted is locally uniform. □

The following two theorems will be essential in the developments below.

Theorem 6.2 (Treil [51]). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $H^\infty_{\mathcal{H}_1\to\mathcal{H}_2}$ denote the set of all bounded analytic function on the open unit disk $\mathbb{D}$ whose values are in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Let $F \in H^\infty_{\mathcal{H}_1\to\mathcal{H}_2}$ and suppose that there is a $\delta > 0$ such that $F^*(z)F(z) \geq \delta I$ for all $z \in \mathbb{D}$. If there is a constant operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that*

$$\sup_{z\in\mathbb{D}} \|A - F^*(z)F(z)\|_1 < \infty,$$

where $\|\cdot\|_1$ denotes the trace-norm, then there is a $G \in H^\infty_{\mathcal{H}_2\to\mathcal{H}_1}$ such that $G(z)F(z) = I$ for all $z \in \mathbb{D}$.

Theorem 6.3 (Shargorodsky [43]). *Let Ω_0 be a connected open subset of $\mathbb{C}$ and Z a Banach space. Suppose that $F : \Omega_0 \to Z$ is an analytic vector valued function, $\|F(z)\| \leq M$ for all z in an open subset $\Omega \subset \Omega_0$, and $\|F(z_0)\| < M$ for some $z_0 \in \Omega_0$. Then $\|F(z)\| < M$ for all $z \in \Omega$.*

We are now ready to prove the main theorems.

Proof. (**Proof of Theorem 3.4 and 3.5**) Note that if $T \in \Delta$ it follows that, for compact $K \subset \mathbb{C}$ intersecting $\overline{\sigma_{n,\epsilon}(T)}$ or $\sigma(T)$ we have

$$\sigma(T) \cap K = \lim_{\epsilon\to 0} \overline{\sigma_{n,\epsilon}(T)} \cap K, \qquad \overline{\omega_\epsilon(\sigma(T))} \cap K = \lim_{n\to\infty} \overline{\sigma_{n,\epsilon}(T)} \cap K,$$

(the first assertion is obvious and the second follows from Theorem 5.1) thus, it suffices to show, in both cases, the bound on $SC_{\text{ind}}(\Xi_1)$. We will first show that if $\Delta = \mathcal{B}(\mathcal{H})$ then $SC_{\text{ind}}(\Xi_1) \leq 2$, and then use this to show that if Δ is defined as in (3.2) then $SC_{\text{ind}}(\Xi_1) \leq 3$. Let P_n be the projection onto $\text{span}\{e_1, \ldots, e_n\}$ and $x_{ij} = \langle Te_j, e_i\rangle$ for $T \in \mathcal{B}(\mathcal{H})$. Also, define the set

$$\Theta_k = \{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z = r\delta, r \in \mathbb{Z}, |r| \leq k\}, \qquad \delta = \sqrt{\frac{1}{k}}, \tag{6.3}$$

and define the set of estimating functions Γ_{n_1,n_2} and Γ_{n_1} in the following way. Let

$$\begin{aligned}
\Gamma_{n_1,n_2}(\{x_{ij}\}) &= \{z \in \Theta_{n_2} : \nexists\, L \in LT_{\text{pos}}(P_{n_1}\mathcal{H}), T_{\epsilon,n_1,n_2}(z) = LL^*\} \\
&\quad \cup \{z \in \Theta_{n_2} : \nexists\, L \in LT_{\text{pos}}(P_{n_1}\mathcal{H}), \widetilde{T}_{\epsilon,n_1,n_2}(z) = LL^*\}, \\
\Gamma_{n_1}(\{x_{ij}\}) &= \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(T_{\epsilon,n_1}(z)) \neq \emptyset\} \\
&\quad \cup \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(\widetilde{T}_{\epsilon,n_1}(z)) \neq \emptyset\},
\end{aligned} \tag{6.4}$$

where $LT_{\text{pos}}(P_m\mathcal{H})$ denotes the set of lower triangular matrices in $\mathcal{B}(P_m\mathcal{H})$ (with respect to $\{e_j\}$) with strictly positive diagonal elements and

$$\begin{aligned}
T_{\epsilon,n_1,n_2}(z) &= T_{n_1,n_2}(z) - \epsilon^{2^{n+1}} I, \\
\widetilde{T}_{\epsilon,n_1,n_2}(z) &= \widetilde{T}_{n_1,n_2}(z) - \epsilon^{2^{n+1}} I, \\
T_{\epsilon,n_1}(z) &= T_{n_1}(z) - \epsilon^{2^{n+1}} I, \\
T_{\epsilon,n_1}(z) &= T_{n_1}(z) - \epsilon^{2^{n+1}} I,
\end{aligned} \tag{6.5}$$

where T_{n_1,n_2}, $\widetilde{T}_{n_1,n_2}$, T_{n_1} and $\widetilde{T}_{n_1}$ are defined as in (6.2). Note that, clearly, from the definition, Γ_{n_1,n_2} depends only on $\{x_{ij}\}_{i,j\leq n_2}$. We claim that $\Gamma_{n_1,n_2}(\{x_{ij}\})$ can be evaluated using only finitely many arithmetic operations and radicals of elements in $\{x_{ij}\}_{i,j\leq n_2}$. Indeed, $T_{\epsilon,n_1,n_2}(z)$ and $\widetilde{T}_{\epsilon,n_1,n_2}(z)$ are both in $\mathcal{B}(P_{n_1}\mathcal{H})$. Also, $a_{ij} = \langle T_{\epsilon,n_1,n_2}(z)e_j, e_i\rangle$ and $\tilde{a}_{ij} = \langle \widetilde{T}_{\epsilon,n_1,n_2}(z)e_j, e_i\rangle$, for $i, j \leq n_1$, are, by the definition of $T_{\epsilon,n_1,n_2}(z)$ and $\widetilde{T}_{\epsilon,n_1,n_2}(z)$, polynomials in $\{x_{ij}\}_{i,j\leq n_2}$. Since the existence of $L \in LT_{\mathrm{pos}}(P_{n_1})$ such that $T_{\epsilon,n_1,n_2}(z) = LL^*$ can be determined using finitely many arithmetic operations and radicals of $\{a_{ij}\}_{i,j\leq n_1}$ (this is known as the Cholesky decomposition), similar reasoning holds for $\widetilde{T}_{\epsilon,n_1,n_2}(z)$ and the fact that Θ_{n_2} is finite, the assertion follows.

Step I: We will show that for any compact ball $K \subset \mathbb{C}$ such that $\Gamma_{n_1,n_2}(\{x_{ij}\}) \cap K^o \neq \emptyset$, then

$$d_H(\Gamma_{n_1,n_2}(\{x_{ij}\}) \cap K, \Gamma_{n_1}(\{x_{ij}\}) \cap K) \longrightarrow 0, \quad n_2 \to \infty.$$

Note that since $d_H(\Theta_{n_2} \cap K, K) \to 0$, as $n_2 \to \infty$, and by the observations that for $n_2 \geq n_1$ we have

$$\begin{aligned}
&\{z \in \mathbb{C} : \nexists L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), T_{\epsilon,n_1,n_2}(z) = LL^*\} \\
&\qquad \cup \{z \in \mathbb{C} : \nexists L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), \widetilde{T}_{\epsilon,n_1,n_2}(z) = LL^*\} \\
&= \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(T_{\epsilon,n_1,n_2}(z)) \neq \emptyset\} \cup \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(\widetilde{T}_{\epsilon,n_1,n_2}(z)) \neq \emptyset\} \\
&= \{z \in \mathbb{C} : \gamma_{n,n_1,n_2}(z) \leq \epsilon\},
\end{aligned}$$

where γ_{n,n_1,n_2} is defined in (6.1), and

$$\begin{aligned}
\{z \in \mathbb{C} : \gamma_{n,n_1}(z) \leq \epsilon\} = \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(T_{\epsilon,n_1}(z)) \neq \emptyset\} \\
\cup \{z \in \mathbb{C} : (-\infty, 0] \cap \in \sigma(\widetilde{T}_{\epsilon,n_1}(z)) \neq \emptyset\}
\end{aligned} \tag{6.6}$$

where γ_{n,n_1} is defined in (6.1), the assertion will follow if we can demonstrate that

$$d_H(\{z \in \mathbb{C} : \gamma_{n,n_1,n_2}(z) \leq \epsilon\} \cap K, \{z \in \mathbb{C} : \gamma_{n,n_1}(z) \leq \epsilon\} \cap K) \longrightarrow 0, \tag{6.7}$$

as $n_2 \to \infty$. Now, by Proposition 6.1 it follows that $\gamma_{n,n_1,n_2} \to \gamma_{n,n_1}$ locally uniformly hence, by Proposition 4.1, (6.7) will follow if we can prove the following.

Claim: We claim that

$$\mathrm{cl}(\{z \in \mathbb{C} : \gamma_{n,n_1}(z) < \epsilon\}) = \{z \in \mathbb{C} : \gamma_{n,n_1}(z) \leq \epsilon\}. \tag{6.8}$$

Now, letting ζ_{1,n_1} and ζ_{2,n_1} be defined by $\zeta_{1,n_1}(z) = \Phi_{n,n_1}(T, z)$ and $\zeta_{2,n_1}(z) = \Phi_{n,n_1}(T^*, \bar{z})$, where Φ_{n,n_1} is defined as in Definition 4.3. Then

$$\gamma_{n,n_1} = \min[\zeta_{1,n_1}, \zeta_{2,n_1}].$$

Thus, (6.8) will follow if we can show that

$$\mathrm{cl}(\{z \in \mathbb{C} : \zeta_{j,n_1}(z) < \epsilon\}) = \{z \in \mathbb{C} : \zeta_{j,n_1}(z) \leq \epsilon\}, \qquad j = 1, 2. \tag{6.9}$$

We will demonstrate the latter, but before we do so we need to establish some facts about the set of points where ζ_{1,n_1} does not vanish. Let

$$\Omega = \{z \in \mathbb{C} : \zeta_{1,n_1}(z) \neq 0\},$$

then Ω is obviously open and we claim that $\mathbb{C} \setminus \Omega$ is finite. To see that we argue by contradiction and suppose that ζ_{1,n_1} vanishes at infinitely many points. If that was the case we would have

$$\inf_{\|\xi\|=1, \xi\in\mathcal{H}} \|(T - z)^{2^n} P_{n_1}\xi\| = 0, \tag{6.10}$$

for infinitely many zs. This is indeed impossible because, since P_{n_1} has finite rank, there is a finite dimensional subspace $\mathcal{H}_1 \subset \mathcal{H}$ such that $\mathrm{ran}(T-z)^{2^n-1}P_{n_1} \subset \mathcal{H}_1$ for all $z \in \mathbb{C}$. Thus , if E is the projection onto $\mathcal{H}_1$ then, by (6.10), $\inf_{\eta\in\mathcal{H}_1} \|(ETE - zE)\eta\| = 0$ for infinitely many zs. But the infimum in the equation above is actually attained since $\mathcal{H}_1$ is finite dimensional and hence the finite rank operator ETE must have infinitely many eigenvalues and this is impossible. Armed with this fact we return to the task of showing (6.9). To do this for $j = 1$ we argue by contradiction and suppose that there is a

$$z_0 \notin \mathrm{cl}(\{z \in \mathbb{C} : \zeta_{1,n_1}(z) < \epsilon\}) \tag{6.11}$$

such that $\zeta_{1,n_1}(z_0) = \epsilon$. This implies that there is a neighborhood θ around z_0 such that $\zeta_{1,n_1}(z) \geq \epsilon$ for $z \in \theta$. We will now demonstrate that this is impossible. First note that by the definition of ζ_{1,n_1} we can make $\zeta_{1,n_1}(z)$ arbitrary large for large $|z|$. In particular, we can find an open set $\tilde{\theta} \subset \Omega$ such that $\zeta_{1,n_1}(z) > \epsilon$ for $z \in \tilde{\theta}$. Now choose a simply connected open set $\Omega_0 \subset \Omega$ such that $\theta \cup \tilde{\theta} \subset \Omega_0$ and ζ_{1,n_1} does not vanish on $\mathrm{cl}(\Omega_0)$. Note that this is possible by the fact that $\mathbb{C} \setminus \Omega$ is finite. Now define, for $z \in \Omega_0$, the operator

$$F(z) : P_{n_1}\mathcal{H} \to \mathcal{H}, \qquad F(z) = (T - z)^{2^n} P_{n_1}.$$

Now, obviously F is holomorphic. Note that, by continuity of ζ_{1,n_1} and the choice of Ω_0, there is a $\delta > 0$ such that

$$\inf_{z \in \Omega_0} \zeta_{1,n_1}(z) \geq \delta.$$

By possibly composing F with a holomorphic function we may assume that $\Omega_0 = \mathbb{D}$, the open disk with radius one centered at the origin. Hence we get that $F \in H^{\infty}_{P_{n_1}\mathcal{H} \to \mathcal{H}}$ and $F^*(z)F(z) \geq \delta I$, for all $z \in \mathbb{D}$, where I is the identity on $P_{n_1}\mathcal{H}$. Obviously, since P_{n_1} is a finite rank projection, it follows that

$$\sup_{z \in \mathbb{D}} \|F^*(z)F(z)\|_1 < \infty,$$

where $\|\cdot\|_1$ denotes the trace norm. Thus, we may appeal to Theorem 6.2 and deduce that there is a $G \in H^{\infty}_{\mathcal{H} \to P_{n_1}\mathcal{H}}$ such that $G(z)F(z) = I$ for all $z \in \mathbb{D}$. Again, by possibly composing with another holomorphic function (and with a slight abuse of notation) we have a holomorphic function G on Ω_0 such that $G(z) : \mathcal{H} \to P_{n_1}\mathcal{H}$ and

$$1/\zeta_{1,n_1}(z) = 1/(\inf_{\xi \in P_{n_1}\mathcal{H}, \|\xi\|=1} \|F(z)\xi\|) = \|G(z)\|, \qquad z \in \Omega_0.$$

Then, by the reasoning above, it follows that $\|G(z)\| \leq 1/\epsilon$ for $z \in \theta$ and $\|G(z)\| < 1/\epsilon$ for $z \in \tilde{\theta}$. This implies, by Theorem 6.3, that $\|G(z)\| < 1/\epsilon$ for $z \in \theta$, but $\|G(z_0)\| = 1/\epsilon$ and $z_0 \in \theta$ (recall (6.11)) and we have finally reached the desired contradiction. By a similar argument one can show (6.9) for $j = 2$ and hence we are done with step I.

Step II: We will show that for any compact ball $K \subset \mathbb{C}$ such that $\overline{\sigma_{n,\epsilon}(T)} \cap K^o \neq \emptyset$, then

$$d_H(\Gamma_{n_1}(\{x_{ij}\}) \cap K, \overline{\sigma_{n,\epsilon}(T)} \cap K) \longrightarrow 0, \quad n_1 \to \infty.$$

But, by (6.6) and Theorem 4.4 (ii), this will follow if

$$d_H(\{z \in \mathbb{C} : \gamma_{n,n_1}(z) \leq \epsilon\} \cap K, \{z \in \mathbb{C} : \gamma_n(z) \leq \epsilon\} \cap K) \longrightarrow 0, \quad n_1 \to \infty,$$

where γ_n is defined in (4.3), and by Theorem 4.4 (iii) and Proposition 4.1 this is true if $\gamma_{n,n_1} \to \gamma_n$ locally uniformly, which in fact was established in Proposition 6.1. Now, adding Step I and Step II together we have shown that $SC_{\mathrm{ind}}(\Xi_1) \leq 2$ for $\Xi_1 : \Delta \to \Omega$ when $\Delta = \mathcal{B}(\mathcal{H})$, and we will now use this to establish the assertion of the theorem.

Step III: We will now show that if Δ is defined as in (3.2) then $SC_{\mathrm{ind}}(\Xi_1) \leq 3$. Suppose that we have $T = W + A$, where W is a weighted shift and A is bounded. Letting $x_{ij} = \langle Te_j, e_i \rangle$ we will define the set of estimating functions $\Gamma_{n_1,\ldots,n_3}, \ldots, \Gamma_{n_1}$ in the following way. Now, for $\xi \in \mathcal{H}$ we may without loss of generality assume that $(W\xi)_j = x_{j,j+k}\xi_j$ for some integer k. Define a new set $\{\tilde{x}_{ij}(n)\}$, depending on an integer n, in the following way: $\tilde{x}_{j,j+k}(n) = n$ if $|x_{j,j+k}| > n$ and $\tilde{x}_{ij}(n) = x_{ij}$ elsewhere. Note that $\{\tilde{x}_{ij}(n)\}$ gives rise to a bounded operator S_n whose matrix elements are $\{\tilde{x}_{ij}(n)\}$. Thus we can define

$$\Gamma_{n_1,\ldots,n_3}(\{x_{ij}\}) = \Gamma_{n_2,n_3}(\{\tilde{x}_{ij}(n_1)\}),$$

where Γ_{n_2,n_3} is defined as in (6.4). If we let $\Gamma_{n_1}(\{x_{ij}\}) = \Xi_1(S_{n_1})$, and since we have shown above that Γ_{n_2,n_3} and Γ_{n_2} is a set of estimating functions for $\Xi_1 : \mathcal{B}(\mathcal{H}) \to \Omega$, it follows that $\Gamma_{n_1,\ldots,n_3}, \ldots, \Gamma_{n_1}$ is a set of estimating functions for Ξ_1 if we can show that

$$\lim_{n_1 \to \infty} \Xi_1(S_{n_1}) = \Xi_1(T).$$

Note that, by Theorem 5.3 (and assumption), the latter will follow if we can show that

$$S_n \xrightarrow{\hat{\delta}} T, \qquad n \to \infty. \tag{6.12}$$

Define the operator W_n by $(W_n\xi)_j = \tilde{x}_{j,j+k}(n)\xi_j$ for $\xi \in \mathcal{H}$. Then $S_n = W_n + A$. Thus, by Theorem 2.2, (6.12) will follow if we can show that $\delta(W_n, W) \to 0$ and $\delta(W, W_n) \to 0$ as $n \to \infty$. To show the former we need to demonstrate that

$$\sup_{\varphi \in G(W_n), \|\varphi\| \leq 1} \inf_{\psi \in G(W)} \|\varphi - \psi\| \longrightarrow 0, \qquad n \to \infty,$$

where $G(W)$ denotes the graph of W as defined in (2.1). Let $\varphi \in G(W_n)$ such that $\|\varphi\| \leq 1$. Then there is a $\xi \in \mathcal{H}$ such that $\varphi = (\xi, W_n\xi)$ and $\|W_n\xi\| + \|\xi\| \leq 1$. Now, choose $\eta \in \mathcal{D}(W)$ in the following way:

$$\eta_j = \begin{cases} \xi_j & \text{if} \quad \tilde{x}_{j,j+k}(n) = x_{j,j+k} \\ \frac{\tilde{x}_{j,j+k}(n)}{x_{j,j+k}}\xi_j & \text{if} \quad \tilde{x}_{j,j+k}(n) \neq x_{j,j+k}. \end{cases}$$

Let also $\Theta = \{j \in \mathbb{N} : \eta_j = \xi_j\}$ and $\theta = \{j \in \mathbb{N} : \eta_j \neq \xi_j\}$. Then,

$$\begin{aligned} \|\xi - \eta\| + &\|W_n\xi - W\eta\| \\ &= \sum_{j\in\Theta} |\xi_j - \eta_j|^2 + \sum_{j\in\theta} |\xi_j - \eta_j|^2 \\ &\quad + \sum_{j\in\Theta} |\tilde{x}_{j,j+k}(n)\xi_j - x_{j,j+k}\eta_j|^2 + \sum_{j\in\theta} |\tilde{x}_{j,j+k}(n)\xi_j - x_{j,j+k}\eta_j|^2 \\ &= \sum_{j\in\theta} |\xi_j - \eta_j|^2 + \sum_{j\in\theta} |\tilde{x}_{j,j+k}(n)\xi_j - x_{j,j+k}\eta_j|^2. \end{aligned}$$

Now $\sum_{j\in\theta} |\tilde{x}_{j,j+k}(n)|^2|\xi_j|^2 \leq 1$ and $\tilde{x}_{j,j+k}(n) = n$ for $j \in \theta$ so $\sum_{j\in\theta} |\xi_{j+k}|^2 \leq 1/n^2$. So by the fact that $|\tilde{x}_{j,j+k}(n)/x_{j+k}| \leq 1$ and the choice of η it follows that

$$\sum_{j\in\theta} |\xi_j - \eta_j|^2 \leq 4/n^2.$$

Also, $\sum_{j\in\theta} |\tilde{x}_{j,j+k}(n)\xi_j - x_{j,j+k}\eta_j|^2 = 0$, by the choice of η, and thus $\|\xi - \eta\| + \|W_n\xi - W\eta\| \leq 2/n$. Hence $\inf_{\psi\in G(W)} \|\varphi - \psi\| \leq 2/n$ and so since φ was arbitrary we have

$$\sup_{\varphi \in G(A_N), \|\varphi\| \leq 1} \inf_{\psi \in G(A)} \|\varphi - \psi\| \leq 2/n \longrightarrow 0, \qquad n \to \infty.$$

The fact that $\delta(W, W_n) \to 0$ as $n \to \infty$ follows by similar reasoning. □

Proof. (**Proof of Theorem 3.8**) Arguing as in the proof of Theorem 3.5, it suffices to show that $SC_{\mathrm{ind}}(\Xi_1) \leq 2$. Let P_m and $\widetilde{P}_m$ be the projections onto $\mathrm{span}\{e_j\}_{j=1}^m$ and $\mathrm{span}\{\tilde{e}_j\}_{j=1}^m$ respectively and define

$$S_m : \Delta \times \mathbb{C} \to \mathcal{B}(P_m\mathcal{H}, \mathcal{H}), \qquad \widetilde{S}_m : \Delta \times \mathbb{C} \to \mathcal{B}(\widetilde{P}_m\mathcal{H}, \mathcal{H})$$

by

$$S_m(T, z) = (TE_1 - z)P_m, \qquad \widetilde{S}_m(T, z) = (TE_2 - \bar{z})\widetilde{P}_m,$$

where $E_1 : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$ and $E_2 : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$ are the projections onto the first and second component, respectively. (Note that there is a slight abuse of notation here since TE_1 actually denotes $TE_1\lceil_{\mathcal{H}}$, similarly with TE_2, however, this should be clear from the context.) Also, define

$$S_{m,k} : \Delta \times \mathbb{C} \to \mathcal{B}(P_m\mathcal{H}, \mathcal{H}), \qquad \widetilde{S}_{m,k} : \Delta \times \mathbb{C} \to \mathcal{B}(\widetilde{P}_m\mathcal{H}, \mathcal{H})$$

by

$$S_{m,k}(T, z) = (P_kTE_1P_k - z)P_m \qquad \widetilde{S}_m(T, z) = (\widetilde{P}_kTE_2\widetilde{P}_k - \bar{z})\widetilde{P}_m.$$

Now, for $T \in \Delta$, let $\{x_{ij}\}$ be some ordering of the matrix elements

$$\{\langle T_1e_j, e_i\rangle\} \cup \{\langle T_2\tilde{e}_j, \tilde{e}_i\rangle\}_{i,j\in\mathbb{N}},$$

and define the estimating functions Γ_{n_1,n_2} and Γ_{n_1} by

$$\begin{aligned}\Gamma_{n_1,n_2}(\{x_{ij}\}) &= \{z \in \Theta_{n_2} : \nexists L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), T_{\epsilon,n_1,n_2}(z) = LL^*\} \\ &\qquad \cup \{z \in \Theta_{n_2} : \nexists L \in LT_{\mathrm{pos}}(\widetilde{P}_{n_1}\mathcal{H}), \widetilde{T}_{\epsilon,n_1,n_2}(z) = LL^*\}, \\ \Gamma_{n_1}(\{x_{ij}\}) &= \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(T_{\epsilon,n_1}(z)) \neq 0\} \\ &\qquad \cup \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(\widetilde{T}_{\epsilon,n_1}(z)) \neq \emptyset\},\end{aligned}$$

where Θ_{n_2} is defined as in (6.3) and

$$T_{\epsilon,n_1,n_2}(z) = S_{n_1,n_2}(z)^* S_{n_1,n_2}(z) - \epsilon^2 I, \quad \widetilde{T}_{\epsilon,n_1,n_2}(z) = \widetilde{S}_{n_1,n_2}(z)^* \widetilde{S}_{n_1,n_2}(z) - \epsilon^2 I$$

and $T_{\epsilon,n_1}(z) = S_{n_1}(z)^* S_{n_1}(z) - \epsilon^2 I$, $\widetilde{T}_{\epsilon,n_1}(z) = \widetilde{S}_{n_1}(z)^* \widetilde{S}_{n_1}(z) - \epsilon^2 I$. As argued in the proof of Theorem 3.5, Γ_{n_1,n_2} depends on only finitely many elements in $\{x_{ij}\}$, and its evaluation requires finitely many arithmetic operations and radicals of the matrix elements $\{x_{ij}\}$. We are now ready to prove:

Step I. We will show that

$$\Gamma_{n_1}(\{x_{ij}\}) = \lim_{n_2 \to \infty} \Gamma_{n_1,n_2}(\{x_{ij}\}).$$

Before we can do that, we must establish a couple of facts first. Now, let $\Phi_m : \Delta \times \mathbb{C} \to \mathbb{R}$, $\widetilde{\Phi}_m : \Delta \times \mathbb{C} \to \mathbb{R}$, $\Phi_{m,k} : \Delta \times \mathbb{C} \to \mathbb{R}$ and $\widetilde{\Phi}_{m,k} : \Delta \times \mathbb{C} \to \mathbb{R}$ be defined by

$$\begin{aligned}\Phi_m(T, z) &= \min\{\sqrt{\lambda} : \lambda \in \sigma(S_m(T, z)^* S_m(T, z))\}, \\ \widetilde{\Phi}_m(T, z) &= \min\{\sqrt{\lambda} : \lambda \in \sigma(\widetilde{S}_m(T, z)^* \widetilde{S}_m(T, z))\}, \\ \Phi_{m,k}(T, z) &= \min\{\sqrt{\lambda} : \lambda \in \sigma(S_{m,k}(T, z)^* S_{m,k}(T, z))\}, \\ \widetilde{\Phi}_{m,k}(T, z) &= \min\{\sqrt{\lambda} : \lambda \in \sigma(\widetilde{S}_{m,k}(T, z)^* \widetilde{S}_{m,k}(T, z))\}.\end{aligned}$$

Claim: We claim that

$$\{z \in \mathbb{C} : \Phi_m(T, z) \leq \epsilon\} = \mathrm{cl}(\{z \in \mathbb{C} : \Phi_m(T, z) < \epsilon\}). \tag{6.13}$$

Indeed, this is the case, and the proof is almost identical to the argument used in the proof of Theorem 3.5. Let

$$\Omega = \{z \in \mathbb{C} : \Phi_m(T, z) \neq 0\},$$

then Ω is obviously open and we claim that $\mathbb{C} \setminus \Omega$ is finite. To see that, we argue by contradiction and suppose that $\Phi_m(T, \cdot)$ vanishes at infinitely many points. If that was the case we would have

$$\inf_{\|\xi\|=1, \xi \in \mathcal{H}} \|(T_1 - z)P_m \xi\| = 0 \tag{6.14}$$

for infinitely many zs. But the infimum in (6.14) is attained since P_m has finite rank, so this implies that the operator $P_m T_1 \lceil_{P_m\mathcal{H}}$ has infinitely many eigenvalues. This is, of course, impossible since P_m has finite rank. Armed with this fact we return to the task of showing (6.13). Observe that since P_m has finite rank we can make $\inf_{\|\xi\|=1, \xi \in \mathcal{H}} \|(T_1 - z)P_m \xi\|$ arbitrary large for large $|z|$, and in particular, $\Phi_m(T, \cdot)$ can be made arbitrary large as long as $|z|$ is large. Using this we may argue exactly as in the proof of Theorem 3.5 and deduce that if there is a

$$z_0 \notin \mathrm{cl}(\{z \in \mathbb{C} : \Phi_m(T, z) < \epsilon\})$$

such that $\Phi_m(T, z_0) = \epsilon$ then there is an open connected set $\Omega_0 \subset \Omega$ containing z_0 and an operator valued holomorphic function G on Ω_0 such that we have $G(z) : \mathcal{H} \to P_m\mathcal{H}$,

$$1/\Phi_m(T, z) = \|G(z)\|, \qquad z \in \Omega_0,$$

and $\|G(z_1)\| < 1/\epsilon$ for some $z_1 \in \Omega_0$. By the assumption on z_0, there is a neighborhood θ around z_0 such that

$$\|G(z)\| \leq 1/\epsilon, \qquad z \in \theta$$

and since $\|G(z_1)\| < 1/\epsilon$ it follows, by Theorem 6.3, that $\|G(z)\| < 1/\epsilon$ for all $z \in \theta$. But $\|G(z_0)\| = 1/\epsilon$ and this is a contradiction.

Note that similar reasoning gives that

$$\{z \in \mathbb{C} : \widetilde{\Phi}_m(T, z) \leq \epsilon\} = \mathrm{cl}(\{z \in \mathbb{C} : \widetilde{\Phi}_m(T, z) < \epsilon\}). \tag{6.15}$$

So, by observing that

$$\begin{aligned} \Gamma_{n_1,n_2}(\{x_{ij}\}) &= \{z \in \Theta_{n_2} : \min[\Phi_{n_1,n_2}(T, z), \widetilde{\Phi}_{n_1,n_2}(T, z)] \leq \epsilon\}, \\ \Gamma_{n_1}(\{x_{ij}\}) &= \{z \in \mathbb{C} : \min[\Phi_{n_1}(T, z), \widetilde{\Phi}_{n_1}(T, z)] \leq \epsilon\} \end{aligned} \tag{6.16}$$

it suffices to show, by Proposition 4.1 that

$$\min[\Phi_{n_1,n_2}(T, z), \widetilde{\Phi}_{n_1,n_2}(T, z)] \to \min[\Phi_{n_1}(T, z), \widetilde{\Phi}_{n_1}(T, z)]$$

locally uniformly as $n_2 \to \infty$, which again will follow if we can show that the mappings

$$\begin{aligned} z &\mapsto \langle S_{n_1,n_2}(T, z)^* S_{n_1,n_2}(T, z)e_j, e_i\rangle \longrightarrow z \mapsto \langle S_{n_1}(T, z)^* S_{n_1}(T, z)e_j, e_i\rangle \\ z &\mapsto \langle \widetilde{S}_{n_1,n_2}(T, z)^* \widetilde{S}_{n_1,n_2}(T, z)\tilde{e}_j, \tilde{e}_i\rangle \longrightarrow z \mapsto \langle \widetilde{S}_{n_1}(T, z)^* \widetilde{S}_{n_1}(T, z)\tilde{e}_j, \tilde{e}_i\rangle \end{aligned} \tag{6.17}$$

locally uniformly as $n_2 \to \infty$, where $e_j, e_i \in P_{n_1}\mathcal{H}$ and $\tilde{e}_j, \tilde{e}_i \in \widetilde{P}_{n_1}\mathcal{H}$. Note that for $k \geq m$ we have $\langle S_{n_1,n_2}(T, z)^* S_{n_1,n_2}(T, z)e_j, e_i\rangle = \langle P_{n_2}(T - z)e_j, P_{n_2}(T - z)e_i\rangle$, yielding the first part of (6.17), and similar reasoning yields the second part.

Step II: We will show that

$$\lim_{n_1\to\infty} \Gamma_{n_1}(\{x_{ij}\}) = \overline{\sigma_\epsilon(T_1)}. \tag{6.18}$$

To do that we will first demonstrate the following;

$$\gamma_1(z) = \lim_{n_1\to\infty} \Phi_{n_1}(T, z), \qquad \gamma_2(z) = \lim_{n_1\to\infty} \widetilde{\Phi}_{n_1}(T, z)$$

exist, the convergence is monotonically from above and locally uniform and

$$\sigma_\epsilon(T_1) = \{z \in \mathbb{C} : \min[\gamma_1(z), \gamma_2(z)] < \epsilon\}. \tag{6.19}$$

Now, note that

$$\Phi_{n_1}(T, z) = \min_{\xi\in P_{n_1}\mathcal{H}} \|(T_1 - z)\xi\|, \qquad \widetilde{\Phi}_{n_1}(T, z) = \min_{\xi\in \widetilde{P}_{n_1}\mathcal{H}} \|(T_1 - z)^*\xi\|.$$

So, by the assumption that $\mathrm{span}\{e_j\}_{j\in\mathbb{N}}$ is a core for T_1 and $\mathrm{span}\{\tilde{e}_j\}_{j\in\mathbb{N}}$ is a core for T_2, it follows that the limits exist and that

$$\gamma_1(z) = \inf\{\lambda : \lambda \in \sigma(|(T_1 - z)|\}, \qquad \gamma_2(z) = \inf\{\lambda : \lambda \in \sigma(|(T_1 - z)^*|)\}.$$

By Dini's theorem it follows that the convergence is as asserted. Using this fact and by arguing as in the proof of Theorem 5.3 we get (6.19). The previous reasoning implies that

$$\min[\Phi_{n_1}(T, z), \widetilde{\Phi}_{n_1}(T, z)] \longrightarrow \min[\gamma_1(z), \gamma_2(z)]$$

monotonically from above and locally uniformly as $n_1 \to \infty$. So, by Proposition 4.4 and (6.19), it follows that, for a compact ball K such that $\overline{\sigma_\epsilon(T_1)} \cap K^o \neq \emptyset$, we have

$$\mathrm{cl}(\{z \in \mathbb{C} : \min[\Phi_{n_1}(T, z), \widetilde{\Phi}_{n_1}(T, z)] < \epsilon\}) \cap K \longrightarrow \overline{\sigma_\epsilon(T_1)} \cap K,$$

as $n_1 \to \infty$. But by (6.13),(6.16) and (6.15) it follows that

$$\Gamma_{n_1}(\{x_{ij}\}) = \mathrm{cl}(\{z \in \mathbb{C} : \min[\Phi_{n_1}(T, z), \widetilde{\Phi}_{n_1}(T, z)] < \epsilon\}),$$

and hence (6.18) follows. $\square$

Proof. (**Proof of Theorem 3.6**) As in the proof of Theorem 3.5 it suffices to demonstrate that $SC_{\mathrm{ind}}(\Xi_1) = 1$. Now, obviously we have $SC_{\mathrm{ind}}(\Xi) > 0$, so it suffices to show that $SC_{\mathrm{ind}}(\Xi) \leq 1$. We follow the proof of Theorem 3.5 closely. Let P_n be the projection onto $\mathrm{span}\{e_1, \ldots, e_n\}$ and $x_{ij} = \langle Te_j, e_i\rangle$ for $T \in \Delta$. For $k \in \mathbb{N}$ define T^k inductively by $T^k\xi = T(T^{k-1}\xi)$ on

$$\mathcal{D}(T^k) = \{\xi : \xi \in \mathcal{D}(T^{k-1}), T^{k-1}\xi \in \mathcal{D}(T)\},$$

and define $\mathcal{D}((T^*)^k)$ similarly. Then it is easy to see that $\bigcup_m P_m\mathcal{H} \subset \mathcal{D}(T^k)$, so T^k is densely defined. The fact that T^k is closed is well known [29](p. 603), and it follows (by a straightforward argument using the assumptions (ii) and (iii)) that $\bigcup_m P_m\mathcal{H}$ is a core for T^k. Similarly, we get that $(T^*)^k$ is closed and densely defined and that $\bigcup_m P_m\mathcal{H} \subset \mathcal{D}((T^*)^k)$ is a core for $(T^*)^k$. Using this, it is easy to see that we can, for integers m, k, define $T_{\epsilon,m,k}(z) = T_{m,k}(z) - \epsilon^{2^{n+1}}I$ and $\widetilde{T}_{\epsilon,m,k}(z) = \widetilde{T}_{m,k}(z) - \epsilon^{2^{n+1}}I$, where $T_{m,k}(z)$ and $\widetilde{T}_{m,k}(z)$ are defined in (6.2). Let, for $k \in \mathbb{N}$, Θ_k be defined as in (6.3) and

$$\begin{aligned} \Psi_k = \{z \in \mathbb{C} : \nexists\, L \in LT_{\mathrm{pos}}(P_k\mathcal{H}), T_{\epsilon,k,2^nd+k}(z) = LL^*\} \\ \cup \{z \in \mathbb{C} : \nexists\, L \in LT_{\mathrm{pos}}(P_k\mathcal{H}), \widetilde{T}_{\epsilon,k,2^nd+k}(z) = LL^*\}, \end{aligned} \tag{6.20}$$

where $LT_{\mathrm{pos}}(P_m\mathcal{H})$ denotes the set of lower triangular matrices in $P_m\mathcal{H}$ (with respect to $\{e_j\}$) with strictly positive diagonal elements. Now, define Γ_k by

$$\Gamma_k(\{x_{ij}\}) = \Psi_k \cap \Theta_k.$$

By the same reasoning as in the proof of Theorem 3.5, it follows that Γ_{n_1,n_2} depends only on finitely many of the x_{ij}s and requires only finitely many arithmetic operations and radicals of $\{x_{ij}\}$ for its evaluation. Now, to show that

$$\Xi(T) = \lim_{k\to\infty} \Gamma_k(\{x_{ij}\}),$$

we need to show that for any compact ball K such that $\overline{\sigma_{n,\epsilon}(T)} \cap K^o \neq \emptyset$ then

$$d_H(\overline{\sigma_{n,\epsilon}(T)} \cap K, \Gamma_k(\{x_{ij}\}) \cap K) \longrightarrow 0, \qquad k \to \infty.$$

But, since obviously $d_H(\Theta_k \cap K, K) \to 0$ as $k \to \infty$ it suffices to show that

$$d_H(\Psi_k \cap K, \overline{\sigma_{n,\epsilon}(T)} \cap K) \longrightarrow 0. \tag{6.21}$$

To prove that, note that by the reasoning in the beginning of the proof we may define $\Phi_{n,m} : \Delta \times \mathbb{C} \to \mathbb{R}$ by

$$\Phi_{n,m}(S, z) = \min\left\{\lambda^{1/2^{n+1}} : \lambda \in \sigma\left(P_m((S-z)^*)^{2^n}(S-z)^{2^n}\lceil_{P_m\mathcal{H}}\right)\right\}.$$

Let

$$\begin{aligned} \gamma_{n,k} &= \min[\Phi_{n,k}(T, \cdot), \Phi_{n,k}(T^*, \cdot)] \\ \gamma_{n,k,m} &= \min[\Phi_{n,k}(P_mTP_m, \cdot), \Phi_{n,k}(P_mT^*P_m, \cdot)]. \end{aligned}$$

Before we can continue with the proof of (6.21) we need the following fact.

Claim: We claim that $\Psi_k = \{z \in \mathbb{C} : \gamma_{n,k}(z) \leq \epsilon\}$. To deduce the claim it suffices to show that

$$\gamma_{n,k}(z) = \gamma_{n,k,2^nd+k}(z), \qquad z \in \mathbb{C}, \tag{6.22}$$

and why becomes clear after we make the observation that we have

$$\begin{aligned} \Psi_k &= \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(T_{\epsilon,k,2^nd+k}(z)) \neq \emptyset\} \\ &\qquad \cup \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(\widetilde{T}_{\epsilon,k,2^nd+k}(z)) \neq \emptyset\} \\ &= \{z \in \mathbb{C} : \gamma_{n,k,2^nd+k}(z) \leq \epsilon\}. \end{aligned}$$

Now (6.22) will follow if we can prove that

$$\begin{aligned} &\langle((T-z)^*)^{2^n}(T-z)^{2^n}\xi, \eta\rangle \\ &\qquad = \langle(P_{2^nd+k}(T-z)P_{2^nd+k})^*)^{2^n}(P_{2^nd+k}(T-z)P_{2^nd+k})^{2^n}\xi, \eta\rangle. \\ &\langle(T-z)^{2^n}((T-z)^*)^{2^n}\xi, \eta\rangle \\ &\qquad = \langle(P_{2^nd+k}(T-z)P_{2^nd+k})^{2^n}(P_{2^nd+k}(T-z)P_{2^nd+k})^*)^{2^n}\xi, \eta\rangle, \end{aligned}$$

for $\xi, \eta \in P_k\mathcal{H}$. To show the latter it is easy to see that it suffices to show that

$$\begin{aligned} (P_{2^n d+k} T P_{2^n d+k})^l \xi &= T^l \xi, \qquad \xi \in P_k\mathcal{H}, \quad l \leq 2^n, \\ (P_{2^n d+k} T^* P_{2^n d+k})^l \xi &= T^l \xi, \qquad \xi \in P_k\mathcal{H}, \quad l \leq 2^n. \end{aligned} \tag{6.23}$$

To show the first part of (6.23), let $\mu \in \mathbb{N}$ such that $\mu > d$, and note that, by assumption, we can write $T\lceil_{\bigcup_m P_m\mathcal{H}}$ as (with a slight abuse of notation)

$$T = P_\mu T P_\mu + P_\mu^\perp T P_\mu^\perp + \sum_{j=-d}^{d-1} \zeta_j \otimes e_{\mu-j},$$

where $\zeta_j \in (P_{\mu+d} - P_{\mu-d})\mathcal{H}$. Now this gives us that, for $l \in \mathbb{N}$,

$$\begin{aligned} T^l = (P_\mu T P_\mu)^l &+ \text{terms of the form} \\ &(P_\mu^\perp T P_\mu^\perp + \sum_{j=-d}^{d-1} \zeta_j \otimes e_{\mu-j})^{p_1} \times (P_\mu T P_\mu)^{q_1} \\ &\times (P_\mu^\perp T P_\mu^\perp + \sum_{j=-d}^{d-1} \zeta_j \otimes e_{\mu-j})^{p_2} \times (P_\mu T P_\mu)^{q_2} \times \cdots \\ &\times (P_\mu^\perp T P_\mu^\perp + \sum_{j=-d}^{d-1} \zeta_j \otimes e_{\mu-j})^{p_t} \times (P_\mu T P_\mu)^{q_t}, \end{aligned}$$

where $q_i \leq l-1$ and $p_i \leq l$. Note that since $T \in \Delta$ (using assumption (ii)) it is straightforward to show that

$$\langle (P_\mu T P_\mu)^q e_r, e_j \rangle = 0, \qquad r \leq k, \quad j > qd + k,$$

for any integer q. Hence,

$$(P_{2^n d+k}^\perp T P_{2^n d+k}^\perp + \sum_{j=-d}^{d-1} \zeta_j \otimes e_{2^n d+k-j})^p \times (P_{2^n d+k} T P_{2^n d+k})^q e_r = 0, \tag{6.24}$$

for $r \leq k$, $q \leq 2^n - 1$ and $p \leq 2^n$ yielding the first part of (6.23). The second part of (6.23) follows by similar reasoning.

Armed with the claim we have reduced the problem to showing that if K is a compact ball such that K^o intersects $\overline{\sigma_{n,\epsilon}(T)}$, then

$$\lim_{k\to\infty} \{z \in \mathbb{C} : \gamma_{n,k}(z) \leq \epsilon\} \cap K = \overline{\sigma_{n,\epsilon}(T)} \cap K. \tag{6.25}$$

Now, the fact that $T \in \Delta$ and the reasoning in the beginning of the proof allows us to define

$$\begin{aligned} \gamma_n(z) = \min \Big[&\inf \Big\{ \lambda^{2^{n+1}} : \lambda \in \sigma\Big(|(T-z)^{2^n}|\Big) \Big\}, \\ &\inf \Big\{ \lambda^{2^{n+1}} : \lambda \in \sigma\Big(|((T-z)^*)^{2^n}|\Big) \Big\} \Big]. \end{aligned}$$

Note that, by arguing similarly as in the proof of (ii) and (iii) in Theorem 4.4, we deduce that $\sigma_{n,\epsilon}(T) = \{z \in \mathbb{C} : \gamma_n(z) < \epsilon\}$. By arguing as in Proposition 6.1, using the fact that $\bigcup_m P_m\mathcal{H}$ is a core for T^k and $(T^*)^k$ we deduce that $\gamma_{n,k} \to \gamma_n$ locally uniformly and monotonically from above. By arguing as in the proof of Theorem 3.8 we deduce that

$$\mathrm{cl}(\{z \in \mathbb{C} : \gamma_{n,k}(z) < \epsilon\}) = \{z \in \mathbb{C} : \gamma_{n,k}(z) \leq \epsilon\}.$$

Thus, using Proposition 4.1 we conclude that (6.25) is true, and we are done. □

7 Other Types of Pseudospectra

The disadvantage of the n-pseudospectrum is that even though one can estimate the spectrum by taking n very large, n may have to be too large for practical purposes. Thus, since we only have the estimate for $T \in \mathcal{C}(\mathcal{H}), \epsilon > 0$ that $\sigma(T) \subset \sigma_{n,\epsilon}(T)$, it is important to get a "lower" bound on $\sigma(T)$ i.e. we want to find a set $\Omega \subset \mathbb{C}$ such that $\Omega \subset \sigma(T)$. A candidate for this is described in the following.

Definition 7.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and Φ_0 be defined as in Definition 4.3. Let $\zeta_1(z) = \Phi_0(T, z)$ $\zeta_2(z) = \Phi_0(T^*, \bar{z})$. Now let $\epsilon > 0$ and define the ϵ-residual pseudospectrum to be the set*

$$\sigma_{\mathrm{res},\epsilon}(T) = \{z : \zeta_1(z) > \epsilon,\, \zeta_2(z) = 0\}$$

and the adjoint ϵ-residual pseudospectrum to be the set

$$\sigma_{\mathrm{res}^*,\epsilon}(T) = \{z : \zeta_1(z) = 0,\, \zeta_2(z) > \epsilon\}.$$

Theorem 7.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $\{T_k\} \subset \mathcal{B}(\mathcal{H})$ such that $T_k \to T$ in norm, as $k \to \infty$. Then for $\epsilon > 0$ we have the following,*

(i) $\sigma(T) \supset \bigcup_{\epsilon>0} \sigma_{\mathrm{res},\epsilon}(T) \cup \sigma_{\mathrm{res}^*,\epsilon}(T)$

(ii) $\mathrm{cl}(\{z \in \mathbb{C} : \zeta_1(z) < \epsilon\}) = \{z \in \mathbb{C} : \zeta_1(z) \le \epsilon\}$

(iii) $\mathrm{cl}(\{z \in \mathbb{C} : \zeta_2(z) < \epsilon\}) = \{z \in \mathbb{C} : \zeta_2(z) \le \epsilon\}$

(iv) For any compact ball $K \subset \mathbb{C}$ such that $\mathrm{cl}(\sigma_{\mathrm{res},\epsilon}(T)) \cap K^o \neq \emptyset$ it follows that

$$d_H(\mathrm{cl}(\sigma_{\mathrm{res},\epsilon}(T_k)) \cap K, \mathrm{cl}(\sigma_{\mathrm{res},\epsilon}(T)) \cap K) \longrightarrow 0, \qquad k \to \infty.$$

(v) For any compact ball $K \subset \mathbb{C}$ such that $\sigma_{\mathrm{res}^*,\epsilon}(T) \cap K^o \neq \emptyset$ it follows that

$$d_H(\mathrm{cl}(\sigma_{\mathrm{res}^*,\epsilon}(T_k)) \cap K, \mathrm{cl}(\sigma_{\mathrm{res}^*,\epsilon}(T) \cap K)) \longrightarrow 0, \qquad k \to \infty.$$

Proof. Note that (i) follows by arguing as in the proof of Theorem 4.4, so we will not be repeating that reasoning here. Now, we will show (ii), namely, that

$$\{z \in \mathbb{C} : \zeta_1(z) \le \epsilon\} = \mathrm{cl}(\{z \in \mathbb{C} : \zeta_1(z) < \epsilon\}). \tag{7.1}$$

We argue by contradiction. Suppose that there is a $z_0 \in \mathbb{C} \setminus \mathrm{cl}(\{z \in \mathbb{C} : \zeta_1(z) < \epsilon\})$ such that $\zeta_1(z_0) = \epsilon$. Then, there is a neighborhood ω around z_0 such that $\zeta_1(z) \ge \epsilon$ for $z \in \omega$. We claim that this is impossible. Indeed, let φ be defined on ω by $\varphi(z) = 1/\zeta_1(z)$. Now

$$\varphi(z) = 1/ \inf_{\|\xi\|=1, \xi \in \mathcal{H}} \|(T - z)\xi\|,$$

so $T - z$ is bounded from below by ϵ for $z \in \omega$. Let $\mathcal{H}_1 = \mathrm{ran}(T - z_0)$ and let $\widetilde{\mathcal{H}}$ be an infinite dimensional Hilbert space. Choose an isomorphism $V : \widetilde{\mathcal{H}} \to \mathcal{H}_1^{\perp} \oplus \widetilde{\mathcal{H}}$, and define the following operator

$$\widetilde{T}_c = (T - z_0) \oplus cV : \mathcal{H} \oplus \widetilde{\mathcal{H}} \to \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp} \oplus \widetilde{\mathcal{H}}, \qquad c \in \mathbb{R}.$$

Note that $\widetilde{T}_c$ is invertible and for sufficiently large c we have

$$\varphi(z_0) = \left(\inf_{\|\xi\|=1, \xi \in \mathcal{H}} \|\widetilde{T}\xi\| \right)^{-1}.$$

Moreover, for z sufficiently close to z_0 it follows that

$$\varphi(z) = \left(\inf_{\|\xi\|=1, \xi \in \mathcal{H}} \|\widetilde{T}_c - (z_0 - z)\xi\| \right)^{-1}.$$

Let $G(z)$ be the inverse of $\widetilde{T}_c - (z_0 - z)$ for z in a neighborhood $\tilde{\omega}$ around z_0. Then $\varphi(z) = \|G(z)\|$. Now $\varphi(z_0) = 1/\epsilon$ and $\varphi(z) \le 1/\epsilon$ for $z \in \tilde{\omega}$. But, clearly, $G'(z)$ is invertible for all $z \in \tilde{\omega}$ so by Theorem 4.2 it follows that $\|G(z)\| < 1/\epsilon$ for $z \in \tilde{\omega}$, contradicting $\varphi(z_0) = 1/\epsilon$ and we have shown (7.1). To show (iii) one argues almost exactly as in the proof of (ii).

We will now prove (iv). Firstly, to see the fact that $d_H(\sigma_{\mathrm{res},\epsilon}(T_k) \cap K, \sigma_{\mathrm{res},\epsilon}(T) \cap K) \to 0$, as $k \to \infty$, define Φ_0 as in Definition 4.3 and let $\zeta_{1,k}(z) = \Phi_0(T_k, z)$. Note that $\zeta_{1,k} \to \zeta_1$ locally uniformly as $k \to \infty$, by reasoning as in (4.5). Secondly, note that, for $\delta \in (0, \epsilon)$, we have

$$\mathrm{cl}(\{z \in \mathbb{C} : \zeta_1(z) > \epsilon,\ \zeta_2(z) \leq \delta\}) = \mathrm{cl}(\{z \in \mathbb{C} : \zeta_1(z) > \epsilon,\ \zeta_2(z) = 0\}).$$

So if we define $\zeta_{2,k}(z) = \Phi_0(T_k^*, \bar{z})$, it suffices to show that

$$d_H(\mathrm{cl}(\{z \in \mathbb{C} : \zeta_{1,k}(z) > \epsilon\}) \cap K, \mathrm{cl}(\{z \in \mathbb{C} : \zeta_1(z) > \epsilon\}) \cap K) \longrightarrow 0, \quad k \to \infty \tag{7.2}$$

and, by (ii), that $d_H(\{z \in \mathbb{C} : \zeta_{2,k}(z) \leq \delta\} \cap K, \{z \in \mathbb{C} : \zeta_2(z) \leq \delta\} \cap K) \to 0$ as $k \to \infty$. The latter follows from arguing similarly to the proof of Theorem 3.5, and hence we will concentrate on the former. Now, it is easy to see, by the definition of the Hausdorff metric and (ii), that (7.2) follows if we can show that

$$d_H(\{z \in \mathbb{C} : \zeta_{1,k}(z) \leq \epsilon\}, \{z \in \mathbb{C} : \zeta_1(z) \leq \epsilon\}) \longrightarrow 0, \quad k \to \infty,$$

but the latter follows by the locally uniform convergence of $\{\zeta_{1,k}\}$ and Proposition 4.1. Also, (v) follows by similar reasoning, and we are done. ☐

Theorem 7.3. *Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for $\mathcal{H}$ and define $\Xi_1, \Xi_2 : \mathcal{B}(\mathcal{H}) \to \Omega$, for $\epsilon > 0$, by $\Xi_1(T) = \mathrm{cl}(\sigma_{\mathrm{res},\epsilon}(T))$ and $\Xi_2(T) = \mathrm{cl}(\sigma_{\mathrm{res}^*,\epsilon}(T))$. Then $SC_{\mathrm{ind}}(\Xi_1) \leq 2$ and $SC_{\mathrm{ind}}(\Xi_2) \leq 2$.*

Proof. To show that $SC_{\mathrm{ind}}(\Xi_1) \leq 2$ let Θ_k be defined as in (6.3) and define the estimating functions Γ_{n_1,n_2} and Γ_{n_1} in the following way. Define P_n to be the projection onto $\mathrm{span}\{e_1, \dots, e_n\}$, choose $\delta \in (0, \epsilon)$ and define

$$\begin{aligned}
\Gamma_{n_1,n_2}(\{x_{ij}\}) &= \{z \in \Theta_{n_2} : \exists\, L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), T_{\epsilon,n_1,n_2}(z) = LL^*\} \\
&\qquad \cap \{z \in \Theta_{n_2} : \nexists\, L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), \widetilde{T}_{\delta,n_1,n_2}(z) = LL^*\}, \\
\Gamma_{n_1}(\{x_{ij}\}) &= \mathrm{cl}(\{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(T_{\epsilon,n_1}(z)) = \emptyset\}) \\
&\qquad \cap \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(\widetilde{T}_{\delta,n_1}(z)) \neq \emptyset\},
\end{aligned}$$

where T_{ϵ,n_1,n_2}, $\widetilde{T}_{\delta,n_1,n_2}$, T_{ϵ,n_1} and $\widetilde{T}_{\delta,n_1}$ as defined as in (6.5). As the rest of the proof is just epsilon away from the proof of Theorem 3.5 we will just sketch the ideas. By letting $\zeta_{1,n_1}(z) = \Phi_{0,n_1}(T, z)$, $\zeta_{2,n_1}(z) = \Phi_{0,n_1}(T^*, \bar{z})$ and

$$\zeta_{1,n_1,n_2}(z) = \Phi_{0,n_1}(P_{n_2}TP_{n_2}, z), \quad \zeta_{2,n_1,n_2}(z) = \Phi_{0,n_1}(P_{n_2}T^*P_{n_2}, \bar{z}),$$

where Φ_0 is defined as in Definition 4.3, one observes that

$$\begin{aligned}
&\{z \in \Theta_{n_2} : \zeta_{1,n_1,n_2}(z) > \epsilon, \zeta_{2,n_1,n_2}(z) \leq \delta\} \\
&\qquad = \{z \in \mathbb{C} : \exists\, L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), T_{\epsilon,n_1,n_2}(z) = LL^*\} \\
&\qquad\qquad \cap \{z \in \mathbb{C} : \nexists\, L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), \widetilde{T}_{\delta,n_1,n_2}(z) = LL^*\},
\end{aligned} \tag{7.3}$$

and

$$\Gamma_{n_1}(\{x_{ij}\}) = \mathrm{cl}(\{z : \zeta_{1,n_1}(z) > \epsilon, \zeta_{2,n_1}(z) \leq \delta\}).$$

Now, let ζ_1 and ζ_2 be defined as in Definition 7.1. By using (ii) in Theorem 7.2 and reasoning as in the proof of Theorem 3.5 (StepI and StepII) using arguments similar to the last part of the proof of Theorem 7.2 one deduces that, for compact ball $K \subset \mathbb{C}$ with K^o intersecting the appropriate sets,

$$\begin{aligned}
\mathrm{cl}(\{z \in \mathbb{C} : \zeta_{1,n_1}(z) > \epsilon\}) \cap K &\longrightarrow \mathrm{cl}(\{z \in \mathbb{C} : \zeta_1(z) > \epsilon\}) \cap K \\
\{z \in \mathbb{C} : \zeta_{2,n_1}(z) \leq \delta\} \cap K &\longrightarrow \{z \in \mathbb{C} : \zeta_2(z) \leq \delta\} \cap K, \quad n_1 \to \infty,
\end{aligned}$$

$$\begin{aligned}
\{z \in \Theta_{n_2} : \zeta_{1,n_1,n_2}(z) > \epsilon\} \cap K &\longrightarrow \mathrm{cl}(\{z \in \mathbb{C} : \zeta_{1,n_1}(z) > \epsilon\}) \cap K \\
\{z \in \Theta_{n_2} : \zeta_{2,n_1,n_2}(z) \leq \delta\} \cap K &\longrightarrow \{z \in \mathbb{C} : \zeta_{2,n_1}(z) \leq \delta\} \cap K, \quad n_2 \to \infty,
\end{aligned}$$

hence

$$\mathrm{cl}(\{z \in \mathbb{C} : \zeta_{1,n_1}(z) > \epsilon, \zeta_{2,n_1}(z) \leq \delta\}) \cap K \longrightarrow \mathrm{cl}(\{z : \zeta_1(z) > \epsilon, \zeta_2(z) \leq \delta\}) \cap K$$

as $n_1 \to \infty$, and

$$\begin{aligned}\{z \in \Theta_{n_2} : \zeta_{1,n_1,n_2}(z) > \epsilon, \zeta_{2,n_1,n_2}(z) \leq \delta\} \cap K \\ \longrightarrow \mathrm{cl}(\{z \in \mathbb{C} : \zeta_{1,n_1}(z) > \epsilon, \zeta_{2,n_1,n_2}(z) \leq \delta\}) \cap K\end{aligned}$$

as $n_2 \to \infty$. But

$$\mathrm{cl}(\{z : \zeta_1(z) > \epsilon, \zeta_2(z) \leq \delta\}) = \mathrm{cl}(\{z : \zeta_1(z) > \epsilon, \zeta_2(z) = 0\}) = \mathrm{cl}(\sigma_{\mathrm{res},\epsilon}(T)),$$

and hence we have shown that $SC_{\mathrm{ind}}(\Xi_1) \leq 2$. The fact that $SC_{\mathrm{ind}}(\Xi_2) \leq 2$ follows by similar reasoning. □

8 Applications to Schrödinger and Dirac Operators

Non-Hermitian quantum mechanics has been an increasingly popular field in the last decades [35, 36, 8, 50]. As the importance of non-Hermitian operators in physics has been established, the spectral theory of such operators has been given a substantial amount of attention [21, 23, 24, 16] (Note also that non-Hermitian spectral problems in quantum mechanics occur in the theory of Resonances [52, 40]). Since the spectral theory of non-Hermitian operators is very different from the self-adjoint case, very little is known in general, and the same is true for the theory of approximating spectra. In fact it is an open problem how to approximate the spectrum and the pseudospectrum of an arbitrary Schrödinger operator. In this section we will show how to use the theory from the previous sections to get some insight on how to estimate spectra and pseudospectra of non-hermitian Schrödinger and Dirac operators with bounded potential. Let

$$P_j = -i\frac{\partial}{dx_j} \qquad Q_j = \text{multiplication by } x_j$$

with their appropriate domains in $\mathcal{H} = L^2(\mathbb{R}^d)$. Let $v \in L^\infty(\mathbb{R}^d)$ be a complex valued, continuous function, and define the Schrödinger operator

$$H = \frac{1}{2}\sum_{1\leq j\leq d} P_j^2 + v(Q_1, \ldots, Q_d), \quad \mathcal{D}(H) = W_{2,2}(\mathbb{R}^d),$$

where $W_{2,2}(\mathbb{R}^d)$ is the Sobolev space of functions whose second derivative (in the distributional sense) is square integrable.

Similarly we can define the Dirac operator. Let $\mathcal{H} = \bigoplus_{k=1}^4 L^2(\mathbb{R}^3)$ and define (formally) $\tilde{P}_j$ on $\mathcal{H}$ by

$$\tilde{P}_j = \bigoplus_{k=1}^{4} P_j, \qquad P_j = -i\frac{\partial}{\partial x_j}, \qquad j = 1, 2, 3,$$

where P_j is formally defined on $L^2(\mathbb{R}^3)$. Let

$$H_0 = \sum_{j=1}^{3} \alpha_j \tilde{P}_j + \beta,$$

where α_j and β are 4-by-4 matrices satisfying the commutation relation

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I, \qquad j, k = 1, 2, 3, 4, \qquad \alpha_4 = \beta. \tag{8.1}$$

Then it is well known that H_0 is self-adjoint on $\bigoplus_{k=1}^4 W_{2,1}(\mathbb{R}^3)$ where

$$W_{2,1}(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : \mathcal{F}f \in L^2_1(\mathbb{R}^3)\}$$

and $L^2_1(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : (1+|\cdot|^2)^{1/2} f \in L^2(\mathbb{R}^3)\}$. Let $v \in L^\infty(\mathbb{R}^d)$ and define the Dirac operator

$$H_D = H_0 + \bigoplus_{k=1}^{4} v(Q_1, Q_2, Q_3), \qquad \mathcal{D}(H) = \bigoplus_{k=1}^{4} W_{2,1}(\mathbb{R}^3).$$

Note that H is closed since v is bounded. It is easy to see that

$$H^* = \frac{1}{2} \sum_{1 \le j \le d} P_j^2 + \bar{v}(Q_1, \ldots, Q_d), \quad \mathcal{D}(H^*) = W_{2,2}(\mathbb{R}^d)$$

and

$$H_D^* = H_0 + \bigoplus_{k=1}^{4} \bar{v}(Q_1, Q_2, Q_3), \qquad \mathcal{D}(H_D^*) = \bigoplus_{k=1}^{4} W_{2,1}(\mathbb{R}^3).$$

Thus, in order to estimate the pseudospectra of H and H_D, we may follow the ideas in the proof of Theorem 3.8. We will give a description of this for H and note that the procedure is exactly the same for H_D. Choose a basis $\{\varphi_j\}_{j\in\mathbb{N}}$ for $W_{2,2}(\mathbb{R}^d)$ that is orthonormal in $L^2(\mathbb{R}^d)$, and let P_n be the projection onto $\mathrm{span}\{\varphi_j\}_{j=1}^n$. Now let $\{x_{ij}\}$ be defined by $x_{ij} = \langle H\varphi_j, \varphi_i\rangle$ and note that if we let $\tilde{x}_{ij} = \langle H^*\varphi_j, \varphi_i\rangle$ then $\tilde{x}_{ij} = \bar{x}_{ji}$. This allows us to define the set of estimating functions in the following way. Let $\epsilon > 0$ and define

$$\begin{aligned}\Gamma_{n_1,n_2}(\{x_{ij}\}) = \{z \in \Theta_{n_2} : \nexists\, L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), T_{\epsilon,n_1,n_2}(z) = LL^*\} \\ \cup \{z \in \Theta_{n_2} : \nexists\, L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), \widetilde{T}_{\epsilon,n_1,n_2}(z) = LL^*\}\end{aligned}$$

and

$$\begin{aligned}&\Gamma_{n_1}(\{x_{ij}\}) \\ &= \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(T_{\epsilon,n_1}(z)) \neq \emptyset\} \cup \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(\widetilde{T}_{\epsilon,n_1}(z)) \neq \emptyset\},\end{aligned}$$

where where Θ_{n_2} is defined as in (6.3) and

$$T_{\epsilon,n_1,n_2}(z) = S_{n_1}(P_{n_2}HP_{n_2}, z)^* S_{n_1}(P_{n_2}HP_{n_2}, z) - \epsilon^2 I,$$

$$\widetilde{T}_{\epsilon,n_1,n_2}(z) = S_{n_1}(P_{n_2}H^*P_{n_2}, z)^* S_{n_1}(P_{n_2}H^*P_{n_2}, z) - \epsilon^2 I$$

and $T_{\epsilon,n_1}(z) = S_{n_1}(H,z)^*S_{n_1}(H,z) - \epsilon^2 I$, $\widetilde{T}_{\epsilon,n_1}(z) = \widetilde{S}_{n_1}(H^*,z)^*\widetilde{S}_{n_1}(H^*z) - \epsilon^2 I$, where $S_m : \Delta \times \mathbb{C} \to \mathcal{B}(P_m\mathcal{H}, \mathcal{H})$ is defined by $S_m(T,z) = (T-z)P_m$ and Δ denotes the set of closed operators having $W_{2,2}(\mathbb{R}^d)$ as their domain. Arguing as in the proof of Theorem 3.8 one deduces that

$$\sigma_\epsilon(H) = \lim_{n_1\to\infty} \Gamma_{n_1}(\{x_{ij}\}), \qquad \Gamma_{n_1}(\{x_{ij}\}) = \lim_{n_2\to\infty} \Gamma_{n_1,n_2}(\{x_{ij}\}).$$

Hence we get the following corollaries to Theorem 3.8.

Corollary 8.1. *Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a (not necessarily orthonormal) basis for $W_{2,2}(\mathbb{R}^d)$ that is orthonormal in $L^2(\mathbb{R}^d)$ and let Δ denote the set of Schrödinger operators on $L^2(\mathbb{R}^d)$ with potential function in $L^\infty(\mathbb{R}^d)$. Let $\epsilon > 0$, $\Xi_1 : \Delta \to \Omega$ and $\Xi_2 : \Delta \to \Omega$ be defined by $\Xi_1(H) = \overline{\sigma_\epsilon(H)}$ and $\Xi_2(H) = \sigma(H)$. Then $SC_{\mathrm{ind}}(\Xi_1) \le 2$ and $SC_{\mathrm{ind}}(\Xi_2) \le 3$.*

Corollary 8.2. *Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a (not necessarily orthonormal) basis for the space $\bigoplus_{k=1}^4 W_{2,1}(\mathbb{R}^3)$ that is orthonormal in $\bigoplus_{k=1}^4 L^2(\mathbb{R}^3)$, and let Δ denote the set of Dirac operators on the Hilbert space $\bigoplus_{k=1}^4 L^2(\mathbb{R}^3)$ with bounded potential function. Let $\epsilon > 0$, $\Xi_1 : \Delta \to \Omega$ and $\Xi_2 : \Delta \to \Omega$ be defined by $\Xi_1(H_D) = \overline{\sigma_\epsilon(H_D)}$ and $\Xi_2(T) = \sigma(H_D)$. Then $SC_{\mathrm{ind}}(\Xi_1) \le 2$ and $SC_{\mathrm{ind}}(\Xi_2) \le 3$.*

Remark 8.1 As the proof of Theorem 3.8, and hence also the proofs of Corollaries 8.1 and 8.2, are constructive, we have a constructive way of recovering spectra and pseudospectra of a large class of important operators in mathematical physics and hence the previous results may have impact in applications.

9 Convergence of Densities

9.1 Arveson's Szegö-type Theorem

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C^*-algebra with a unique tracial state. Then a self-adjoint operator $A \in \mathcal{A}$ determines a natural probability measure μ_A on $\mathbb{R}$ by

$$\int_{\mathbb{R}} f(x)\, d\mu_A(x) = \tau(f(A)), \quad f \in C_0(\mathbb{R}).$$

Also, if τ is faithful then $\mathrm{supp}(\mu_A) = \sigma(A)$ and one refers to μ_A as the spectral distribution (we will follow the setup in [3] and assume that $\mathcal{A}$ is unital). Now, suppose that $\{P_n\}$ is an increasing sequence of finite rank projections converging strongly to the identity. As we have seen above we can approximate the spectrum of A by using the techniques demonstrated in Section 6. We now turn the attention to the task of approximating μ_A. Define the tracial state

$$\tau_n(B) = \frac{1}{d_n}\mathrm{trace}(P_n B), \qquad d_n = \dim(P_n\mathcal{H}).$$

Now τ_n restricts to the normalised trace on $P_n\mathcal{B}(\mathcal{H})P_n$ and similar to τ induces a measure $\mu_{P_nA\lceil_{P_n\mathcal{H}}}$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} f(x)\, d\mu_{P_nA\lceil_{P_n\mathcal{H}}}(x) = \tau_n(f(P_nA\lceil_{P_n\mathcal{H}})), \quad f \in C_0(\mathbb{R}). \tag{9.1}$$

The question is then: what is the relationship between $\mu_{P_nA\lceil_{P_n\mathcal{H}}}$ and μ_A. In particular under which assumptions (if any) can one guarantee that

$$\mu_{P_nA\lceil_{P_n\mathcal{H}}} \overset{\text{weak}^*}{\longrightarrow} \mu_A, \quad n \to \infty.$$

This has been investigated in [3, 6, 5, 33]. The crucial ingredient in Arveson's framework is the definition of the degree of an operator with respect to a certain filtration $\mathcal{F} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$.

Definition 9.1. *(i) A filtration of $\mathcal{H}$ is a sequence $\mathcal{F} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$ of finite dimensional subspaces of $\mathcal{H}$ such that $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ and*

$$\overline{\bigcup_{n=1}^{\infty} \mathcal{H}_n} = \mathcal{H}.$$

(ii) Let $\mathcal{F} = \{\mathcal{H}_n\}$ be a filtration of $\mathcal{H}$ and let P_n be the projection onto $\mathcal{H}_n$. The degree of an operator $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$deg(A) = \sup_{n\geq 1} \mathrm{rank}(P_nA - AP_n).$$

Definition 9.2. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C^*-algebra. An $\mathcal{A}$-filtration is a filtration of $\mathcal{H}$ such that the $*$-subalgebra of all finite degree operators in $\mathcal{A}$ is norm dense in $\mathcal{A}$.*

Proposition 9.3 (Arveson [3])**.** *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C^*-algebra with a unique tracial state τ and suppose that $\{\mathcal{H}_n\}$ is an $\mathcal{A}$-filtration. Let τ_n be the state of $\mathcal{A}$ defined by*

$$\tau_n(A) = \frac{1}{d_n}\mathrm{trace}(P_nA), \qquad d_n = \dim(\mathcal{H}_n).$$

Then

$$\tau_n(A) \to \tau(A), \qquad \textit{for all} \quad A \in \mathcal{A}.$$

Proposition 9.4 (Arveson [3])**.** *Let $\mathcal{F} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$ be a filtration of $\mathcal{H}$, let P_n be the projection onto $\mathcal{H}_n$ and let $A_1, A_2, \ldots, A_p$ be a finite set of operators in $\mathcal{B}(\mathcal{H})$. Then for every $n = 1, 2, \ldots$ we have*

$$\mathrm{trace}|P_nA_1A_2\ldots A_pP_n - P_nA_1P_nA_2P_n\ldots P_nA_pP_n| \leq \|A_1\|\ldots\|A_p\|\sum_{k=1}^{p}\mathrm{deg}A_k.$$

Theorem 9.5 (Arveson [3]). *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C^*-algebra with a unique tracial state τ, and let $\mathcal{F} = \{\mathcal{H}_n\}$ be an $\mathcal{A}$-filtration. For a self-adjoint operator $A \in \mathcal{A}$ denote the spectral distribution by μ_A and let $\mu_{P_nA\lceil_{P_n\mathcal{H}}}$ be defined as in (9.1). Then*

$$\mu_{P_nA\lceil_{P_n\mathcal{H}}} \overset{\text{weak}^*}{\longrightarrow} \mu_A, \quad n \to \infty.$$

Our next goal is to prove an analogue of Theorem 9.5 for non-normal operators. But as there is no spectral distribution for non-normal operators we first need to introduce the Brown measure.

9.2 The Brown Measure

Let $\mathcal{M}$ be a finite von Neumann algebra of operators on $\mathcal{H}$ with a faithful, normal tracial state τ. Let $T \in \mathcal{M}$, then the Fuglede-Kadison determinant $\Delta(T)$ [30] is defined as

$$\Delta(T) = \exp\left(\int_0^\infty \log t \, d\mu_{|T|}(t)\right),$$

where

$$\mu_{|T|}(\omega) = \tau(E_{|T|}(\omega)), \qquad \omega \in \text{Borel}(\mathbb{R}),$$

and $E_{|T|}$ denotes the spectral projection measure corresponding to $|T|$. Now define

$$f(z) = \log(\Delta(T - z)), \qquad z \in \mathbb{C}. \tag{9.2}$$

It can be shown [31] that f is subharmonic and therefore gives rise to a measure (see Section 3 in [37])

$$d\mu_T = \frac{1}{2\pi}\nabla^2 f \, dm,$$

where m denotes the Lebesgue measure on $\mathbb{R}^2$ and $\nabla^2 f$ is understood to be in the distributional sense i.e. $\int \varphi \, d\mu_T = \frac{1}{2\pi}\int f \nabla^2\varphi \, dm$, for $\varphi \in C_c^\infty(\mathbb{R}^2)$. The measure μ_T satisfies $\text{supp}(\mu_T) \subset \sigma(T)$ and is often referred to as Brown's spectral distribution measure. Now the inclusion $\text{supp}(\mu_T) \subset \sigma(T)$ can be proper, but (by Remark 4.4 in [17]) if $\lambda \in \sigma(T)$ is isolated then $\mu_T(\{\lambda\}) \neq 0$. Thus, knowing μ_T would be a nice tool for locating isolated eigenvalues of T.

Note that if $T \in \mathcal{M}$ is normal, then $\mu_T = \tau \circ E_T$, and also, if $\mathcal{M} = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ then the Fuglede-Kadison determinant and the Brown measure is defined for $T \in \mathcal{M}$ and

$$\Delta(T) = |\text{det}T|^{\frac{1}{n}}, \qquad \mu_T = \frac{1}{n}\sum_{j=1}^n \delta_{\lambda_j},$$

where δ_{λ_j} denotes the point measure at λ_j and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of T, repeated according to multiplicity.

Our approach is to extend Arveson's ideas regarding approximating the spectral distribution of self-adjoint operators to Browns spectral distribution. Let $\mathcal{F}$ be a filtration with corresponding projections $\{P_n\}$, and define the tracial state

$$\tau_n(B) = \frac{1}{d_n}\text{trace}(P_nB), \quad B \in \mathcal{B}(\mathcal{H}), \quad d_n = \dim(P_n\mathcal{H}).$$

In order to approximate f defined in (9.2), it could be tempting to define, for $z \in \mathbb{C}$ and $T \in \mathcal{B}(\mathcal{H})$, a measure by

$$\mu_{|P_n(T-z)P_n|}(\omega) = \tau_n(E_{|P_n(T-z)P_n|}(\omega)), \qquad \omega \in \text{Borel}(\mathbb{R}),$$

but knowing how bad the spectrum of P_nTP_n may approximate $\sigma(T)$ when T is non-self-adjoint we abandon that idea immediately and instead define the measure $\mu_{T,z,n}$ by

$$\mu_{T,z,n}(\omega) = \tau_n(E_{P_n(T-z)^*(T-z)\lceil_{P_n\mathcal{H}}}(\omega)), \quad \omega \in \text{Borel}(\mathbb{R}). \tag{9.3}$$

Using this measure we obtain the following results.

Theorem 9.6. *Let $\mathcal{M}$ be a finite von Neumann algebra of operators on $\mathcal{H}$ with a faithful, normal tracial state τ. Suppose that $\mathcal{A} \subset \mathcal{M}$ is a C^*-algebra such that the restriction of τ to $\mathcal{A}$ is the unique tracial state on $\mathcal{A}$, and that $\{\mathcal{H}_n\}$ is an $\mathcal{A}$-filtration with corresponding projections $\{P_n\}$. Define the tracial state τ_n by*

$$\tau_n(B) = \frac{1}{d_n}\mathrm{trace}(P_n B), \quad B \in \mathcal{M}, \quad d_n = \dim(P_n\mathcal{H}).$$

For $z \in \mathbb{C}$ and $T \in \mathcal{A}$, define the measure $\mu_{T,z,n}$ as in (9.3). Let

$$f_n(z) = \frac{1}{2}\int_0^\infty \log t \, d\mu_{T,z,n}(t)$$

and $d\nu_n = \frac{1}{2\pi}\nabla^2 f_n dm$, where m is Lebesgue measure on $\mathbb{R}^2$. Then ν_n defines a positive Borel measure on $\mathbb{R}^2$ satisfying $\nu_n(\mathbb{C}) \leq 1$. Moreover, there exists a positive Borel measure ν on $\mathbb{R}^2$ with $\mathrm{supp}(\nu) \subset \sigma(T)$ and a subsequence $\{\nu_{n_k}\}$ such that

$$\nu_{n_k} \xrightarrow{\mathrm{weak}^*} \nu, \qquad k \to \infty.$$

Theorem 9.7. *Suppose the assumptions in Theorem 9.6 are true and that $T \in \mathcal{A}$.*

(i) Then, if $\rho : \mathbb{C} \to \mathbb{C}$ defined by

$$\rho(z) = \begin{cases} \log(1/\|(T-z)^{-1}\|) & z \in \mathbb{C} \setminus \sigma(T) \\ -\infty & z \in \sigma(T) \end{cases}$$

is locally integrable, the measure ν from Theorem 9.6 is equal to the Brown measure μ_T, and

$$\nu_n \xrightarrow{\mathrm{weak}^*} \mu_T, \qquad n \to \infty,$$

where ν_n is defined as in Theorem 9.6.

(ii) Suppose that $\omega \subset \mathbb{C}$ is an open set such that $\omega \cap \sigma(T) = \{\lambda_1, \dots \lambda_k\}$, where λ_j is an isolated eigenvalue. Suppose also that there is an $\alpha > 0$ such that

$$\inf_{z \in \partial D(\lambda_j, r)} 1/(\|(T-z)^{-1}\|) \geq r^\alpha$$

for all sufficiently small r, where $D(\lambda_j, r)$ denotes the disk with center λ_j and radius r. Then

$$\nu_n\lceil_\omega \xrightarrow{\mathrm{weak}^*} \mu_T\lceil_\omega, \qquad n \to \infty.$$

Proof. (**Proof of Theorem 9.6**) The proof will be done in several steps.

Step I. We first need to show that ν_n indeed is a positive Borel measure. To prove that, it suffices, by Lemma 3.6 and Section 3.5 in [37], to demonstrate that f_n is subharmonic. To do that, let $\epsilon > 0$ and define

$$g_{n,\epsilon}(z) = \frac{1}{2}\tau_n(\log(P_n(T-z)^*(T-z)P_n + \epsilon I)).$$

We claim that $g_{n,\epsilon}$ is subharmonic. The method we use here is quite close to the techniques used in [31]. Note that $g_{n,\epsilon}$ is infinitely smooth. Indeed, since

$$z \mapsto P_n(T-z)^*(T-z)P_n + \epsilon I$$

is obviously infinitely smooth and so is log on $\{z : \mathfrak{Re} z \geq \epsilon\}$ so

$$z \mapsto \mathrm{trace}\left(\log(P_n(T-z)^*(T-z)P_n + \epsilon I)\lceil_{P_n\mathcal{H}}\right)$$

is infinitely smooth, thus $g_{n,\epsilon}$ is infinitely smooth. Thus, we need to show that $\nabla^2 g_{n,\epsilon} = 0$. This we will do using brute force computations. Using the standard notation

$$\frac{\partial}{\partial\lambda} = \frac{1}{2}\left(\frac{\partial}{\partial\lambda_1} - i\frac{\partial}{\partial\lambda_2}\right) \quad \text{and} \quad \frac{\partial}{\partial\bar{\lambda}} = \frac{1}{2}\left(\frac{\partial}{\partial\lambda_1} + i\frac{\partial}{\partial\lambda_2}\right)$$

and letting $z = \lambda_1 + i\lambda_2$ we have

$$\nabla^2 g_{n,\epsilon} = \left(\frac{\partial^2}{\partial \lambda_1^2} + \frac{\partial^2}{\partial \lambda_2^2}\right) g_{n,\epsilon} = 4\frac{\partial^2}{\partial\bar{\lambda}\partial\lambda} g_{n,\epsilon}.$$

Let $\varphi(z) = P_n(T-z)^*(T-z)P_n + \epsilon I$. By the definition of the derivative, linearity and boundedness of τ_n we have that

$$\frac{\partial^2 g_{n,\epsilon}}{\partial\bar{\lambda}\partial\lambda} = \frac{1}{2}\frac{\partial^2 \tau_n(\log\circ\varphi)}{\partial\bar{\lambda}\partial\lambda} = \frac{1}{2}\tau_n\left(\frac{\partial^2 \log\circ\varphi}{\partial\bar{\lambda}\partial\lambda}\right)$$

so it is straightforward to show that

$$\begin{aligned}\frac{\partial^2 g_{n,\epsilon}}{\partial\bar{\lambda}\partial\lambda} &= \frac{1}{2}\tau_n\left(-\varphi^{-1}\frac{\partial\varphi}{\partial\bar{\lambda}}\varphi^{-1}\frac{\partial\varphi}{\partial\lambda} + \varphi^{-1}\frac{\partial^2\varphi}{\partial\bar{\lambda}\partial\lambda}\right)\\ &= \frac{1}{2}\tau_n\left(\varphi^{-1/2}\left(-\frac{\partial\varphi}{\partial\bar{\lambda}}\varphi^{-1}\frac{\partial\varphi}{\partial\lambda} + \frac{\partial^2\varphi}{\partial\bar{\lambda}\partial\lambda}\right)\varphi^{-1/2}\right).\end{aligned}$$

Thus, it suffices to show that $-\frac{\partial\varphi}{\partial\bar{\lambda}}\varphi^{-1}\frac{\partial\varphi}{\partial\lambda} + \frac{\partial^2\varphi}{\partial\bar{\lambda}\partial\lambda}$ is positive. Now,

$$\frac{\partial\varphi}{\partial\lambda} = -P_n(T-z)^*P_n, \qquad \frac{\partial\varphi}{\partial\bar{\lambda}} = -P_n(T-z)P_n, \qquad \frac{\partial^2\varphi}{\partial\bar{\lambda}\partial\lambda} = P_n.$$

Thus, we can compute

$$\begin{aligned}&-\frac{\partial\varphi}{\partial\bar{\lambda}}\varphi^{-1}\frac{\partial\varphi}{\partial\lambda} + \frac{\partial^2\varphi}{\partial\hat{\lambda}\partial\lambda}\\ &\quad= -P_n(T-z)P_n(P_n(T-z)^*(T-z)P_n + \epsilon I)^{-1}P_n(T-z)^*P_n + P_n\\ &\quad= -P_nB(B^*B+\epsilon I)^{-1}B^*P_n + P_n, \qquad\qquad B = (T-z)P_n\\ &\quad= -P_n((BB^*+\epsilon I)^{-1}BB^* + I)P_n\\ &\quad= -P_n(-\epsilon(BB^*+\epsilon I)^{-1})P_n\\ &\quad= \epsilon P_n((T-z)P_n(T-z)^* + \epsilon I)^{-1}P_n,\end{aligned} \tag{9.4}$$

which is clearly positive. Observe also that

$$f_n(z) = \frac{1}{2}\tau_n(\log(P_n(T-z)^*(T-z)P_n)) = \frac{1}{2}\int_0^\infty \log t\, d\mu_{T,z,n}(t)$$

and

$$g_{n,\epsilon}(z) = \frac{1}{2}\int_0^\infty \log(t+\epsilon)\, d\mu_{T,z,n}(t).$$

In particular $g_{n,\epsilon}$ decreases pointwise to f_n as $\epsilon \to 0$. Thus, f_n must be subharmonic or identically $-\infty$. But $f_n(z) > -\infty$ for $z \notin \sigma(T)$, and thus f_n must be subharmonic.

Step II. We will now show that $\nu_n(\mathbb{C}) \le 1$ for all n. Define

$$\psi_R(z) = \begin{cases}\log R & |z| \le 1\\ \log(\frac{R}{|z|}) & 1 < |z| < R\\ 0 & |z| \ge R.\end{cases}$$

Then, since $\frac{1}{\log R}\psi_R$ increases monotonically to 1, it follows by monotone convergence that

$$\nu_n(\mathbb{C}) = \lim_{R\to\infty}\int_{\mathbb{C}} \frac{1}{\log R}\psi_R\, d\nu_n.$$

Now, by Lemma 2.12 in [31] it is true that

$$\int_{\mathbb{C}} \frac{1}{\log R}\psi_R\, d\nu_n = \frac{1}{\log R}\left(\frac{1}{2\pi}(\int_0^{2\pi} f_n(Re^{i\theta})\, d\theta - \int_0^{2\pi} f_n(e^{i\theta})\, d\theta)\right).$$

Thus, it suffices to show that $\lim_{R\to\infty} \frac{1}{2\pi \log R}(\int_0^{2\pi} f_n(Re^{i\theta})\, d\theta \leq 1$. Now,

$$
\begin{aligned}
&\frac{1}{2}\pi \log R \int_0^{2\pi} f_n(Re^{i\theta})\, d\theta \\
&\qquad = \frac{1}{4\pi \log R} \int_0^{2\pi} \tau_n(\log(|P_n(T - Re^{i\theta})^*(T - Re^{i\theta})P_n|))\, d\theta \\
&\qquad \leq \frac{1}{2\log R} \|\tau_n\| \log(\sup_{\theta\in[0,2\pi]} \||P_n(T - Re^{i\theta})^*(T - Re^{i\theta})P_n|\|) \\
&\qquad \leq \frac{1}{2\log R} \log((\|T\| + R)^2) \longrightarrow 1, \quad R \to \infty.
\end{aligned}
$$

Step III. The existence of ν now follows from the weak* compactness of the unit ball of $C_0(\mathbb{C})^*$ since we have proved in Step II that $\{\nu_n\}$ is uniformly bounded as elements in $C_0(\mathbb{C})^*$.

We are left with the task of proving that

$$
\operatorname{supp}(\nu) \subset \sigma(T), \tag{9.5}
$$

and this will be done in Step IV and V.

Step IV. We will show that $f_n(z) \to f(z)$ when $z \notin \sigma(T)$ and f is defined in (9.2). To prove that we need to demonstrate that

$$
\lim_{n\to\infty} \frac{1}{2} \int_0^\infty \log t\, d\mu_{T,z,n}(t) = \int_0^\infty \log t\, d\mu_{|(T-z)|}(t), \quad z \notin \sigma(T). \tag{9.6}
$$

Before we can prove (9.6) we need the following observation. Note that since $z \notin \sigma(T)$ then there is an $\epsilon > 0$ and $M < \infty$ such that

$$
\sigma(|T - z|^2) \subset [\epsilon, M], \qquad \sigma(P_n(T - z)^*(T - z)\lceil_{P_n\mathcal{H}}) \subset [\epsilon, M]. \tag{9.7}
$$

Indeed, letting

$$
\epsilon = (\inf_{\|\xi\|=1,\xi\in\mathcal{H}} \langle (T - z)^*(T - z)\xi, \xi\rangle)^{1/2}
$$

and

$$
\epsilon_n = (\inf_{\|\xi\|=1,\xi\in\mathcal{H}} \langle (P_n(T - z)^*(T - z)P_n\xi, \xi\rangle)^{1/2}
$$

then $\sigma(|T - z|) \subset [\epsilon, \infty)$ and $\sigma(P_n(T - z)^*(T - z)P_n) \subset [\epsilon_n, \infty)$ so

$$
\begin{aligned}
\mu_{|T-z|}([0, \epsilon)) &= \tau(E_{|T-z|}([0, \epsilon))) = 0 \\
\mu_{T,z,n}([0, \epsilon)) &= \tau_n(E_{|P_n(T-z)^*(T-z)P_n}([0, \epsilon_n))) = 0,
\end{aligned}
$$

since $(E_{|T-z|}([0, \epsilon)) = E_{P_n(T-z)^*(T-z)P_n}([0, \epsilon_n)) = 0$. Also,

$$
\begin{aligned}
\epsilon_n &= (\inf_{\|\xi\|=1,\xi\in\mathcal{H}} \langle (P_n(T - z)^*(T - z)P_n\xi, \xi\rangle)^{1/2} \\
&= (\inf_{\|\xi\|=1,\xi\in\mathcal{H}_n} \langle (T - z)^*(T - z)\xi, \xi\rangle)^{1/2} \\
&\geq (\inf_{\|\xi\|=1,\xi\in\mathcal{H}} \langle (T - z)^*(T - z)\xi, \xi\rangle)^{1/2} \\
&= \epsilon.
\end{aligned}
$$

Thus, since

$$
\epsilon = (\inf_{\|\xi\|=1,\xi\in\mathcal{H}} \langle (T - z)^*(T - z)\xi, \xi\rangle)^{1/2} = 1/\|(T - z)^{-1}\| > 0
$$

and T is bounded then (9.7) follows. We can now return to the task of proving (9.6). Now, using (9.7), we have that

$$
\begin{aligned}
f_n(z) &= \frac{1}{2} \int_0^\infty \log t\, d\mu_{T,z,n}(t) = \tau_n(\chi_{[\epsilon,M]} \log \circ g(P_n(T - z)^*(T - z)\lceil_{P_n\mathcal{H}})) \\
f(z) &= \int_0^\infty \log t\, d\mu_{|T-z|}(t) = \tau(\chi_{[\epsilon,M]} \log \circ g((T - z)^*(T - z))),
\end{aligned}
$$

where $g(t) = \sqrt{t}$, $t \in [0, \infty)$. Thus, we are left with the task of showing that

$$\lim_{n\to\infty} \tau_n((\chi_{[\epsilon,M]} \log \circ g(P_n(T-z)^*(T-z)\lceil_{P_n\mathcal{H}})) = \tau((\chi_{[\epsilon,M]} \log \circ g)((T-z)^*(T-z))).$$

But, by the uniqueness of τ and Proposition 9.3 we have that

$$\lim_{n\to\infty} \tau_n(B) = \tau(B), \qquad B \in \mathcal{A},$$

thus our problem is reduced to showing

$$\begin{aligned}\lim_{n\to\infty} |\tau_n((\chi_{[\epsilon,M]} \log \circ\, g((T-z)^*(T-z))) \\ - \tau_n((\chi_{[\epsilon,M]} \log \circ g)(P_n(T-z)^*(T-z)P_n))| = 0.\end{aligned} \tag{9.8}$$

Thus, by the fact that the norm of the linear functionals

$$\begin{aligned} f \in C[\epsilon, M] \mapsto &\tau_n(f((T-z)^*(T-z))) \\ &- \tau_n(f((P_n(T-z)^*(T-z)P_n)^*(P_n(T-z)^*(T-z)P_n))) \end{aligned}$$

is bounded by 2, the Stone-Weierstrass Theorem, (9.7) and linearity of τ_n it is true that (9.8) follows if we can show that

$$\lim_{n\to\infty} |\tau_n(((T-z)^*(T-z))^p) - \tau_n(((P_n(T-z)P_n)^*(P_n(T-z)P_n))^p)| = 0$$

for $p = 1, 2, \ldots$. Also, since the sequence of p-linear forms

$$B_n(T_1, T_2, \ldots, T_{2p}) = \tau_n(T_1T_2\cdots T_{2n}) - \tau_n(P_nT_1P_nT_2P_n\cdots P_nT_{2n}), \quad T_j \in \mathcal{A}$$

is uniformly bounded (by 2) we may assume that T and T^* have finite degree. By Proposition 9.4 we have that

$$\begin{aligned}&|\tau_n(((T-z)^*(T-z))^p) - \tau_n(((P_n(T-z)P_n)^*(P_n(T-z)P_n))^p)| \\ &\qquad \le \|T-z\|^p \|(T-z)^*\|^p \frac{1}{d_n} p(\deg(T) + \deg(T^*)) \longrightarrow 0, \quad n \to \infty,\end{aligned}$$

where $d_n = \dim(\mathcal{H}_n)$, and thus we have shown Step IV.

Step V. We claim that

$$\int_{\mathbb{R}^2} f_n \nabla^2 \varphi \, dm \longrightarrow \int_{\mathbb{R}^2} f \nabla^2 \varphi \, dm, \quad n \to \infty, \quad \varphi \in C_c^\infty, \tag{9.9}$$

when $\operatorname{supp}(\varphi) \subset \mathbb{C} \setminus \sigma(T)$. Let $\delta > 0$ and

$$\Omega_\delta = \{z \in \mathbb{C} : \operatorname{dist}(z, \sigma(T)) \le \delta\}.$$

We claim that there is a constant $C > -\infty$ such that

$$\inf\{f_n(z) : z \in \mathbb{C} \setminus \Omega_\delta\} \ge C. \tag{9.10}$$

Indeed, this is the case. Firstly, observe that for $z \notin \sigma(T)$ it follows that

$$f_n(z) \ge \frac{1}{2} \int_0^1 \log t \, d\mu_{T,z,n}(t),$$

thus (9.10) will follow if we can show that there is an $\epsilon > 0$ such that

$$\operatorname{supp}(\mu_{T,z,n}) \subset [\epsilon, \infty) \quad \text{for all} \quad z \in \mathbb{C} \setminus \Omega_\delta.$$

Secondly, note that

$$\inf\{1/\|(T-z)^{-1}\| : z \in \mathbb{C} \setminus \Omega_\delta\} > 0.$$

So let

$$\epsilon = \inf_{z\in\mathbb{C}\setminus\Omega_\delta}\Big(\inf_{\|\xi\|=1,\xi\in\mathcal{H}}\langle (T-z)^*(T-z)\xi,\xi\rangle\Big)^{1/2} = \inf\{1/\|(T-z)^{-1}\| : z\in\mathbb{C}\setminus\Omega_\delta\}.$$

Then, as argued in Step IV, we have that

$$\mu_{T,z,n}([0,\epsilon)) = \tau_n(E_{P_n(T-z)^*(T-z)P_n}([0,\epsilon_n))) = 0,$$

since $\sigma(|P_n(T-z)^*(T-z)P_n|)\subset[\epsilon_n,\infty)$, where

$$\epsilon_n = \inf_{z\in\mathbb{C}\setminus\Omega_\delta}\Big(\inf_{\|\xi\|=1,\xi\in\mathcal{H}}\langle (P_n(T-z)^*(T-z)P_n)\xi,\xi\rangle\Big)^{1/2}\geq\epsilon$$

Pick $\delta>0$ so small that $\mathrm{supp}(\varphi)\subset\mathbb{C}\setminus\Omega_\delta$. Let

$$g(z) = \begin{cases}\inf\{f_n(z) : z\in\mathbb{C}\setminus\Omega_\delta\} & z\in\mathbb{C}\setminus\Omega_\delta\\ 0 & z\in\Omega_\delta.\end{cases}$$

Then, by the reasoning above, g is integrable and dominates $\{f_n\}$ from below. Hence, (9.9) follows by Step IV and dominated convergence.

Note that (9.5) follows from Step V and the fact that $\mathrm{supp}(\mu_T)\subset\sigma(T)$, and thus we have proved the theorem. □

Proof. (**Proof of Theorem 9.7**) To prove (i) we need to show that

$$\int_{\mathbb{R}^2} f_n\nabla^2\varphi\,dm \longrightarrow \int_{\mathbb{R}^2} f\nabla^2\varphi\,dm,\quad n\to\infty,\quad \varphi\in C_c^\infty, \tag{9.11}$$

where f is defined in (9.2). Now, for $z\notin\sigma(T)$ we have

$$\begin{aligned}
f_n(z) &\geq \inf_{n\in\mathbb{N}}\tau_n(\log(P_n(T-z)^*(T-z)\lceil_{P_n\mathcal{H}}))\\
&= \frac{1}{d_n}\sum_j^{d_n}\lambda_j(\log(P_n(T-z)^*(T-z)\lceil_{P_n\mathcal{H}}))\\
&= \frac{1}{d_n}\sum_j^{d_n}\log(\lambda_j(P_n(T-z)^*(T-z)\lceil_{P_n\mathcal{H}}))\\
&\geq \frac{1}{d_n}\sum_j^{d_n}\log(\min_{j\in\{1,\dots,d_n\}}\{\lambda_j(P_n(T-z)^*(T-z)\lceil_{P_n\mathcal{H}}\}))\\
&= \log((\inf_{\|\xi\|=1,\xi\in\mathcal{H}}\langle P_n(T-z)^*(T-z)P_n\xi,\xi\rangle)^{1/2})\\
&\geq \log((\inf_{\|\xi\|=1,\xi\in\mathcal{H}}\langle (T-z)^*(T-z)\xi,\xi\rangle)^{1/2})\\
&= \log(1/\|(T-z)^{-1}\|),
\end{aligned}$$

where $d_n=\dim(\mathcal{H}_n)$ and $\lambda_j(B)$ denotes the j-th eigenvalue of $B\in\mathcal{B}(\mathcal{H}_n)$ according to some ordering, where the eigenvalues of B are repeated according to multiplicity (obviously, the ordering is irrelevant in this context). Hence, f_n is dominated from below by ρ and since ρ is integrable, (9.11) follows by Step IV in the proof of Theorem 9.6 and dominated convergence.

Now (ii) follows by noting that $z\mapsto\log(|z|^\alpha)$ is locally integrable and arguing as in the proof of (i) using dominated convergence. □

10 Examples

In this section we will demonstrate that the abstract framework developed in the previous sections is indeed applicable in actual computations. For more examples see [34].

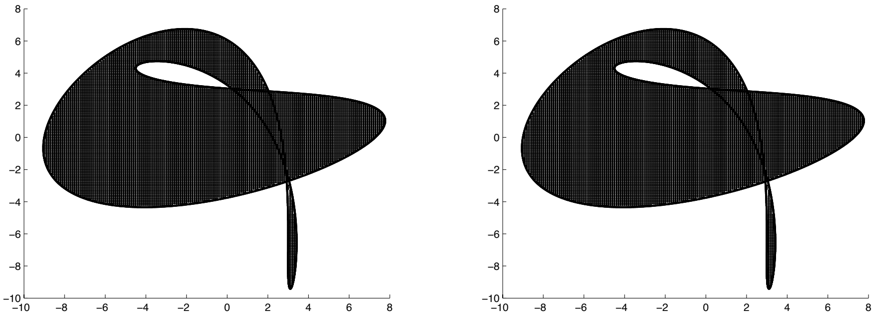

Figure 1: The left figure shows $\Gamma_k(\{x_{ij}\})$ with $n = 1$, $\epsilon = 0.005$ and $k = 1500$, where $\Gamma_k(\{x_{ij}\}$ is defined in (10.2). The right figure shows $\sigma(T)$, where T is the Toeplitz operator with symbol defined in (10.1).

10.1 Toeplitz Operators

Toeplitz operators are excellent test objects in computational spectral theory since their spectral theory is very well understood. In particular, the spectral theory of banded Toeplitz operators is completely understood [13, 12] and determined by the symbols of the operators. Recall from Theorem 3.6 that the Solvability Complexity Index for the n-pseudospectra of banded operators is equal to one. Recall also that the constructive proof of Theorem 3.6 gives us a numerical algorithm for computing n-pseudospectra. In particular, if T is a Toeplitz operator with symbol

$$f(z) = 2iz^{-3} + 5z^{-2} + 2iz - 2iz^2 + 2z^3, \tag{10.1}$$

$x_{ij} = \langle Te_j, e_i\rangle$, $i, j \in \mathbb{N}$ and $\epsilon > 0$ we can, for integers m, k, n, define $T_{\epsilon,m,k}(z) = T_{m,k}(z) - \epsilon^{2^{n+1}} I$ and $\widetilde{T}_{\epsilon,m,k}(z) = \widetilde{T}_{m,k}(z) - \epsilon^{2^{n+1}} I$, where $T_{m,k}(z)$ and $\widetilde{T}_{m,k}(z)$ are defined in (6.2) and $z \in \mathbb{C}$. Let, for $k \in \mathbb{N}$, Θ_k be defined as in (6.3) and

$$\begin{aligned} \Gamma_k(\{x_{ij}\}) = \{z \in \Theta_k : \nexists L \in LT_{\text{pos}}(P_k\mathcal{H}), T_{\epsilon,k,2^n d+k}(z) = LL^*\} \\ \cup \{z \in \Theta_k : \nexists L \in LT_{\text{pos}}(P_k\mathcal{H}), \widetilde{T}_{\epsilon,k,2^n d+k}(z) = LL^*\}, \end{aligned} \tag{10.2}$$

where $LT_{\text{pos}}(P_m\mathcal{H})$ denotes the set of lower triangular matrices in $\mathcal{B}(P_m\mathcal{H})$ (with respect to $\{e_j\}$) with strictly positive diagonal elements and d is the bandwith of T (in this case $d = 3$), then

$$\lim_{k\to\infty} \Gamma_k(\{x_{ij}\}) = \overline{\sigma_{n,\epsilon}(T)}.$$

In Figure 1 we have plotted $\Gamma_k(\{x_{ij}\})$ for $k = 1500$, $n = 1$ (we have computed an approximation to the 1-pseudospectrum) and $\epsilon = 0.005$ together with the spectrum of T. One observes that at least up to the resolution of the image, the two plots are indistinguishable.

10.2 The Fourier Transform

Another test example is the Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R})$. The spectrum of $\mathcal{F}$ is of course $\sigma(\mathcal{F}) = \{1, -1, i, -i\}$. In this example we have chosen a basis for $L^2(\mathbb{R})$ by first considering a basic Gabor basis, namely, a basis of the form

$$e^{2\pi imx}\chi_{[0,1]}(x - n), \qquad m, n \in \mathbb{Z},$$

(where χ is the characteristic function) and then chosen some enumeration of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{N}$ to obtain a basis $\{\varphi_j\}$ that is just indexed over $\mathbb{N}$. Letting T be the infinite matrix defined by $T_{ij} = \langle \mathcal{F}\varphi_j, \varphi_i\rangle$, we can apply the techniques from the constructive proof of Theorem 3.4 to find a set of estimating functions for the spectrum of $\mathcal{F}$. In particular, by letting $\{x_{ij}\}$ denote the matrix elements of T and by recalling (6.4) we know that for $n \in \mathbb{Z}_+$ and $\epsilon > 0$ we get that

$$\overline{\sigma_{n,\epsilon}(\mathcal{F})} = \lim_{n_1\to\infty} \lim_{n_2\to\infty} \Gamma_{n_1,n_2}(\{x_{ij}\}),$$

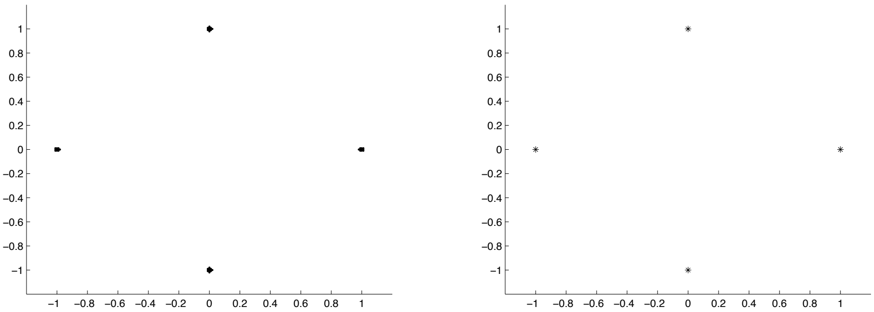

Figure 2: The left figure shows $\Gamma_{n_1,n_2}(\{x_{ij}\})$ with $n = 1$,$\epsilon = 0.02$, $n_1 = 500$ and $n_2 = 1300$ where $\Gamma_{n_1,n_2}(\{x_{ij}\})$ is defined in (10.3). The right figure shows the spectrum of the Fourier transform.

where

$$\begin{aligned}\Gamma_{n_1,n_2}(\{x_{ij}\}) = \{z \in \Theta_{n_2} : \nexists L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), T_{\epsilon,n_1,n_2}(z) = LL^*\} \\ \cup \{z \in \Theta_{n_2} : \nexists L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}), \widetilde{T}_{\epsilon,n_1,n_2}(z) = LL^*\},\end{aligned} \tag{10.3}$$

and

$$\begin{aligned}T_{\epsilon,n_1,n_2}(z) &= T_{n_1,n_2}(z) - \epsilon^{2^{n+1}} I, \\ \widetilde{T}_{\epsilon,n_1,n_2}(z) &= \widetilde{T}_{n_1,n_2}(z) - \epsilon^{2^{n+1}} I,\end{aligned}$$

Θ_k is defined as in (6.3) and T_{n_1,n_2}, $\widetilde{T}_{n_1,n_2}$ are defined in (6.2). In Figure 2 we have plotted $\Gamma_{n_1,n_2}(\{x_{ij}\})$ for $n = 1$, $\epsilon = 0.02$, $n_1 = 500$ and $n_2 = 1300$ together with the spectrum of $\mathcal{F}$.

10.3 The Operator $\Phi(Q)$ for $\Phi \in L^\infty(\mathbb{R})$

When constructing other examples, the functional calculus and the spectral mapping theorem come in handy. By defining Q on $L^2(\mathbb{R})$ (on its appropriate domain) by $(Qf)(x) = xf(x)$ we obviously have that $\sigma(\Phi(Q)) = \operatorname{ess\,ran}(\Phi)$ (the essential range) for $\Phi \in L^\infty(\mathbb{R})$. In this example we let

$$\Phi(x) = e^{ix}\chi_{[-\pi/2,\pi/2]}(x), \qquad x \in \mathbb{R},$$

where χ is the characteristic function. Then the spectrum is obviously

$$\sigma(\Phi(Q)) = \{z \in \mathbb{C} : \mathfrak{Re}(z) \geq 0,\, |z| = 1\} \cup \{0\}.$$

In this example we have chosen a basis for $L^2(\mathbb{R})$ by first choosing a basis $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ for $j,k \in \mathbb{Z}$ and ψ is the Haar wavelet, and then some enumeration of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{N}$ to obtain a basis $\{\varphi_j\}$ that is just indexed over $\mathbb{N}$. Letting T be the infinite matrix defined by $T_{ij} = \langle \Phi(Q)\varphi_j, \varphi_i \rangle$ and $\{x_{ij}\}$ denote the matrix elements of T we can use $\Gamma_{n_1,n_2}(\{x_{ij}\})$ from (10.3) exactly as in the previous example. In Figure 3 we have plotted $\Gamma_{n_1,n_2}(\{x_{ij}\})$ with $\epsilon = 0.06$, $n = 2$ and $n_2 = 8000$, $n_1 = 1600$ as well as $\omega_\epsilon(\sigma(\Phi(Q)))$ (the ϵ-neighborhood of the spectrum).

10.4 The Residual Pseudospectrum

In this final example we recall the computational tool (residual pseudospectra from Section 7) for estimating the spectrum both from "above" and "below", meaning that for $T \in \mathcal{B}(\mathcal{H})$ we have

$$\sigma_{\mathrm{res},\epsilon}(T) \cup \sigma_{\mathrm{res}^*,\epsilon}(T) \subset \sigma(T) \subset \sigma_{n,\epsilon}(T). \tag{10.4}$$

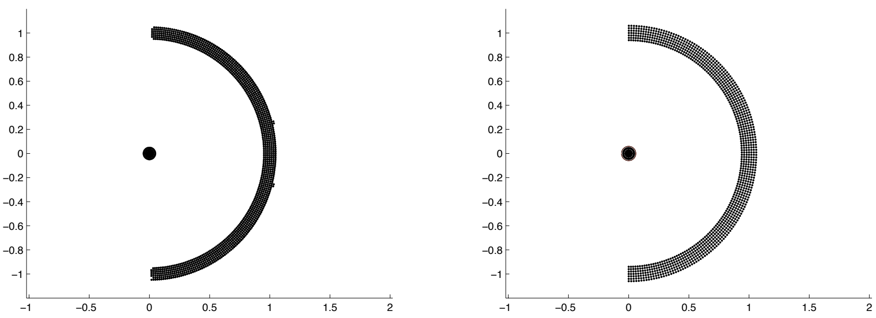

Figure 3: The left figure shows $\Gamma_{n_1,n_2}(\{x_{ij}\})$ with $\epsilon = 0.06$, $n = 2$ and $n_2 = 8000$, $n_1 = 1600$. The right figure shows $\omega_\epsilon(\sigma(\Phi(Q)))$.

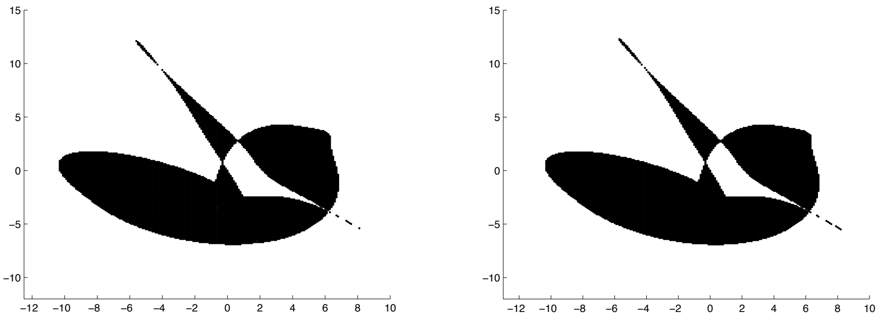

Figure 4: The figure shows $\sigma_{\mathrm{res},\epsilon}(T) \cup \sigma_{\mathrm{res}^*,\epsilon}(T)$ (left) and $\sigma_\epsilon(T)$ for $\epsilon = 0.01$ (right).

Consider the infinite matrix

$$T = \begin{pmatrix} 0 & a & b & c & 0 & 0 & \dots \\ d & 0 & a & b & c & 0 & \dots \\ f & e & 0 & a & b & c & \dots \\ g & f & d & 0 & a & b & \dots \\ 0 & g & f & e & 0 & a & \dots \\ 0 & 0 & g & f & d & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a = 1+2i$, $b = -1$, $c = 5+i$, $d = -2$, $e = 1+2i$, $f = -4$, $g = -1-2i$. In Figure 4 we have plotted $\sigma_{\mathrm{res},\epsilon}(T) \cup \sigma_{\mathrm{res}^*,\epsilon}(T)$ and $\sigma_\epsilon(T)$ for $\epsilon = 0.01$ (the computational techniques used are the ones from Section 7)). In view of (10.4) one observes that this computation gives a rather precise estimate of the spectrum of T.

11 Concluding Remarks

We have shown that it is possible to construct/compute spectra of arbitrary linear operators from the matrix elements, and the Solvability Complexity Index has been introduced as a tool for determining how complex such a construction may be. The first question that arises is then: What is the Solvability Complexity Index for spectra of different classes of operators? We have so far only presented upper bounds, and this suggests that the theory is far from complete. Let us for simplicity consider bounded operators. Could it be that the Solvability Complexity Index for the spectrum, when considering all bounded operators, is actually one? This cannot be ruled out, although, we strongly believe that this is not the case. However, suppose for a moment that it is indeed one, what would that mean? That means that there exists an algorithm that could handle all (bounded) spectral problems and it would require just one limit. If one could give a constructive proof and actually display such an algorithms, that would be a spectacular result.

Although desirable, such a spectacular outcome seems a little too good to be true. Note also that a trained eye will immediately spot that any attempts on reducing the bound on the Solvability Complexity Index by clever use of subsequences, with the type of estimating functions used in this paper, is doomed to fail.

We would like to emphasize that the definition of the Solvability Complexity Index in this paper is a first attempt to shed light on the rather intricate general computational spectral problem, and further thoughts towards a deeper understanding of this and related notions should be the subject of future work.

As a motivation for future work on the Solvability Complexity Index we mention that better bounds than three may be established for certain subclasses of operators (we have already seen this in Theorem 3.6). We may consider for example Toeplitz operators with continuous symbols. For this subclass of operators the Solvability Complexity Index of the spectrum is equal to one. We will sketch the ideas. Let T be a Toeplitz operator with continuous symbol. Equip the complex plane with a grid of step size $1/n$ and consider the n-th partial sum $S_n a(z)$ of the symbol $a(z)$ of the operator T. Evaluating $S_n a(z)$ at the n-th roots of unity gives n points in the plane, and one can in finitely many steps find all points on the grid whose winding number with respect to the piecewise linear curve determined by the n points is nonzero (this requires an argument, but is fairly straightforward). Denoting this set by Ω_n, and then observing that $\Omega_n \to \sigma(T)$ yields the result.

Similarly, one can show that the Solvability Complexity Index of the pseudospectra of Toeplitz operators with continuous symbols is one. This is done by letting P_n be the usual projection onto the span of the first n basis elements, and then using the fact (from [9]) that $\sigma_\epsilon(P_n T\lceil_{P_n\mathcal{H}}) \to \sigma_\epsilon(T)$. A similar argument holds for compact operators, however, we omit the details as more careful analysis of these questions will appear in future papers.

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