Stable nonuniform sampling with weighted Fourier frames and recovery in arbitrary spaces

Ben Adcock
Simon Fraser University
Email: ben_adcock@sfu.ca

Milana Gataric
University of Cambridge
Email: m.gataric@maths.cam.ac.uk

Anders C. Hansen
University of Cambridge
Email: a.hansen@damtp.cam.ac.uk

***Abstract*—We present recently devised approach for recovery of compactly supported multivariate functions from nonuniform samples of their Fourier transforms. This is the so-called nonuniform generalized sampling (NUGS), based on a generalized sampling framework which permits an arbitrary choice of the reconstruction space and where nonuniform sampling is modeled via weighted Fourier frames. We establish a sharp sampling density which is sufficient to guarantee stable recovery, without imposing any separation condition on the sampling points. In particular for the stable NUGS recovery, we also provide sufficient sampling bandwidths in the case of one-dimensional wavelet reconstructions and show sufficient linear scaling of the sampling bandwidth and the number of wavelets.**

I. Introduction

Let $E \subseteq \mathbb{R}^d$ be a compact set in the space domain and H a Hilbert space of L^2-functions with a support in E. Let $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the standard L^2-norm and L^2-inner product, respectively. For a point in the frequency domain $\omega \in \hat{\mathbb{R}}^d$, define

$$e_\omega(x) = \mathrm{e}^{\mathrm{i}2\pi\omega\cdot x}\chi_E(x), \quad x \in \mathbb{R}^d,$$

and the Fourier transform of $f \in \mathrm{H}$ at $\omega \in \hat{\mathbb{R}}^d$ by $\hat{f}(\omega) = \langle f, e_\omega \rangle$. The problem we address is that of recovering an unknown function $f \in \mathrm{H}$ given the set of samples

$$\{\hat{f}(\omega) : \omega \in \Omega\},$$

where $\Omega \in \hat{\mathbb{R}}^d$ is a countable set of sampling points not necessarily taken on a Cartesian grid.

The problem of function recovery from a finite set of pointwise measurements of its Fourier transform is ubiquitous in applications such as Magnetic Resonance Imaging (MRI), Computed Tomography (CT), seismology and electron microscopy, just to name a few. In these applications, Fourier samples are often acquired along a non-Cartesian sampling pattern such as spirals or radial lines. In contrast to Cartesian sampling which leans on the fundamental Nyquist–Shannon theorem, nonuniform sampling is typically studied within the concept of Fourier frames, see [1] and references therein. The set $\{e_\omega : \omega \in \Omega\}$ forms a Fourier frame for H if there exist constants $A, B > 0$ such that for all $f \in \mathrm{H}$

$$A\|f\|^2 \leq \sum_{\omega\in\Omega} |\hat{f}(\omega)|^2 \leq B\|f\|^2.$$

We refer to such a system as a *classical* Fourier frame.

If $\{e_\omega : \omega \in \Omega\}$ is a classical Fourier frame, then Ω necessarily cannot have a clustering point, i.e. Ω must be separated [2]. However, once a spiral or radial sampling trajectory is used, as the number of sampling points increases, clustering points at lower frequencies appear and as a result the upper frame bound B blows up. To compensate for arbitrary clustering of sampling points, which often needs to be facilitated in practice, it is common to use weights, also known as density compensation factors. For some weights $\mu_\omega > 0$, the set $\left\{\sqrt{\mu_\omega}e_\omega : \omega \in \Omega\right\}$ forms a *weighted* Fourier frame for H if there exist constants $A, B > 0$ such that for all $f \in \mathrm{H}$

$$A\|f\|^2 \leq \sum_{\omega\in\Omega} \mu_\omega |\hat{f}(\omega)|^2 \leq B\|f\|^2.$$

Weighted Fourier frames allow stable sampling with a substantially more general type of nonuniform sampling pattern.

Provided one has a weighted or a classical Fourier frame, stable function recovery is possible and can be carried out via a number of different algorithms [1]. Therefore, it is crucial to understand conditions under which a set of sampling points is guaranteed to give rise to a Fourier frame. In one dimension, there is an almost full characterization of classical Fourier frames in terms of Beurling density and separation of the sampling points, mainly due to work of Beurling, Landau, Jaffard and Seip, see the review paper [3]. In higher dimensions, however, characterization of Fourier frames complicates considerably. A classical result due to Beurling [4] provides a Nyquist criterion for nonuniform sampling of L^2-functions supported in the unit Euclidean ball in terms of density and separation of the sampling points. Unfortunately, the separation condition prevents a consideration of general sampling patterns, such as those mentioned above.

The separation of the sampling points rising to a classical Fourier frame is assumed only to control the upper frame bound. However, if it does not hold, the Fourier samples need to be appropriately weighted and once nontrivial weights $\mu_\omega > 0$ are introduced, existence of a lower frame bound is no longer guaranteed by Beurling's result. Nevertheless, in the first result of this paper, we demonstrate how the separation condition from Beurling's result can be successfully removed by using weights corresponding to the volumes of the Voronoi cells of the sampling points.

Weighted Fourier frames have been used previously in Gröchenig's work [5] where he provides a sufficient sampling density without necessity for the separation condition. Moreover, he provides explicit frame bounds. However, these bounds and the density condition cease to be sharp in higher dimensions. We improve this in two ways: our first above mentioned result provides the universal sampling density, and our second result improves explicit estimates for the frame bounds, which in certain cases are also dimension independent.

Once Fourier frames are characterized and conditions for stable reconstruction are provided, it is important to construct a good approximation to f from finite Fourier data. In this paper we use the approach of generalized sampling (GS), where the sampling system is described by weighted Fourier frames and recovery is carried out in an arbitrary reconstruction space. This is the so-called nonuniform generalized sampling (NUGS), developed by the authors in [6]. NUGS is essentially a special instance of GS, a more general approach of sampling and reconstructing in abstract Hilbert spaces introduced by Adcock and Hansen [7]. The major advantage of this approach is free choice of the reconstruction space T which can be tailored to a specific application by using a favorable space.

Among popular methods for reconstruction from nonuniform Fourier data, one can find gridding [8], iterative techniques [9], the ACT algorithm [10] and methods based on inversion of the frame operator [11]. NUGS is closest to the efficient algorithm from the recent work of Guerquin–Kern, Häberlin, Pruessmann and Unser [12] where the equivalent model is considered for wavelet reconstruction spaces. However, NUGS is a more general framework for reconstruction in arbitrary finite-dimensional spaces together with guarantees for its convergence and robustness leaning on the theory of weighed Fourier frames.

II. Weighted Fourier frames

A classical result due to Beurling [4] provides a sufficient condition for sampling points to give rise to a Fourier frame for the space H of L^2-functions supported in the unit Euclidean ball. Namely, if a countable set $\Omega \subseteq \hat{\mathbb{R}}^d$ is separated, i.e. if there exists $\eta > 0$ such that

$$\forall \omega, \lambda \in \Omega, \quad \omega \neq \lambda, \quad |\omega - \lambda| \geq \eta,$$

and also if Ω satisfies the following density condition

$$\delta = \sup_{z \in \hat{\mathbb{R}}^d} \inf_{\omega \in \Omega} |\omega - z| < \frac{1}{4},$$

then the family of functions $\{e_\omega : \omega \in \Omega\}$ is a Fourier frame for H. Moreover, if $\delta \geq 1/4$ then a lower frame bound does not necessarily exist, implying the sharpness of the density bound. This was generalized to arbitrary compact, convex and symmetric supports $E \subseteq \mathbb{R}^d$ by Benedetto and Wu [13] and also by Olevskii and Ulanovskii [14].

In [5] (see also [15]), Gröchenig provides a sufficient sampling density in order to have a weighted Fourier frame for L^2-functions supported in the unit cube. Due to use of weights, unlike Beurling's result, Gröchenig's result does not assume the separation condition. The weights are defined as measures of Voronoi regions with respect to the Euclidean norm, namely for $\omega \in \Omega$, $\mu_\omega = \int_{\hat{\mathbb{R}}^d} \chi_{V_\omega}(x) \,\mathrm{d}x$, where

$$V_\omega = \left\{ z \in \hat{\mathbb{R}}^d \ : \ \forall \lambda \in \Omega, \ \lambda \neq \omega, \ |z - \omega| \leq |z - \lambda| \right\}.$$

Although in one dimension the density condition is the same as Beurling's and therefore sharp, in higher dimensions it reads

$$\delta = \sup_{z \in \hat{\mathbb{R}}^d} \inf_{\omega \in \Omega} |\omega - z| < \frac{\log 2}{2\pi d}. \tag{II.1}$$

However, Gröchenig additionally provides explicit estimates for the corresponding frame bounds

$$A \geq 2 - \mathrm{e}^{2\pi\delta d}, \quad B \leq \mathrm{e}^{2\pi\delta d},$$

but which unfortunately also deteriorate with dimension.

By using Voronoi weights, our first result removes the separation condition from Beurling's result while it keeps the sampling density sharp.

Theorem II.1. [16] *Let $\mathrm{H} = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \operatorname{supp}(f) \subseteq E\}$, where $E \subseteq \mathbb{R}^d$ is compact, convex and symmetric. If a countable set $\Omega \subseteq \hat{\mathbb{R}}^d$ has density*

$$\delta_{E^\circ} = \sup_{z \in \hat{\mathbb{R}}^d} \inf_{\omega \in \Omega} |\omega - z|_{E^\circ} < \frac{1}{4} \tag{II.2}$$

where E° is the polar set of E, then $\{\sqrt{\mu_\omega} e_\omega\}_{\omega \in \Omega}$ is a weighted Fourier frame for H with the weights $\mu_\omega > 0$ defined as the measures of Voronoi regions with respect to $|\cdot|_{E^\circ}$.

Although Theorem II.1 indeed guarantees frames under the sharp density condition (II.2), note that, the same as Beurling's result, it does not provide explicit frame bounds, which can be useful in stability and convergence analysis of a given reconstruction algorithm. Employing very similar techniques used to derive explicit frame bounds for weighted Fourier frames in [5], our next theorem gives the frame bounds under an improved density condition compared to the one given by Gröchenig's result.

Theorem II.2. [16] *Let $\mathrm{H} = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \operatorname{supp}(f) \subseteq E\}$, where $E \subseteq \mathbb{R}^d$ is compact. Suppose that $|\cdot|_*$ is an arbitrary norm on $\mathbb{R}^d$ and $c^* > 0$ the smallest constant such that $|\cdot| \leq c^* |\cdot|_*$, where $|\cdot|$ denotes the Euclidean norm. Let $\Omega \subseteq \mathbb{R}^d$ be such that*

$$\delta_* = \sup_{z \in \hat{\mathbb{R}}^d} \inf_{\omega \in \Omega} |\omega - z|_* < \frac{\log(2)}{2\pi m_E c^*}, \tag{II.3}$$

where $m_E = \sup_{x \in E} |x|$. Then $\{\sqrt{\mu_\omega} e_\omega\}_{\omega \in \Omega}$ is a weighted Fourier frame for H with the weights defined as the measures of Voronoi regions with respect to the norm $|\cdot|_*$. The weighted Fourier frame bounds $A, B > 0$ satisfy

$$\sqrt{A} \geq 2 - \mathrm{e}^{2\pi\delta_* c^* m_E}, \quad \sqrt{B} \leq \mathrm{e}^{2\pi\delta_* c^* m_E}.$$

We firstly note that if the density and Voronoi regions are defined in the Euclidean norm, i.e. if $|\cdot|_* = |\cdot|$, which is typically the case in practice, then $c^* = 1$. Also, if E is taken to be the unit Euclidean ball, which corresponds to Beurling's setting, then $m_E = 1$. In this particular case, the dimension dependence of (II.1) is completely removed and the density condition (II.3) reads $\delta < \log 2/(2\pi) \approx 0.11$.

Besides removing dimension dependence for supports contained in the unit spheres, Theorem II.2 also directly improves the setting in the original Gröchenig's result for the unit cubes. Namely, if E is the unit cube then $m_E = \sqrt{d}$, and therefore, the linear dependence of density (II.1) with dimension is reduced to the square-root dependence.

III. NUGS IN MULTIPLE DIMENSIONS

We have seen the conditions on a countable set of points under which stable sampling is possible. In practice, we are typically faced with a finite set of sample points and therefore the remaining challenge is to construct a stable and efficient reconstruction using only finite Fourier data. In this section, we construct an approximation $G(f) \in \mathrm{T}$ to the function $f \in \mathrm{H}$ using only $\{\hat{f}(\omega) : \omega \in \Omega_K\}$, where $\mathrm{T} \subseteq \mathrm{H}$ is any finite-dimensional reconstruction space, and $\Omega_K \subseteq \hat{\mathbb{R}}^d$ a finite countable set of sampling points with the density defined as

$$\delta_*^K = \sup_{z \in Z_K} \inf_{\omega \in \Omega_K} |\omega - z|_*,$$

for a given norm $|\cdot|_*$. The set $Z_K \subseteq \hat{\mathbb{R}}^d$ is chosen such that $\Omega_K \subseteq Z_K$ and $g\chi_{Z_K} \to g$, for $g \in \mathrm{L}^2(\hat{\mathbb{R}}^d)$, as $K \to \infty$, where K is referred to as the sampling bandwidth. The aim is to ensure that $G(f)$ is stable and close to the orthogonal projection $\mathcal{P}_{\mathrm{T}} f$, i.e. for any $f, h \in \mathrm{H}$, we want $G(f)$ which satisfies

$$\|f - G(f + h)\| \leq C_G \left(\|f - \mathcal{P}_{\mathrm{T}} f\| + \|h\|\right), \tag{III.1}$$

for some constant $C_G > 0$. To this end, define the NUGS reconstruction $G(f) \in \mathrm{T}$ by the weighted least-squares fit

$$G(f) = \underset{g \in \mathrm{T}}{\operatorname{argmin}} \sum_{\omega \in \Omega_K} \mu_\omega^K \left| \hat{f}(\omega) - \hat{g}(\omega) \right|^2, \tag{III.2}$$

where $\mu_\omega^K > 0$ is the measure of the Voronoi region

$$V_{\omega,*}^K = \{z \in Z_K \ : \ \forall \lambda \in \Omega,\ \lambda \neq \omega,\ |z - \omega|_* \leq |z - \lambda|_*\}.$$

Using Theorem II.2, we get the following result for NUGS.

Theorem III.1. [16] *Let $\mathrm{T} \subseteq \mathrm{H} = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \mathrm{supp}(f) \subseteq E\}$ be finite-dimensional, $E \subseteq \mathbb{R}^d$ compact, and let Ω_K be a sampling set such that for all K*

$$\delta_*^K < \frac{\log(2)}{2\pi m_E c^*},$$

where $m_E = \sup_{x \in E} |x|$ and $c^* > 0$ is the smallest constant such that $|\cdot| \leq c^* |\cdot|_*$. Let also $\epsilon \in (0, 2 - \exp(2\pi m_E \delta_* c^*))$. If $K > 0$ is large enough so that

$$R_K(T) = \sup_{\substack{f \in \mathrm{T} \\ \|f\|=1}} \|\hat{f} - \hat{f}\chi_{Z_K}\| \leq \sqrt{\epsilon(2 - \epsilon)},$$

then $G(f)$ defined by (III.2) exists uniquely and satisfies (III.1) with

$$C_G \leq \frac{\exp\left(2\pi m_E \delta_*^K c^*\right)}{2 - \exp\left(2\pi m_E \delta_*^K c^*\right) - \epsilon}.$$

Since T is finite dimensional, the residual $R_K(\mathrm{T})$ converges to zero when $K \to \infty$ and hence there always exists K such that $R_K(\mathrm{T})$ is small enough. Therefore, this theorem guarantees stable and optimal recovery in an arbitrary finite-dimensional T, with the explicit bound on the reconstruction constant C_G, provided that the sampling scheme is sufficiently dense and wide in the frequency domain.

The boundedness of the reconstruction constant C_G under the sharp density condition can be provided by use of Theorem II.1. However, the use of this theorem trades the explicitness of the bound, since it deploys non-explicit frame bounds A and B. In that case the guarantees for stability of $G(f)$ are formulated in terms of the following K-residual

$$\tilde{R}_K(\Omega_K, \mathrm{T}) = \sup_{\substack{f \in \mathrm{T} \\ \|f\|=1}} \sqrt{\sum_{\omega \in \Omega'} \mu_\omega |\hat{f}(\omega)|^2},$$

where Ω' is a minimal subsequence of Ω such that $\hat{\mathbb{R}}^d \backslash Z_K \subseteq \bigcup_{\omega \in \Omega'} V_{\omega,*}$, and Ω is a sequence such that $\Omega_K \subseteq \Omega$ and such that it yields a weighted Fourier frame. Note that the existence of a sequence Ω only imposes that Ω_K has sufficient density δ_*^K. Also, note that the residual $\tilde{R}_K$ again converges to zero as $K \to \infty$, but it now depends on both T and Ω_K.

By Theorem II.1, we get the following result for NUGS.

Theorem III.2. [16] *Let $\mathrm{T} \subseteq \mathrm{H} = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \mathrm{supp}(f) \subseteq E\}$ be finite-dimensional, $E \subseteq \mathbb{R}^d$ compact convex and symmetric, and Ω_K a sampling set such that for all K*

$$\delta_{E^\circ}^K < \frac{1}{4}.$$

Denote by A and B the frame bounds corresponding to the weighed Fourier frame arising from Ω, $\Omega_K \subseteq \Omega$, and let $\epsilon \in (0, A)$. If $K > 0$ is large enough so that

$$\tilde{R}_K(\Omega_N, \mathrm{T}) \leq \sqrt{\epsilon},$$

then $G(f)$ defined by (III.2) exists uniquely and satisfies (III.1) with

$$C_G \leq \sqrt{\frac{B}{A - \epsilon}}.$$

Although the density conditions in these theorems are explicit, it is not yet stated how large the sampling bandwidth K needs to be. Nevertheless, this is possible to determine by analysing the residuals R_K or $\tilde{R}_K$. In particular, since the residual R_K depends only on a particular choice of the space T, once T is fixed, it is possible to determine scaling of K and $\dim(\mathrm{T})$ which gives sufficiently small R_K and therefore the stable and optimal recovery from any sufficiently dense sampling set Ω_K. This in return provides the so-called stable sampling rate, which we analyze in the one-dimensional case in the following section. For uniform samples, the stable sampling rates for wavelet reconstructions were analyzed in [17] in one dimension, and in [18] in higher dimensions. For polynomial reconstructions see [19] and [20].

IV. NUGS IN ONE DIMENSION

In one dimension we take $Z_K \subseteq \hat{\mathbb{R}}$ to be the interval $[-K, K]$, and hence, the K-residual for a finite-dimensional T becomes

$$R_K(\mathrm{T}) = \sup\left\{\|\hat{f}\|_{\hat{\mathbb{R}} \backslash [-K,K]} : f \in \mathrm{T}, \|f\| = 1\right\}.$$

Now, by using Gröchenig's one-dimensional sharp result on weighted Fourier frames adopted to finite sampling sequences, in [6] the authors obtained the following one-dimensional version of Theorem III.1.

Theorem IV.1. [6] *Let* $\mathrm{T} \subseteq \mathrm{H} = \{f \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(f) \subseteq E\}$ *be finite-dimensional,* $E \subseteq \mathbb{R}^d$ *compact, and let* $\Omega_K \subseteq [-K, K]$ *be a sampling set such that for all* K*, it satisfies*

$$\delta^K < \frac{1}{4m_E},$$

where $m_E = \sup_{x \in E} |x|$*. Let* $\epsilon \in \left(0, 1 - 4m_E\delta^K\right)$*. If* $K > 0$ *is large enough so that* $R_K(T) \leq \sqrt{\epsilon(2-\epsilon)}$*, then* $G(f)$ *defined by (III.2) exists uniquely and satisfies (III.1) with*

$$C_G \leq \frac{1 + 4m_E\delta^K}{1 - 4m_E\delta^K - \epsilon}.$$

A. Wavelets

In the case when T consists of the first M terms of a wavelet basis, in [6] the authors show that the sampling bandwidth K only needs to scale linearly with M in order to ensure sufficiently small residual $R_K(\mathrm{T})$.

Let now $E = [-1, 1]$ and define a basis on H following the boundary wavelet construction from [21]. Namely, assume that the scaling function ϕ and the wavelet ψ are supported on $[-p+1, p]$, and define

$$\phi_{j,k}^{\mathrm{int}}(x) = \begin{cases} 2^{j/2}\phi(2^j x - k) & -2^j + p \leq k < 2^j - p \\ 2^{j/2}\phi_k^{\mathrm{left}}(2^j x) & -2^j \leq k < -2^j + p \\ 2^{j/2}\phi_{2^j-k-1}^{\mathrm{right}}(2^j x) & 2^j - p \leq k < 2^j, \end{cases}$$

where ϕ^{left} and ϕ^{right} are particular boundary scaling functions. The wavelet functions $\psi_{j,k}^{\mathrm{int}}$ are defined similarly. We may now form a multiresolution analysis for H from subspaces

$$V_j^{\mathrm{int}} = \mathrm{span}\left\{\phi_{j,k}^{\mathrm{int}} : \ k = -2^j, \dots, 2^j - 1\right\},$$
$$W_j^{\mathrm{int}} = \mathrm{span}\left\{\psi_{j,k}^{\mathrm{int}} : \ k = -2^j, \dots, 2^j - 1\right\}.$$

The appropriate truncation, for $R > J \geq \log_2 p$, gives the finite dimensional reconstruction space

$$\mathrm{T} = V_J^{\mathrm{int}} \oplus W_J^{\mathrm{int}} \oplus \cdots \oplus W_{R-1}^{\mathrm{int}} = V_R^{\mathrm{int}}, \tag{IV.1}$$

such that $\dim(\mathrm{T}) = 2^{R+1}$. Now we have the following.

Lemma IV.2. [6] *Let* T *be the reconstruction space (IV.1) such that for some* $\alpha > 1/2$ *the scaling function* ϕ *satisfies* $|\hat{\phi}(\omega)| \lesssim 1/(1+|\omega|)^\alpha$*,* $\omega \in \hat{\mathbb{R}}$*, and* $\{\phi_{R,k}^{\mathrm{int}} : \ k = -2^R, \dots, 2^R - 1\}$ *forms a Riesz basis for* T*. Then for any* $\gamma > 0$ *there exists a* $c_0(\gamma) > 0$ *such that*

$$R_K(T) \leq \gamma, \quad K \geq c_0(\gamma) 2^{R+1}.$$

Thus, for such choice of the reconstruction space T, if the sampling set $\Omega_K \subseteq [-K, K]$ has density $\delta^K < 1/4$, for all K, then by Theorem IV.1 and Lemma IV.2, only a linear scaling of K with $\dim(\mathrm{T})$ is sufficient to guarantee stable recovery in T with the reconstruction constant

$$C_G \leq \frac{1 + 4\delta^K}{1 - 4\delta^K - \epsilon}.$$

Moreover, due to the result for boundary wavelets given in [22], for smooth functions we obtain optimal convergence rates. Namely, for such T and Ω_K, if $f \in \mathrm{H}^s(0,1)$, $0 \leq s < p$, then

$$\|f - G(f)\| = \mathcal{O}\left(K^{-s}\right),$$

where $G(f) \in \mathrm{T}$ is the NUGS reconstruction based on the sampling set Ω_K.

The linear scaling and error decrease for wavelet reconstructions can be illustrated numerically as it is done in Figure 1. On the other hand, in Figure 2 we show two-dimensional wavelet reconstructions by using NUGS with and without weights.

B. Other spaces

Ensuring stable and optimal recovery in a reconstruction space T in terms of an appropriate scaling of the sampling bandwidth K and $\dim(\mathrm{T})$, can also be done for different finite dimensional reconstruction spaces such as algebraic or trigonometric polynomials, nonequidistant splines or piecewise polynomials. These spaces were analyzed in [23].

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References

[1] A. Aldroubi and K. Gröchenig, "Nonuniform sampling and reconstruction in shift-invariant spaces," *SIAM Rev.*, vol. 43, pp. 585–620, 2001.

[2] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, 1st ed. Academic Press Inc., 2001.

[3] O. Christensen, "Frames, Riesz bases, and discrete Gabor/wavelet expansions," *Bull. Amer. Math. Soc*, vol. 38, no. 3, pp. 273–291, 2001.

[4] A. Beurling, "Local harmonic analysis with some applications to differential operators," in *Some Recent Advances in the Basic Sciences, Vol. 1*. Some Recent Advances in the Basic Sciences, Vol. 1 (Proc. Annual Sci. Conf., Belfer Grad. School Sci., Yeshiva Univ., New York, 1962–1964), 1966, pp. 109–125.

[5] K. Gröchenig, "Reconstruction algorithms in irregular sampling," *Math. Comp.*, vol. 59, pp. 181–194, 1992.

[6] B. Adcock, M. Gataric, and A. C. Hansen, "On stable reconstructions from nonuniform Fourier measurements," *SIAM J. Imaging Sci.*, vol. 7, no. 3, pp. 1690–1723, 2014.

[7] B. Adcock and A. C. Hansen, "A generalized sampling theorem for stable reconstructions in arbitrary bases," *J. Fourier Anal. Appl.*, vol. 18, no. 4, pp. 685–716, 2012.

[8] J. I. Jackson, C. H. Meyer, D. G. Nishimura, and A. Macovski, "Selection of a convolution function for Fourier inversion using gridding," *IEEE Trans. Med. Imaging*, vol. 10, pp. 473–478, 1991.

[9] B. P. Sutton, D. C. Noll, and J. A. Fessler, "Fast, iterative image reconstruction for MRI in the presence of field inhomogeneities," *IEEE Trans. Med. Imaging*, vol. 22, no. 2, pp. 178–188, 2003.

[10] K. Gröchenig, "Irregular sampling, Toeplitz matrices, and the approximation of entire functions of exponential type," *Math. Comp.*, vol. 68, no. 226, pp. 749–765, 1999. [Online]. Available: http://dx.doi.org/10.1090/S0025-5718-99-01029-7

[11] J. J. Benedetto, "Irregular sampling and frames," in *Wavelets*, ser. Wavelet Anal. Appl. Boston, MA: Academic Press, 1992, vol. 2, pp. 445–507.

[12] M. Guerquin-Kern, M. Häberlin, K. P. Pruessmann, and M. Unser, "A fast wavelet-based reconstruction method for Magnetic Resonance Imaging," *IEEE Trans. Med. Imaging*, vol. 30, no. 9, pp. 1649–1660, 2011.

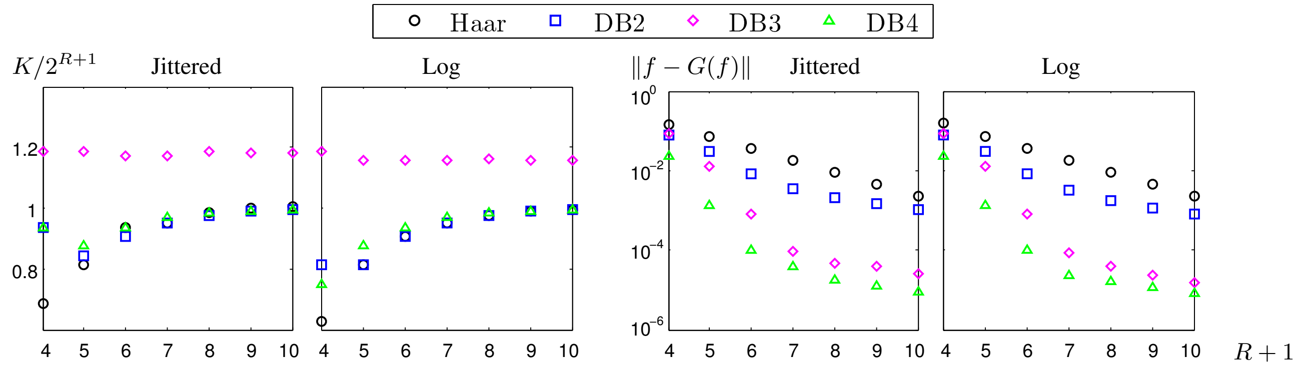

Fig. 1. In the first pair of plots, for a given $R+1$, we compute $K = \min\{K : C_G(K, 2^{R+1}) \leq 3\}$ and plot $K/2^{R+1}$, where $C_G(K, 2^{R+1})$ is the NUGS reconstruction constant corresponding to 2^{R+1} wavelets and Ω_K sampling scheme. In the second pair of plots, for such $R+1$ and K, the error $\|f - G(f)\|$ is plotted where $f = x^2 + x\sin(4\pi x) - \exp(x/2)\cos(3\pi x)^2$. This is done for two typical one-dimensional nonuniform sampling schemes: jittered and log.

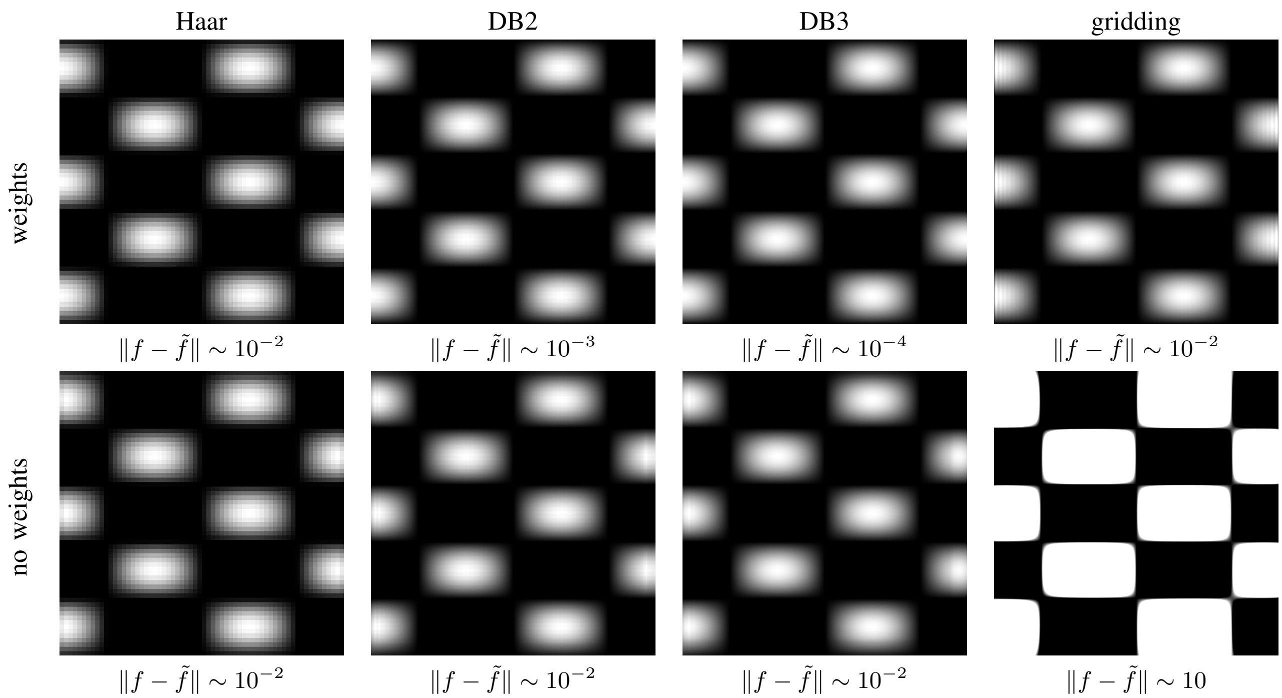

Fig. 2. Reconstructions of the function $f(x,y) = \sin(5/2\pi(x+1))\cos(3/2\pi(y+1))\chi_{[-1,1]^2(x,y)}$ with 64^2 wavelets from the radial sampling scheme taken from the Euclidean ball of radius $K = 64$ with the density measured in ℓ^1-norm strictly less than $1/4$. The lower pictures are reconstructed without using weights and, as demonstrated, the error does not exceed order 10^{-2}. The NUGS reconstruction with Haar, DB2 and DB3 are also compared to the popular MRI reconstruction technique called gridding.

[13] J. J. Benedetto and H. C. Wu, "Non-uniform sampling and spiral MRI reconstruction," *Proc. SPIE*, vol. 4119, pp. 130–141, 2000.

[14] A. Olevskii and A. Ulanovskii, "On multi-dimensional sampling and interpolation," *Anal. Math. Phys.*, vol. 2, no. 2, pp. 149–170, 2012. [Online]. Available: http://dx.doi.org/10.1007/s13324-012-0027-4

[15] K. Gröchenig, "Non-uniform sampling in higher dimensions: from trigonometric polynomials to bandlimited functions," in *Modern Sampling Theory*, J. J. Benedetto, Ed. Birkhöuser Boston, 2001, ch. 7, pp. 155–171.

[16] B. Adcock, M. Gataric, and A. C. Hansen, "Weighted frames of exponentials and stable recovery of multidimensional functions from nonuniform Fourier samples," *arXiv:1405.3111*, 2014.

[17] B. Adcock, A. C. Hansen, and C. Poon, "On optimal wavelet reconstructions from Fourier samples: linearity and universality of the stable sampling rate," *Appl. Comput. Harmon. Anal.*, vol. 36, no. 3, pp. 387–415, 2014.

[18] B. Adcock, A. C. Hansen, G. Kutyniok, and J. Ma, "Linear stable sampling rate: Optimality of 2d wavelet reconstructions from fourier measurements," *SIAM J. Math. Anal.*, vol. 47, no. 2, pp. 1196–1233, 2015.

[19] T. Hrycak and K. Gröchenig, "Pseudospectral Fourier reconstruction with the modified inverse polynomial reconstruction method," *J. Comput. Phys.*, vol. 229, no. 3, pp. 933–946, 2010.

[20] B. Adcock, A. C. Hansen, and A. Shadrin, "A stability barrier for reconstructions from Fourier samples," *SIAM J. Numer. Anal.*, vol. 52, no. 1, pp. 125–139, 2014.

[21] A. Cohen, I. Daubechies, and P. Vial, "Wavelets on the interval and fast wavelet transforms," *Appl. Comput. Harmon. Anal.*, vol. 1, no. 1, pp. 54 – 81, 1993.

[22] S. G. Mallat, *A Wavelet Tour of Signal Processing: The Sparse Way*, 3rd ed. Academic Press, 2009.

[23] B. Adcock, M. Gataric, and A. C. Hansen, "Recovering piecewise smooth functions from nonuniform Fourier measurements," *arXiv:1410.0088*, 2014.