COMPUTING SPECTRA – ON THE SOLVABILITY COMPLEXITY INDEX HIERARCHY AND TOWERS OF ALGORITHMS

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ABSTRACT. This paper establishes the Solvability Complexity Index (SCI) hierarchy to resolve the long-standing computational spectral problem. That is to determine the existence of algorithms that approximate spectra $\mathrm{sp}(A)$ of classes of bounded operators $A = \{a_{ij}\}_{i,j\in\mathbb{N}} \in \mathcal{B}(l^2(\mathbb{N}))$ given the matrix elements $\{a_{ij}\}_{i,j\in\mathbb{N}}$. Similarly, for Schrödinger operators $H = -\Delta + V$, we determine the existence of algorithms that can approximate the spectrum $\mathrm{sp}(H)$ given point samples of the potential function V. Our results are sharp, in the sense that they realise the boundaries of what algorithms can achieve. To show sharpness, we establish the full SCI hierarchy, based on the SCI introduced by one of the authors in [71]. This is a classification hierarchy for all types of problems in computational mathematics that allows for classifications determining the boundaries of what algorithms can achieve. As a consequence, the SCI hierarchy provides classifications of computational problems that can be used in computer-assisted proofs. The SCI hierarchy captures many key computational issues in the history of mathematics e.g. Smale's problem on the existence of iterative generally convergent algorithms for polynomial root finding, the computational spectral problem, inverse problems, optimisation, computational issues in topology, etc.

CONTENTS

1. INTRODUCTION

The Solvability Complexity Index (SCI) hierarchy introduced in this article, including the mathematical framework and the associated algorithms, is now used in a variety of areas due to circulations of earlier

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versions of this paper. This ranges from spectral computations in the sciences [37] to solving open resonance [14, 15] and geometric spectral problems [13, 34, 36], via inverse problems [1, 35], Smale's 9th problem and optimisation [1, 11, 49], to finally limits of AI and Smale's 18th problem [35], see also §2 and [8–10, 21, 22, 32, 33, 44, 45, 47, 58, 94, 120].

Given the many applications in fields such as mathematical physics, analysis, quantum chemistry, statistical mechanics, quantum mechanics, quasicrystals, and optics, the problem of finding algorithms for approximating spectra of operators has both fascinated and frustrated mathematicians. This challenge has persisted since the seminal work by H. Goldstine, F. Murray, and J. von Neumann in the 1950s [64], giving rise to a vast body of literature (see §2.1). In the early 1990s, W. Arveson [10] pointed out that despite the plethora of papers on the subject of computing spectra, the general computational spectral problem remained unsolved:

> *"Unfortunately, there is a dearth of literature on this basic problem, and so far as we have been able to tell, there are no proven techniques."*
>
> — W. Arveson [10] (1994)

See also A. Böttcher's Problem I in [22] on the problem of computing the spectrum. Arveson considered approximating spectra given the matrix elements $\{a_{ij}\}_{i,j\in\mathbb{N}} \in \mathcal{B}(l^2(\mathbb{N}))$ for general operators and discrete Schrödinger operators. However, the situation does not improve significantly in the general Schrödinger case. Notably, despite over 90 years of advancements in quantum mechanics, there is still no known algorithm for computing approximations to the spectrum $\mathrm{sp}(\Delta + V)$ of $-\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$ given point samples from the potential function V, and of $-\Delta_{\mathrm{discrete}} + V$ on lattices.

To resolve these issues, we introduce algorithms that not only approximate spectra and eigenvectors, but also establish upper and lower bounds on where the problems are situated in the SCI hierarchy. These bounds yield sharp classification results, demonstrating the optimality of the proposed algorithms. Below is a summary of the main results written as classification results in the SCI hierarchy that will be explained informally in §3 and in detail in §7. Examples of the resulting hierarchies are shown in Figures 1 and 2.

Theorem 1.1 (Informal summary of Theorem 4.4, 5.3 & 5.5)**.** *Given infinite matrices of the form* $A = \{a_{ij}\}_{i,j\in\mathbb{N}} \in \mathcal{B}(l^2(\mathbb{N}))$*, Schrödinger operators* $-\Delta+V$ *as well as discrete Schrödinger operators* $-\Delta_{\mathrm{discrete}}+V$*, we have the following classifications:*

$$\text{computing } \mathrm{sp}(A),\ A \text{ is diagonal} \quad \in \Sigma_1 \setminus \Delta_1, \tag{1.1}$$

$$\text{computing } \mathrm{sp}(-\Delta+V)\ \&\ \text{approx. eigenvectors, with bounded } V \quad \in \Sigma_1 \setminus \Delta_1, \tag{1.2}$$

$$\text{computing } \mathrm{sp}(-\Delta_{\mathrm{discrete}}+V)\ \&\ \text{approx. eigenvectors, with bounded } V \quad \in \Sigma_1 \setminus \Delta_1, \tag{1.3}$$

$$\text{computing } \mathrm{sp}(A),\ A \text{ is compact} \quad \in \Delta_2 \setminus (\Sigma_1 \cup \Pi_1), \tag{1.4}$$

$$\text{computing } \mathrm{sp}(-\Delta+V) \text{ with } V \text{ blowing up at } \infty \quad \in \Delta_2 \setminus (\Sigma_1 \cup \Pi_1). \tag{1.5}$$

Moreover,

$$\text{computing } \mathrm{sp}(A),\ A \text{ is self-adjoint or normal} \quad \in \Pi_2 \setminus \Delta_2, \tag{1.6}$$

$$\text{computing } \mathrm{sp}(A),\ A \text{ is general} \quad \in \Pi_3 \setminus \Delta_3, \tag{1.7}$$

$$\text{computing } \mathrm{sp}(A)\ \&\ \text{approx. eigenvectors, } A \text{ is normal and banded} \quad \in \Sigma_1 \setminus \Delta_1, \tag{1.8}$$

$$\text{computing } \mathrm{sp}_{\mathrm{ess}}(A),\ A \text{ is self-adjoint or normal} \quad \in \Pi_2 \setminus \Delta_2, \tag{1.9}$$

$$\text{computing } \mathrm{sp}_{\mathrm{ess}}(A),\ A \text{ is general} \quad \in \Pi_3 \setminus \Delta_3. \tag{1.10}$$

1.1. **Interpretation of the main results.** The reader may consult §3 for an informal introduction to the SCI hierarchy; however, below follows an explanation of the main results in view of the computational spectral problem without technical language. The SCI is the smallest number of limits needed to compute the desired property, e.g., the spectrum or essential spectrum.

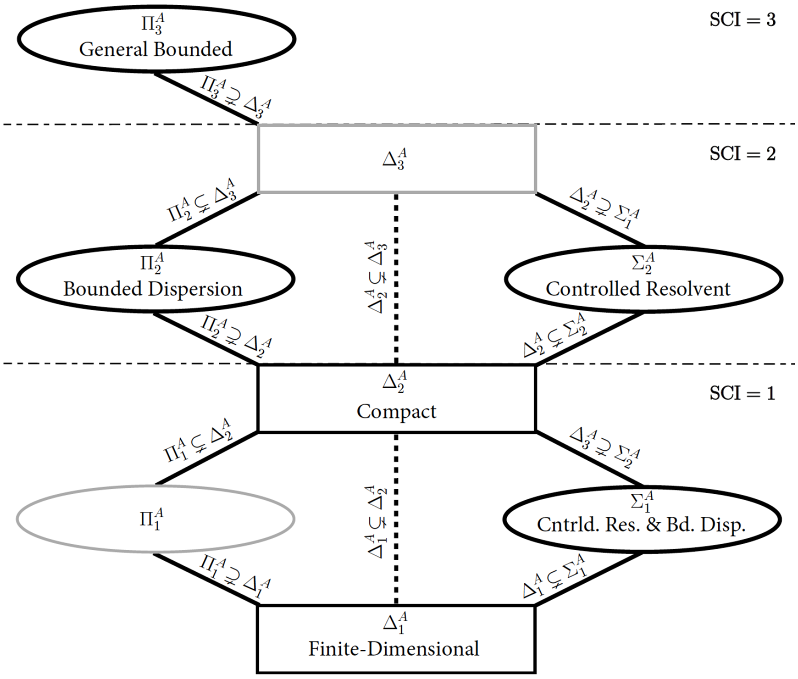

FIGURE 1. Theorem 4.4 (resolving the computational spectral problem): The SCI hierarchy for the computational problem of approximating spectra of bounded infinite matrices acting on $l^2(\mathbb{N})$. Note that $\Sigma_3^A \setminus \Delta_2^A = \emptyset$.

(i) *Error control and algorithms without mistakes – The lower end of the SCI hierarchy*. Note that (1.2) means that there exists an algorithm Γ_n, such that for bounded potentials V, $\Gamma_n(V) \to \mathrm{sp}(-\Delta+V)$ as $n \to \infty$. Moreover, $\Gamma_n(V) \subset \mathcal{N}_{2^{-n}}(\mathrm{sp}(-\Delta+V))$, where $\mathcal{N}_\epsilon(\cdot)$ denotes the ϵ-neighbourhood. In particular, there is an error control, and Γ_n will never make a mistake. In addition, the algorithm can compute the approximating eigenvectors (see §7.2.1 for precise definitions). Furthermore, it is impossible to find an algorithm that will, on input $\epsilon > 0$, compute an ϵ-approximation to $\mathrm{sp}(-\Delta+V)$ even locally (i.e., in the Attouch-Wetts metric, which generalizes the Hausdorff metric to unbounded sets). Due to (1.3) and (1.8), the same positive and negative results hold for discrete Schrödinger operators and banded normal or self-adjoint infinite matrices.

It may seem potentially surprising – given (1.2) and (1.4) – that the problem of computing spectra of compact operators, for which the method has been known for decades, is strictly harder (see (ii) below) than the problem of computing spectra of Schrödinger operators with bounded potentials, which has been open for more than half a century. Given (1.1), algorithms can obtain (by sampling the potential pointwise) as much spectral information from Schrödinger operators with a bounded potential V, as algorithms can obtain from a diagonal infinite matrix – the simplest of the non-trivial infinite-dimensional spectral problems.

(ii) *Computing with one limit, but no error control – The-mid part of the SCI hierarchy*. The problem of computing spectra of $\mathrm{sp}(-\Delta+V)$ with V blowing up at ∞ is as hard as computing spectra of compact operators, and it is strictly harder than computing spectra of diagonal matrices and normal Schrödinger operators with bounded potential. Moreover, despite the existence of having an algorithm Γ_n such that $\Gamma_n(V) \to \mathrm{sp}(-\Delta+V)$, error control is impossible as the problem of computing the spectrum $\notin \Sigma_1 \cup \Pi_1$.

(iii) *Computing with several limits – The higher end of the SCI hierarchy*. Note that (1.7) means that there does exist an algorithm Γ_{n_3,n_2,n_1} depending on integers n_3, n_2, n_1 such that for all $A =$

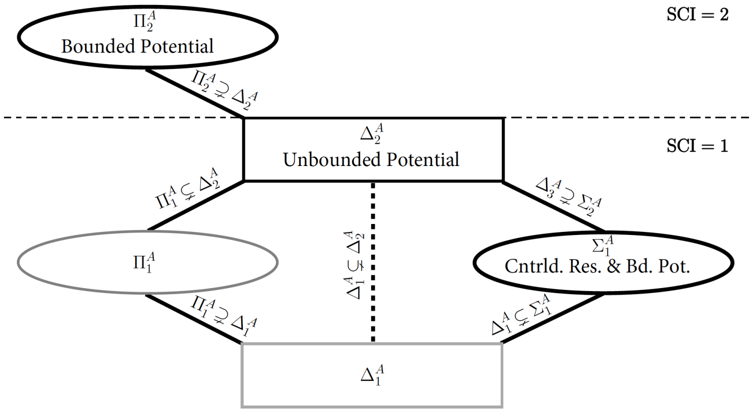

FIGURE 2. Main results (Theorem 5.3 and Theorem 5.5): The SCI hierarchy for the problem of approximating spectra of Schrödinger operators. The Σ_1^A result may be a surprise: One can algorithmically obtain just as much spectral information of such Schrödinger operators (by point sampling the potential) as that of diagonal infinite-matrices (the simplest problem).

$\{a_{ij}\}_{i,j\in\mathbb{N}} \in \mathcal{B}(l^2(\mathbb{N}))$,

$$\lim_{n_3\to\infty}\lim_{n_2\to\infty}\lim_{n_1\to\infty}\Gamma_{n_3,n_2,n_1}(A) = \mathrm{sp}(A), \quad \text{(the SCI} \leq 3),$$

where the last limit is 'from above'. Yet, for any family of algorithms $\{\Gamma_{n_2,n_1}\}$ based on two limits, there is an A such that

$$\lim_{n_2\to\infty}\lim_{n_1\to\infty}\Gamma_{n_2,n_1}(A) \neq \mathrm{sp}(A), \quad \text{(the SCI} > 2\text{, hence SCI} = 3).$$

In the self-adjoint case, however, one needs two limits, and no algorithm can compute the spectrum in one limit. According to (1.9) and (1.10), similar results hold for the essential spectrum.

(iv) *Arveson's comment and the SCI hierarchy.* The phenomenon that the SCI > 1 for many spectral problems is similar to the solution to Smale's problem (see §2.5). Moreover, it explains Arveson's comment – why there have been no known techniques for the general cases – and why it has taken substantial time to resolve the computational spectral problem. Indeed, classical approaches (see §2.1), including the C^*-algebra techniques (see W. Arveson [6–10] and N. Brown [27–30]) also used for the Schrödinger case, yield algorithms based on one limit. By the results above, algorithms based on one limit can never capture the general problem, even in the self-adjoint case. However, classical approaches yield invaluable classification results in the lower part of the SCI hierarchy.

(v) *New algorithms.* The proofs of the upper bounds in the above theorems yield new algorithms allowing for previously untouched problems in the sciences and potentially in computer-assisted proofs. Several examples can be found in §11.

2. THE SCI HIERARCHY IN MATHEMATICS

Our results and the SCI hierarchy have implications and connections to various areas in mathematics. §6 specifically highlights the role of the SCI hierarchy in computer-assisted proofs.

2.1. **Arveson, Böttcher and Schwinger's computational spectral problems.** The literature on computing spectra is extensive; therefore, we can only highlight a small subset of particularly relevant results here. The idea of using computational approaches to obtain spectral information dates back to leading physicists and mathematicians such as E. Schrödinger [102], T. Kato [82], and J. Schwinger [103]. Schwinger introduced finite-dimensional approximations to quantum systems in infinite-dimensional spaces, allowing for spectral computations. These ideas were already present in the work of H. Weyl [121]. The work by H. Goldstine, F. Murray, and J. von Neumann [64] was among the first to establish rigorous convergence results. Their work yields a Δ_1 classification for specific self-adjoint finite-dimensional problems. In [44], T. Digernes, V. S. Varadarajan, and S. R. S. Varadhan proved the convergence of spectra of Schwinger's finite-dimensional discretization matrices for a specific class of Schrödinger operators with certain types of potential. This yields a Δ_2 classification in the SCI hierarchy. Theorems 1.1 and 5.5 imply that their result is sharp, even for a much larger class of problems.

W. Arveson [6–10] and A. Böttcher [21–23] pioneered spectral computations further, both for the general spectral computation problem and for discrete Schrödinger operators (see Arveson [8,9]). Moreover, they established new connections to the C^*-algebra literature (see also the work by N. Brown [27–29]) and Toeplitz theory (see A. Böttcher & B. Silberman [24–26]). Most of these results yield Δ_2 classifications in the SCI hierarchy for special types of self-adjoint spectral problems. For additional information, see the work by N. Brown, K. Dykema, and D. Shlyakhtenko [30]. Arveson [8–10] and Böttcher & Silberman [25,26] also explored spectral computation in terms of densities, which relates to Szegö's work [113] on finite section approximations. A. Laptev and Y. Safarov have also obtained similar results [87] yielding Δ_2 classifications.

Finally, the work initiated by D. Jerison [79] and continued by D. Grieser & D. Jerison [65, 66] on estimating eigenvalues of differential operators become results in the SCI framework on the breakdown epsilon $\epsilon_0 \geq 0$ [11, 35], which determines the best possible accuracy of the computed approximation to a problem. See also §2.4 regarding the role of breakdown epsilon in the proof of blow-up of the 3D Euler equations. In particular, Theorem 1 in [66] shows an upper bound on the breakdown epsilon ϵ_0 for computing the m-th eigenvalue of the Dirichlet Laplacian on certain bounded domains. See D. Grieser & D. Jerison's discussion in [65] on the connection to the computational spectral problem. We end with a list of additional relevant papers that offer just a small glimpse into the vast literature on this topic [4,5,38,39,47,50,69,70,72,89–91,105,119].

2.2. **The Dirac-Schwinger conjecture.** The Dirac–Schwinger conjecture was proven through a series of seminal papers by C. Fefferman and L. Seco [51–59]. In these works, numerical computations are utilized to obtain asymptotic results on the ground state of an atom. Consider the following Schrödinger operator

$$H_{NZ} = \sum_{k=1}^{N}(-\Delta_{x_k} - Z|x_k|^{-1}) + \sum_{1\leq j<k\leq N} |x_j - x_k|^{-1}$$

acting on antisymmetric functions in $\mathrm{L}^2(\mathbb{R}^{3N})$. The ground state energy $E(N, Z)$ for N electrons and a nucleus of charge Z is then defined by $E(N, Z) := \inf\{\lambda \in \mathrm{sp}(H_{NZ})\}$. The ground state energy of an atom is then defined as $E(Z) = \min_{N\geq 1} E(N, Z)$. The key result of C. Fefferman and L. Seco was to show the asymptotic behavior of $E(Z)$ for large Z. In particular, they show the following:

Theorem 2.1 (Fefferman, Seco [58]).

$$E(Z) = -c_0 Z^{7/3} + \frac{1}{8}Z^2 - c_1 Z^{5/3} + \mathcal{O}(Z^{5/3-1/2835}),$$

for some explicitly defined constants c_0 *and* c_1.

Their intricate computer-assisted proof hinges on several problems that are $\notin \Delta_1$ – meaning they are not computable – but are in Σ_1 (see, for example, Algorithm 3.7 and Algorithm 3.8 in [58]). A crucial part of the proof implicitly establishes the Σ_1 classification in the SCI hierarchy. Moreover, the paper [47] by C. Fefferman is based on similar approaches using numerical calculation of eigenvalues. See also the work by R. de la Llave [41, 48] on using computer-assisted proofs for estimating the ground state energy. Computational approaches to spectral theory are, of course, significant outside of computer-assisted proofs. The paper [50] by C. Fefferman and D. H. Phong, which focuses on numerically computing the lowest eigenvalue of pseudo-differential operators, is a great example.

The phenomenon of facing problems $\notin \Delta_1$ in computer-assisted proofs is shared by the program on proving Kepler's conjecture, where implicitly, one shows Σ_1 classification in the SCI hierarchy as part of the proof. See §6 for details. Our main results in Theorem 4.4 and Theorem 5.3 provide the necessary Σ_1 classifications showing that computational spectral problems with any Jacobi operators with known growth of the resolvent can be used in computer-assisted proofs. This is also the case of Schrödinger operators $-\Delta + V$ where V is bounded and of bounded variation. However, by Theorem 5.5, if we only know that V blows up at infinity, the spectral problem $\notin (\Sigma_1 \cup \Pi_1)$ so such a Schrödinger operator cannot be used in a computer-assisted proof unless further assumptions are available.

2.3. **Weinberger's program on computations in topology and geometry.** The book "*Computers, Rigidity, and Moduli*" [120] by S. Weinberger provides a comprehensive illustration of his program (see also [96]). As is pointed out in [120]: "*The main theme of this work is the application of the theory of computation to problems in geometry. My interest is not simply in showing the algorithmic unsolvability of natural questions, but rather in solving geometric existence problems.*" This theme aligns very closely with what is captured by the SCI hierarchy. Indeed, many results in Weinberger's program can be interpreted as classification problems within the SCI hierarchy, with numerous theorems demonstrating either $\notin \Delta_1$ or $\in \Delta_1$ results. As an illustration, Weinberger's theorem (p. 83 in [120]) can be expressed in the SCI language:

Theorem 2.2 (Weinberger [120]). *Let M^n be a nonsimply connected compact manifold whose first homology is trivial and which embeds in S^{n+1}. Then, the computational problem of determining whether any other embedding is isotopic to the given one is $\notin \Delta_1$.*

However, the computational problems in Weinberger's program that are $\notin \Delta_1$ are often situated much higher than Δ_1 in the SCI hierarchy, highlighting the true difficulty in solving geometric existence problems. Notably, the exact location within the SCI hierarchy reveals the true complexity of the problem. Analogously, in computational spectral theory, the general spectral problem resides in $\Pi_3 \setminus \Delta_3$, demonstrating that no spectral problem is more challenging than this. However, the task of computing spectra of Schrödinger operators with bounded potentials falls within $\Sigma_1 \setminus \Delta_1$, low enough in the hierarchy to allow for algorithms that will never make a mistake and provide error control, making them useful in computer-assisted proofs. A key question arises: Which computational problems in Weinberger's program are low enough in the SCI hierarchy to allow for use in computer-assisted proofs, and how high can the computational problems become in the SCI hierarchy?

2.4. **Blow-up of the 3D Euler equation with smooth initial data – Verification of computations.** The problem of proving blow-up of the 3D Euler equation with smooth initial data is considered one of the major open problems in nonlinear PDEs. In connection with this problem, T. Hou poses, in Problem 2 in [49], the question: "*In problems where mathematical analysis precedes a computer-assisted step: How can we ensure that the formulation of the problem is correctly posed so that computability and non-computability of a problem can be determined?*" This question is motivated by the challenge of proving finite-time blow-up of the 3D Euler equation with smooth initial data through a computer-assisted proof – which was recently announced by J. Chen and T. Hou [32, 33] (see also [31]), thereby solving a century-long open problem. Consider the 3D Euler equations

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{2.1}$$

The delicate issue is that a blow-up of the solution to (2.1) would imply instability in terms of unboundedness [117] of the forward operator taking the initial data to the solution at a given time. One might initially think, as suggested in [117], that this means that 3D Euler blow-up is not computable (depending on the problem's formulation). Indeed, the solution operator's unboundedness typically yields the PDE solution's non-computability [99]. This complication could hinder the prospects of a computer-assisted proof, where the validity of the computational step needs to be verifiable. However, the situation is far more nuanced.

Hou further notes: "*It could be misleading to conclude that the 3D Euler blow-up is not computable based on one formulation and one metric that are not suitable to study the potential stable blow-up of the 3D Euler equation*." The SCI hierarchy is designed to respond to Hou's question, as it can accommodate any computational problem in any metric space, addressing the many subtleties of non-computability in computer-assisted proofs. Hou's point is subtle, in particular as the SCI framework identifies phase transitions [11, 35] in non-computable problems (that means $\notin \Delta_1$ in the SCI language) with a so-called breakdown epsilon $\epsilon_0 > 0$ [11, 35]. More precisely, for a computational problem with breakdown epsilon $\epsilon_0 > 0$, one cannot compute an approximation to ϵ-accuracy if $\epsilon < \epsilon_0$. However, it is possible to compute an ϵ-accurate approximation if $\epsilon_0 < \epsilon$ (see [11, 35] as well as Problem 5 (J. Lagarias) in [49] related to Smale's 9th problem and its extensions). In particular, every computer-assisted proof relying on a computational approximation computes this approximation to some accuracy $\epsilon \geq 0$. If this ϵ satisfies $\epsilon > \epsilon_0$, the non-computability of the problem becomes completely irrelevant.

Consider the following main theorem in [32], which yields the finite time blow-up of the 3D Euler equations with smooth initial data. The appropriate definitions needed for Theorem 2.3 can be found in §6.1 in [32].

Theorem 2.3 (Chen, Hou – Theorem 4 in [32]). *Let $(\bar{\theta}_0, \bar{\omega}_0, \bar{\boldsymbol{u}}, \bar{c}_l, \bar{c}_\omega)$ be the approximate self-similar profile constructed in Section 6.4.2 of [32] and $E_* = 5 \cdot 10^{-6}$. Assume that even initial data θ_0 and odd ω_0 of (6.13) in [32] are compactly supported with size $S(0)$ defined in Definition 6.2 in [32] and satisfy*

$$E(\omega_0 - \bar{\omega}, \theta_{0,x} - \bar{\theta}_{0,x}, \theta_{0,y} - \bar{\theta}_{0,y}) < E_*,$$

where E is defined in (2.13) in [32]. For $E_* = 5 \cdot 10^{-6}$, there exists a constant $C(S(0))$ depending on $S(0)$ such that if the initial rescaling factor $C_l(0)$ (6.11) satisfies $C_l(0) < C(S(0))$, we have

$$\|\omega - \bar{\omega}\|_{L^\infty}, \|\theta_x - \bar{\theta}_{0,x}\|_{L^\infty}, \|\theta_y - \bar{\theta}_{0,y}\|_{L^\infty} < 200E_*, \quad |u_x(t,0) - \bar{u}_x(0)|, |\bar{c}_\omega - c_\omega| < 100E_* \tag{2.2}$$

for all time. In particular, we can choose smooth initial data $\omega_0, \theta_0 \in C_c^\infty$ in this class with finite energy $\|\boldsymbol{u}_0\|_{L^2} < \infty$ such that the solution to the physical equations (2.3)-(2.5) in [32] with these initial data blows up in finite time T.

Equation (2.2) establishes that the breakdown epsilon, ϵ_0, for the task of computing the solution to a rescaled 3D Euler equation (as given in (6.13) of [32]) for all time and for a specific set of initial values, is bounded by $200E_*$. Hence, the first part of Theorem 2.3 becomes a statement in the SCI framework. Specifically, $\epsilon_0 \leq 10^{-3}$. The key here is that the bound $\epsilon_0 \leq 10^{-3}$ on the breakdown epsilon is enough to imply the asserted blow-up. In particular, the unboundedness of the forward operator, potentially implying non-computability of the 3D Euler equation, is irrelevant for the computer-assisted proof. The SCI framework elucidates why the apprehensions presented in [117] (regarding the unboundedness of the forward operator for 3D Euler) are not relevant to the work of Chen and Hou. The only thing that matters is that the breakdown epsilon is sufficiently small to imply the asserted blow-up, which Chen and Hou prove.

2.5. **Smale's problem on the existence of iterative generally convergent algorithms.** An example of how the SCI hierarchy encompasses important foundational results is illustrated by the question of computing zeros of polynomials through the iterative application of a rational map, such as Newton's method [108]. The issue with Newton's method is that it may not always converge. This challenge prompted S. Smale to ask whether an alternative algorithm to Newton's method exists [109]: "*Is there any purely iterative generally convergent algorithm for polynomial zero finding?*" He conjectured that the answer is 'no'. C. McMullen addressed this problem in [94] and found that the answer is 'yes' for polynomials of degree three but 'no' for those of higher degrees (see also [95, 110]). However, in [45], P. Doyle and C. McMullen demonstrated a striking phenomenon: the problem can be solved using several limits for quartic and quintic polynomials. Indeed, Smale's question and the results provided by Doyle and McMullen are classification problems within the SCI hierarchy (see the detailed discussion in §10).

3. THE SCI HIERARCHY - AN INFORMAL INTRODUCTION

We give an informal description of the SCI hierarchy to present the main results, and provide detailed definitions in §7. The SCI hierarchy is based on the concept of a computational problem. This is described by a function

$$\Xi : \Omega \to \mathcal{M}$$

that we want to compute, where Ω is some domain, and $(\mathcal{M}, d)$ is a metric space. For example, $\Xi(T) = \mathrm{sp}(T)$ (the spectrum) for some bounded operator $T \in \Omega$ and $\mathcal{M}$ is the collection of non-empty compact subsets of $\mathbb{C}$ equipped with the Hausdorff metric. The SCI was first introduced in the paper *"On the Solvability Complexity Index, the n-pseudospectrum and approximations of spectra of operators"* [71] to introduce the concept of several limits for spectral computations. The SCI of a spectral problem is the smallest number of successive limits needed to compute the solution. However, the main issue was left open in the paper above: is it necessary to use several limits? In other words, could the SCI collapse to one for all spectral problems or, in fact, for all problems in scientific computing? Moreover, as is easily seen, a hierarchy based on only the number of limits needed is not refined enough to capture the boundaries of what is possible in spectral computation.

In this paper, we introduce the general SCI hierarchy (see §7 for the formal definition) for all types of computational problems, and the mainstay of the hierarchy is the Δ_k^α classes. The α is related to the model of computation as explained below. Informally, we have the following description. Given a collection $\mathcal{C}$ of computational problems,

(i) Δ_0^α is the set of problems that can be computed in finite time, the SCI $= 0$.
(ii) Δ_1^α is the set of problems that can be computed using one limit (the SCI $= 1$) with control of the error, i.e., $\exists$ a sequence of algorithms $\{\Gamma_n\}$ such that $d(\Gamma_n(A), \Xi(A)) \leq 2^{-n}$, $\forall A \in \Omega$.
(iii) Δ_2^α is the set of problems that can be computed using one limit (the SCI $= 1$) without error control, i.e., $\exists$ a sequence of algorithms $\{\Gamma_n\}$ such that $\lim_{n\to\infty} \Gamma_n(A) = \Xi(A)$, $\forall A \in \Omega$.
(iv) Δ_{m+1}^α, for $m \in \mathbb{N}$, is the set of problems that can be computed by using m limits, (the SCI $\leq m$), i.e., $\exists$ a family of algorithms $\{\Gamma_{n_m,\ldots,n_1}\}$ such that

$$\lim_{n_m\to\infty} \cdots \lim_{n_1\to\infty} \Gamma_{n_m,\ldots,n_1}(A) = \Xi(A),\ \forall A \in \Omega.$$

In general, this hierarchy can only be refined if there is some extra structure on the metric space $\mathcal{M}$. The hierarchy typically does not collapse, and we have:

$$\Delta_0^\alpha \subsetneq \Delta_1^\alpha \subsetneq \Delta_2^\alpha \subsetneq \ldots \subsetneq \Delta_m^\alpha \subsetneq \ldots. \tag{3.1}$$

However, depending on the collection $\mathcal{C}$ of computational problems, the hierarchy (3.1) may terminate for a finite m or continue for arbitrary large m. For computational spectral problems, the hierarchy terminates; see Figure 1 and Figure 2.

The SCI hierarchy can be refined if the metric space $\mathcal{M}$ allows for convergence from "above" and "below", for example, when considering the Hausdorff metric, which is natural for spectral problems. The motivation behind the refinement is to characterize the intricate classifications of different problems. For example, consider Ω to be the class of all diagonal operators $T \in \mathcal{B}(l^2(\mathbb{N}))$ of the form

$$T = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & \ddots \end{pmatrix}, \qquad a_j \in \mathbb{C}. \tag{3.2}$$

The problem of computing the spectrum $\mathrm{sp}(T)$ of such Ts is trivially not in Δ_1^α. However, one can simply choose an algorithm Γ_n to collect $\{a_j\}_{j=1}^n$ and then one has that $\Gamma_n(T) \to \mathrm{sp}(T)$ as $n \to \infty$. Thus, the problem of computing spectra of operators in Ω is in Δ_2^α. However, we have an extra feature not captured by the hierarchy (3.1). Indeed, we have that

$$\Gamma_n(T) \subset \mathrm{sp}(T), \quad \forall n \in \mathbb{N}.$$

In particular, we have convergence from below, which is much stronger than just convergence since $\Gamma_n(T)$ always produces a correct output. Such convergence becomes incredibly important as it provides an error control from below. Moreover, the hierarchy (3.1) does not capture this important feature. This motivates the Σ_1^α class, which captures the concept of convergence from below. Similarly, the Π_1^α class captures a convergence from above. Informally, for spectral problems, we have the following additions to (3.1):

(1) $\Delta_0^\alpha = \Pi_0^\alpha = \Sigma_0^\alpha$ is the set of problems that can be solved in finite time, the SCI $= 0$.
(2) Σ_1^α: We have $\Delta_1^\alpha \subset \Sigma_1^\alpha \subset \Delta_2^\alpha$ and Σ_1^α is the set of problems for which there exists a sequence of algorithms $\{\Gamma_n\}$ such that for every $A \in \Omega$ we have $\Gamma_n(A) \to \Xi(A)$ as $n \to \infty$. However, $\Gamma_n(A)$ is always contained in the 2^{-n} neighbourhood of $\Xi(A)$.
(3) Π_1^α: We have $\Delta_1^\alpha \subset \Pi_1^\alpha \subset \Delta_2^\alpha$ and Π_1^α is the set of problems for which there exists a sequence of algorithms $\{\Gamma_n\}$ such that for every $A \in \Omega$ we have $\Gamma_n(A) \to \Xi(A)$ as $n \to \infty$. However, the 2^{-n} neighbourhood of $\Gamma_n(A)$ always contains $\Xi(A)$.
(4) Σ_m^α is the set of problems that can be computed by passing to m limits and computing the m-th limit is a Σ_1^α problem.
(5) Π_m^α is the set of problems that can be computed by passing to m limits, and computing the m-th limit is a Π_1^α problem.

Schematically, the general SCI hierarchy can be viewed in the following way:

$$
\begin{array}{ccccccccccccc}
\Pi_0^\alpha & & & & \Pi_1^\alpha & & & & \Pi_2^\alpha & & & & \\
\| & & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\supsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\supsetneq$} & & \rotatebox{-45}{$\subsetneq$} & \\
\Delta_0^\alpha & \subsetneq & \Delta_1^\alpha & \subsetneq & \Sigma_1^\alpha \cup \Pi_1^\alpha & \subsetneq & \Delta_2^\alpha & \subsetneq & \Sigma_2^\alpha \cup \Pi_2^\alpha & \subsetneq & \Delta_3^\alpha & \subsetneq & \cdots \\
\| & & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\supsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\supsetneq$} & & \rotatebox{45}{$\subsetneq$} & \\
\Sigma_0^\alpha & & & & \Sigma_1^\alpha & & & & \Sigma_2^\alpha & & & &
\end{array}
\tag{3.3}
$$

Note that the highlighted Σ_1^α and Π_1^α classes are crucial as they guarantee the existence of algorithms that will never make mistakes. Thus, they become crucial in computer-assisted proofs, see §6.

Remark 3.1 (The general SCI hierarchy)**.** The above sketch of the SCI hierarchy with convergence from below and above is well suited when considering the Hausdorff metric. However, the SCI hierarchy extends immediately to any metric space where there is a total ordering, for example, for $\mathcal{M} = \mathbb{R}$ and for decision problems where $\mathcal{M} = \{0,1\} = \{\mathrm{No}, \mathrm{Yes}\}$. For example, for decision problems, a Σ_1^α classification of a computational problem with domain Ω means that there is a sequence of algorithms $\{\Gamma_n\}$ such that for $A \in \Omega$, $\Gamma_n(A)$ will provide the correct output for large n (however, we do not know how big n must be), but if $\Gamma_n(A) = \mathrm{Yes}$, then the answer to the decision problem is Yes. A similar reversed statement holds for Π_1^α classifications of decision problems.

Remark 3.2 (The meaning of the α, the model of computation)**.** The α in the superscript indicates the model of computation, which is described in §7. For $\alpha = G$, the underlying algorithm is general and can use any tools at its disposal. The purpose is to ensure that lower bounds become universal regardless of the model of computation. The reader may think of a Blum–Shub–Smale (BSS) [19] machine or a Turing machine [116] with access to any oracle, although a general algorithm is even more powerful. However, for $\alpha = A$, only arithmetic operations and comparisons are allowed. In particular, if rational inputs are considered, the algorithm is a Turing machine, and in the case of real inputs, a BSS machine. Hence, a result of the form $\notin \Delta_k^G$ is stronger than $\notin \Delta_k^A$. Indeed, a $\notin \Delta_k^G$ result is universal and holds for any model of computation. Moreover, $\in \Delta_k^A$ is stronger than $\in \Delta_k^G$, and similarly for the Π_k^α and Σ_k^α classes. Note that classical hierarchies, such as the arithmetical hierarchy [98], become special cases of the SCI hierarchy (see Proposition A.5 discussed in the appendix for completeness), and hence we keep the similar notation.

4. Main theorem on the general computational spectral problem

All formal definitions needed can be found in §7, and in particular the definition of a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ in (7.1). For $A \in \Omega$, where Ω is an appropriate domain of operators, we define the problem functions

$$\Xi_{\mathrm{sp}}(A) := \mathrm{sp}(A) \quad \text{(spectrum)}, \qquad \Xi_{\text{e-sp}}(A) := \mathrm{sp}_{\mathrm{ess}}(A) \quad \text{(essential spectrum)} \tag{4.1}$$

$$\Xi_{\mathrm{sp},\epsilon}^N(A) := \mathrm{sp}_{N,\epsilon}(A) \quad (N\text{-pseudospectrum}) \qquad \Xi_{\mathrm{sp}}^z(A) := \textit{Yes} \text{ if } z \in \mathrm{sp}(A), \textit{ No} \text{ otherwise.} \tag{4.2}$$

Here $\mathrm{sp}(A)$ denotes the spectrum, $\mathrm{sp}_{\mathrm{ess}}(A)$ the essential spectrum (invariant under compact perturbations) and $\mathrm{sp}_{N,\epsilon}(A)$ denotes the (N,ϵ)-pseudospectrum [25, 69]

$$\mathrm{sp}_{N,\epsilon}(A) := \mathrm{cl}\left(\left\{z \in \mathbb{C} : \|(A - zI)^{-2^N}\|^{2^{-N}} > 1/\epsilon\right\}\right), \qquad N \in \mathbb{Z}_{\geq 0}, \epsilon > 0, \tag{4.3}$$

where we use the convention that $\|(A - zI)^{-2^N}\| = \infty$ when $z \in \mathrm{sp}(A)$. This set has been popular in spectral theory, analysis of pseudo differential operators and non-Hermitian quantum mechanics. For computing the spectrum/essential spectrum/(N,ϵ)-pseudospectrum, we consider computational problems $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ a la the ones in Example 7.1 in §7 (i.e., with respect to the Hausdorff metric). For the final problem of determining if $z \in \mathrm{sp}(A)$, the metric space becomes the discrete metric on $\{\textit{No}, \textit{Yes}\}$. To avoid trivialities for this final problem, when considering self-adjoint classes of operators, we will restrict to $z \in \mathbb{R}$, and when considering compact operators, we will restrict to $z \neq 0$. The key question then becomes:

> *Given a problem function Ξ of the form* (4.1) *or* (4.2) *with a domain Ω and evaluation set Λ, where in the SCI hierarchy is the computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$?*

Definition 4.1 (Dispersion)**.** We say that the dispersion of an operator $A \in \mathcal{B}(l^2(\mathbb{N}))$ is bounded by the function $f : \mathbb{N} \to \mathbb{N}$ if

$$D_{f,m}(A) := \max\{\|(I - P_{f(m)})AP_m\|, \|P_m A(I - P_{f(m)})\|\} \to 0 \quad \text{as } m \to \infty.$$

Note that for every operator A there is always a function f that bounds its dispersion since AP_m, $P_m A$ are compact and $\{P_n\}$ converges strongly to the identity. However, no function f acts as a uniform bound for all operators. Nevertheless, there are important (sub)classes of operators having well-known uniform bounds, which should be mentioned:

(i) Banded operators with bandwidth less than d: $f(k) = k + d$. More generally, we can consider operators with sparse matrices (only finitely many non-zero entries in each row and column) where f captures the sparsity pattern. For example, for discrete Schrödinger operators on $l^2(\mathbb{Z}^2)$, we can choose an ordering of the lattice sites so that $f(k) - k = \mathcal{O}(\sqrt{k})$.

(ii) Band-dominated and weakly band-dominated operators: $f(k) = 2k$. For definitions and properties of band and band-dominated operators see [88, 100, 104]. Weakly band-dominated operators can be found in [92].

(iii) Laurent/Toeplitz operators with piecewise continuous generating function: $f(k) = k^2$ (cf. [26] and [80, Proposition 5.4]).

(iv) Let $\mathcal{F}$ be a family of bounded operators with a common bound f. Then $\tilde{f}$, given by $\tilde{f}(k) = f(k) + k$, is a common bound for all operators in the Banach algebra which is generated by $\mathcal{F}$.

Without loss of generality, we assume that f is strictly increasing and $f(n) > n$. We are also interested in operators where the control of the growth of the resolvent is bounded.

Definition 4.2 (Controlled growth of the resolvent)**.** Let $g : [0, \infty) \to [0, \infty)$ be a continuous function, vanishing only at $x = 0$ and tending to infinity as $x \to \infty$ with $g(x) \leq x$. We say that a closed operator A with a non-empty spectrum on the Hilbert space $\mathcal{H}$ has controlled growth of the resolvent by g if

$$\|(A - zI)^{-1}\|^{-1} \geq g(\mathrm{dist}(z, \mathrm{sp}(A))) \quad \forall z \in \mathbb{C}, \tag{4.4}$$

where we use the convention $\|B^{-1}\|^{-1} := 0$ if B has no bounded inverse.

Notice that for every bounded operator A there always exists such a g (define $g(\alpha) := \min\{\|(A - zI)^{-1}\|^{-1} : z \in \mathbb{C}$ with $\mathrm{dist}(z, \mathrm{sp}(A)) = \alpha\}$, taking continuity and compactness into account) although there is no g that works for all A.

Remark 4.3 (Assumptions on Λ)**.** To make the "additional knowledge" g available for the algorithms, we assume that Λ also contains the constant functions $g_{i,j} : A \mapsto g(i/j)$ ($i, j \in \mathbb{N}$), which provide the values of g in all positive rational numbers. When considering the case of Δ_1-information and arithmetic algorithms over $\mathbb{Q}$, we assume that g maps $\mathbb{Q}_{\geq 0}$ to $\mathbb{Q}_{\geq 0}$ without loss of generality (e.g. by replacing g with a suitable piecewise linear function). In the case when the dispersion of the operator is known, the values $f(m)$ ($m \in \mathbb{N}$) shall be available to the algorithms as constant evaluation functions. When computing problems with SCI $= 1$ for Ω_f (and Ω_{fg}), our algorithms also require the knowledge of a null sequence $\{c_m\}_{m \in \mathbb{N}} \subset \mathbb{Q}$ such that $D_{f,m}(A) \leq c_m$.

We consider the following domains defined below. In the cases of bounded dispersion or controlled growth of the resolvent, we assume that we are given either f or g as above.

$\Omega_{\mathrm{B}} :=$ bounded operators
$\Omega_{\mathrm{N}} :=$ bounded normal operators,
$\Omega_{\mathrm{SA}} :=$ bounded self-adjoint operators
$\Omega_{\mathrm{C}} :=$ compact operators,
$\Omega_f :=$ bounded oper. w/ dispersion bounded by f
$\Omega_g :=$ bounded oper. w/ contr. res. growth by g.
$\Omega_{fg} := \Omega_f \cap \Omega_g$
$\Omega_{\mathrm{D}} :=$ bounded, diagonal, self-adjoint operators.

Recall that in the case of $\{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{D}}\}$ or $\{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{SA}}\}$ we take z to be real, and in the case of $\{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{C}}\}$ we take $z \neq 0$. Given the different domains, we can now state the main theorem for bounded operators.

Theorem 4.4 (The bounded computational spectral problem)**.** *Given the setup above, we have the following classification results in the SCI hierarchy.*

(i) Spectrum:

$\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ (*all oper.*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{N}}\} \in \Sigma_2^A$ (*normal*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{SA}}\} \in \Sigma_2^A$ (*self-adj.*),
$\Sigma_1^G \cup \Pi_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{C}}\} \in \Delta_2^A$ (*compact*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_f\} \in \Pi_2^A$ (*disp. bound. by f*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma_2^A$ (*resolvent growth bound. by g*),
$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Sigma_1^A$
$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_f \cap \Omega_{\mathrm{N}}\} \in \Sigma_1^{A,\mathrm{eigv}}$.

(ii) Essential spectrum:

$\Delta_3^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ (*all oper.*),
$\Delta_3^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_{\mathrm{N}}\} \in \Pi_3^A$ (*normal*),
$\Delta_3^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_{\mathrm{SA}}\} \in \Pi_3^A$ (*self-adj.*),
$\Delta_2^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_{\mathrm{D}}\} \in \Pi_2^A$ (*self-adj. diag.*),
$\Delta_2^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_f\} \in \Pi_2^A$ (*disp. bound. by f*),
$\Delta_3^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_g\} \in \Pi_3^A$ (*resolvent growth bound. by g*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Pi_2^A$ (*res. growth bound. by g and disp. bound. by f*).

(iii) N-Pseudospectrum:

$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{B}}\} \in \Sigma_2^A$ *(all oper.)*, $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{N}}\} \in \Sigma_2^A$ *(normal)*,

$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{SA}}\} \in \Sigma_2^A$ *(self-adj.)*, $\Sigma_1^G \cup \Pi_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{C}}\} \in \Delta_2^A$ *(compact)*,

$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_f\} \in \Sigma_1^A$ *(disp. bound. by f)*, $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_g\} \in \Sigma_2^A$ *(resolvent growth bound. by g)*,

$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Sigma_1^A$ *(res. growth bound. by g and disp. bound. by f)*.

(iv) Is z in the spectrum?:

$\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ *(all oper.)*, $\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{N}}\} \in \Pi_3^A$ *(normal)*,

$\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{SA}}\} \in \Pi_3^A$ *(self-adj.)*, $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{C}}\} \in \Pi_2^A$ *(compact)*,

$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_f\} \in \Pi_2^A$ *(disp. bound. by f)*, $\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_g\} \in \Pi_3^A$ *(resolvent growth bound. by g)*,

$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{D}}\} \in \Pi_2^A$ *(self-adj. diag.)*, $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Pi_2^A$ *(res. growth bound. by g and disp. bound. by f)*.

Remark 4.5. To gain the Σ_1^A algorithms for $\Xi_{\mathrm{sp},\epsilon}^N$ we need an upper bound for $\|A\|$ when $N > 0$ (without this we gain a Δ_2^A classification). No such knowledge is needed for the other towers of algorithms.

Remark 4.6. The proofs also show that the above lower bounds for compact operators hold when considering self-adjoint compact operators.

5. Main theorems on computational quantum mechanics

Here, we formally state the results summarised in §1. All formal definitions needed can be found in §7, and in particular the definition of a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ in (7.1). We consider the spectral mappings $\Xi_{\mathrm{sp}}, \Xi_{\mathrm{sp},\epsilon}$ from (4.1) for Schrödinger operators:

$$H = -\Delta + V, \qquad V : \mathbb{R}^d \to \mathbb{C}. \tag{5.1}$$

We assume that the information the algorithm can read is point samples $V(x)$ for $x \in \mathbb{Q}^d$. In particular, Λ is as in 7.1 in §7. Moreover, $\mathcal{M}$ is the collection of non-empty closed subsets of $\mathbb{C}$ with the standard Attouch–Wets metric (7.4). If we fix the domain of H such that it is appropriate for a class of potentials V, the spectrum of H is uniquely determined by V. The basic question is therefore:

> *Given a class of Schrödinger operators $-\Delta + V \in \Omega$, let Ξ be either Ξ_{sp} or $\Xi_{\mathrm{sp},\epsilon}$, Λ and $\mathcal{M}$ as above, where in the SCI hierarchy is the computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$?*

Though we have stuck to the Hilbert space $\mathrm{L}^2(\mathbb{R}^d)$ for simplicity, the algorithms we construct can also be adapted for other spaces commonly found in applications such as $\mathrm{L}^2(\mathbb{R}_{>0})$.

Bounded Potentials. We first consider cases with bounded potential. In particular, let $\phi : [0, \infty) \to [0, \infty)$ be some increasing function and $M > 0$, define

$$\Omega_\phi := \{H : \mathcal{D}(H) = \mathrm{W}^{2,2}(\mathbb{R}^d), V \in \mathrm{BV}_\phi(\mathbb{R}^d), \|V\|_\infty \leq M\},$$
$$\Omega_{\phi,g} := \{H \in \Omega_\phi : \|(-\Delta + V - zI)^{-1}\|^{-1} \geq g(\mathrm{dist}(z, \mathrm{sp}(H)))\},$$

where

$$\mathrm{BV}_\phi(\mathbb{R}^d) = \{f : \mathrm{TV}(f_{[-a,a]^d}) \leq \phi(a)\}, \tag{5.2}$$

($f_{[-a,a]^d}$ means f restricted to the box $[-a, a]^d$) with TV being the total variation of a function in the sense of Hardy and Krause (see [97]). Here as in §4, $g : [0, \infty) \to [0, \infty)$ is a continuous strictly increasing function with $g(x) \leq x$, vanishing only at $x = 0$ and tending to infinity as $x \to \infty$.

Note that the set Ω_ϕ requires more than V just being locally of bounded variation. There is a universal upper bound across the class on the growth of the total variation of the potential function as we restrict the function to a larger set. The class $\Omega_{\phi,g}$ includes self-adjoint Schödinger operators in Ω_ϕ; however, it is much larger. We denote the class of self-adjoint Schödinger operators in Ω_ϕ by $\Omega_{\phi,\mathrm{SA}}$.

Remark 5.1 (Assumptions on Λ)**.** In addition to containing the point sampling functions f_x such that $f_x(V) = V(x)$ for $x \in \mathbb{Q}^d$, we have the following. As done in the case of bounded Hilbert space operators discussed in Remark 4.3, the additional knowledge of g, describing the growth of the resolvent, is available for the algorithms by assuming that Λ also contains the constant functions $g_{i,j} : V \mapsto g(i/j)$ $(i, j \in \mathbb{N})$, which provide the values of g in all positive rational numbers (again in the case of Δ_1-information and arithmetic algorithms over $\mathbb{Q}$, we assume that $g(\mathbb{Q}_{\geq 0}) \subset \mathbb{Q}_{\geq 0}$ without loss of generality). Moreover, Λ contains the constant functions $\phi_n : V \mapsto \phi(n)$ for $n \in \mathbb{N}$ and we assume without loss of generality that $\phi(n) \in \mathbb{Q}$.

Remark 5.2 (The upper bounds hold both in the Turing and BSS model)**.** Note that the results in Theorem 5.3 and Theorem 5.5 hold with inexact input (Δ_1 information) as well as with exact input. Hence, our results are valid in both the Turing and the BSS models. To avoid extra notation, we will write $\{\Xi, \Omega\} \in \Delta/\Pi/\Sigma$ rather than the correct notation $\{\Xi, \Omega\}^{\Delta_1} \in \Delta/\Pi/\Sigma$.

Theorem 5.3 (Bounded potential)**.** *Given the above setup, we have the following classification results.*

$$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_\phi\} \in \Pi_2^A, \qquad \Delta_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}, \Omega_\phi\} \in \Sigma_1^A,$$
$$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\phi,g}\} \in \Sigma_1^A, \qquad \Delta_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}, \Omega_{\phi,g}\} \in \Sigma_1^A,$$
$$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\phi,\mathrm{SA}}\} \in \Sigma_1^{A,\mathrm{eigv}}.$$

Remark 5.4. When considering the problem of computing approximate eigenvectors by arithmetic algorithms, we need a suitable way of encoding the space. We choose do to so via computing coefficients of a function with respect to an orthonormal basis in $\mathrm{L}(\mathbb{R}^d)$, where each of these is a simple function consisting of trigonometric and rational functions.

As will be evident from the proof techniques, one can build towers of algorithms for operators with more general classes of potentials (for example, $\mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ or $\mathrm{L}^2(\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$), however, the height of these towers will be higher than the ones considered in this paper. The main future task is to obtain exact values of the SCI of the spectrum, given the different potential classes.

Unbounded Potentials. We obtain a rather intriguing phenomenon for sectorial operators. Namely, the SCI of both the spectrum and the pseudospectrum is one, but no type of error control is possible. In particular, suppose we have non-negative θ_1, θ_2 such that $\theta_1 + \theta_2 < \pi$. Define

$$\Omega_\infty = \{V \in \mathrm{C}(\mathbb{R}^d) : \forall x \arg(V(x)) \in [-\theta_2, \theta_1], |V(x)| \to \infty \text{ as } x \to \infty\}. \tag{5.3}$$

We define the operator H via the minimal operator h as: $H = h^{**}$, $h = -\Delta + V$, $\mathcal{D}(h) = \mathrm{C}_c^\infty(\mathbb{R}^d)$. When $V \in \Omega_\infty$, it follows that H has compact resolvent, a result that we also establish as a part of the proof of the following theorem.

Theorem 5.5 (Unbounded potential)**.** *Given the above setup, we have the following classification results*

$$\Sigma_1^G \cup \Pi_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_\infty\} \in \Delta_2^A, \quad \Sigma_1^G \cup \Pi_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}, \Omega_\infty\} \in \Delta_2^A.$$

Given the compact resolvent of H, it is natural that these problems have the same SCI classification as for compact operators Ω_{C} (see Theorem 4.4 in §4). The continuity assumption on V in Theorem 5.5 ensures that the discretization used converges. However, tweaking the approximation can weaken this assumption to include potentials with certain discontinuities.

6. Computing the non-computable - The role of the SCI hierarchy in computer-assisted proofs

Computer-assisted proofs using numerical approximations have become essential in mathematics. An increasing number of famous conjectures and theorems have been proven using computer-assisted proofs. A highly incomplete list in arbitrary order includes the Dirac–Schwinger conjecture [51–59], the Double-Bubble conjecture [73], Kepler's conjecture (Hilbert's 18th problem) [67, 68], Smale's 14th problem [115], the 290-theorem [18], the weak Goldbach conjecture [76], blow-up of the 3D Euler equation with smooth initial data [32, 33], low dimensional topology [60, 61] etc. In all of these cases, the proofs are based on numerical computations with approximations. Hence, a key question will always be: Given a problem that needs to be computed to secure a computer-assisted proof, can the computation be done with verification that is 100% reliable? Alternatively, asked more broadly:

Question I: Which computational problems are suitable for computer-assisted proofs?

The instinct would normally be that the computational problem must be in Δ_1^A, or computable in the words of Turing. This is not the case. The computer-assisted proof of Dirac–Schwinger conjecture and Kepler's conjecture were done by computing non-computable problems, i.e., $\notin \Delta_1^G$, as explained below. Several cases of important conjectures have been solved by computer-assisted proof, where the computational problem is higher up in the SCI hierarchy than Δ_1^G. Hence, the SCI hierarchy is instrumental in answering Question I above as follows.

(i) *Classifications in the SCI hierarchy - Which problems are safe in computer-assisted proof?* In addition to problems in the class Δ_1^A, problems in the classes Σ_1^A and Π_1^A can be used in computer-assisted proofs regardless of the metric space $\mathcal{M}$ (see Remark 3.1) that induces the different classifications. However, the use of problems in Σ_1^A or Π_1^A depends on the phrasing of the conjecture. For example, suppose the conjectured statement is that spectra of operators in a certain class of self-adjoint discrete Schrödinger operators never intersect a certain open interval I. Such a statement can be falsified given the new Σ_1^A classification of computing spectra of discrete Schrödinger operators. Indeed, suppose one has located a candidate Schrödinger operator H for a counterexample; however, one does not know the spectrum of H. One can use one of the new algorithms realizing the Σ_1^A classification, and if $\mathrm{sp}(H) \cap I \neq \emptyset$, the algorithm will eventually demonstrate this intersection with a 100% guarantee, thus falsifying the conjecture. Similarly, decision problems in Σ_1^A and Π_1^A can be used in computer-assisted proof.

(ii) *A computer-assisted proof typically requires an SCI hierarchy classification.* A computer-assisted proof that relies on numerical computations typically requires proof of a Δ_1^A, Σ_1^A or Π_1^A classification in the SCI hierarchy. Indeed, a mathematician facing a computational problem to complete a computer-assisted proof will likely have to ask: where in the SCI hierarchy is the problem? If this is not already known, then one must prove it, and, as the examples below suggest, this classification is typically done implicitly in the proofs. Sometimes, this is trivial; however, sometimes, this may be very delicate, as in the proof of Kepler's conjecture, and intricate and technical, as in the proof of the Dirac–Schwinger conjecture.

(iii) *Understanding the higher end of the SCI hierarchy helps to answer Question I.* Answering Question I above becomes an infinite classification theory. Hence, given a particular computational problem that is desirable to use in a computer-assisted proof, one may not know the answer to the question whether this problem is in an appropriate class of the SCI hierarchy. However, one may know an upper bound, say Π_3^A. The question is whether extra features of the computational problem would allow for a classification lower in the SCI hierarchy. Existing classification in the SCI hierarchy will, therefore, be invaluable. In fact, the solution to the problem of computing spectra of Schrödinger operators evolved in this way, where initially, there was a crude classification of Π_3^A. By gradually learning which extra assumptions were needed to achieve classifications further down in the hierarchy, we eventually reached the sharp Σ_1^A classification, yielding a classification suitable for computer-assisted proofs.

Below are examples of successful computer-assisted proofs with the corresponding SCI hierarchy classification of the main computational problem.

Dirac–Schwinger conjecture - SCI classification: $\in \Sigma_1^A$, $\notin \Delta_1^G$: We discussed the details in §2.2.

Boolean Pythagorean triples problem - SCI classification: $\in \Pi_1^A$, $\notin \Delta_1^G$: The Boolean Pythagorean triples problem asks if it is possible to color each of the positive integers either red or blue, so that no Pythagorean triple of integers a, b, c, satisfying $a^2 + b^2 = c^2$ are all the same color. For example, in the Pythagorean triple 3, 4 and 5 ($3^2 + 4^2 = 5^2$), if 3 and 4 are coloured red, then 5 must be coloured blue. This is true for integers up to $n = 7824$. The computer-assisted proof, performed by M. Heule, O. Kullmann, and V. Marek (2016) [77], is based on showing that this is not true for $n = 7825$. While it is a combinatorial task checking the problem for any finite set of integers (and hence $\in \Delta_0^A$), it is clearly not $\in \Delta_0^G$ for infinite sets of integers. Nevertheless, the problem is clearly $\in \Pi_1^A$, which is why it was possible to verify the counterexample.

Group theory: $\mathrm{Aut}(\mathbb{F}_5)$ ***has property*** (T) ***- SCI classification :*** $\in \Sigma_1^A$, $\notin \Delta_1^G$: The fact that the automorphism group of the free group on five generators has Kazhdan's property (T), was shown by M. Kaluba, P. Nowak and N. Ozawa [81]. The proof relies on a decision problem involving a minimizer of a semi-definite program (actually a root of a positive definite matrix that is a minimizer). The minimizer is computed using floating-point arithmetic. Hence, it is, at best (if one could do a backward error analysis), equivalent to solving the semi-definite program with inexact input. Computing minimizers to semi-definite programs with inexact yet arbitrary small precision is $\notin \Delta_1^G$ [11]. Showing that the final decision problem used on [81] is $\in \Sigma_1^A$ requires an argument, which we do not repeat here. However, it is similar to the argument above, arguing why Kepler's conjecture could be verified.

Kepler's Conjecture (Hilbert's 18th problem) - SCI classification: $\in \Sigma_1^A$, $\notin \Delta_1^G$: Kepler conjectured that no packing of congruent balls in Euclidean three-space has density greater than that of cubic close packing and hexagonal close packing arrangements. The Flyspeck program, led by T. Hales [67, 68], provides a fully computer-assisted verification, where parts of the numerical computations in the computer-assisted proof are based on deciding the feasibility of about 50000 linear programs with irrational inputs. Such decision problems are $\notin \Delta_1^G$, and in fact generally higher than Σ_1^G, as established using the SCI framework in [11]. However, in the special cases of the linear programs used in the proof of Kepler's conjecture, one establishes the needed Σ_1^A.

Blow-up of the 3D Euler equation – Upper bounds on breakdown-epsilons: An essential part of the proof establishes that the breakdown epsilon, ϵ_0, for computing the solution to a rescaled 3D Euler equation is bounded by 10^{-3}. The details are discussed in 2.4.

7. THE SOLVABILITY COMPLEXITY INDEX HIERARCHY AND TOWERS OF ALGORITHMS

Throughout this paper, we assume the following:

(7.1a) Ω is some set, called the *domain*,

(7.1b) Λ is a set of complex-valued functions on Ω, called the *evaluation* set,

(7.1c) $\mathcal{M}$ is a metric space,

(7.1d) The mapping $\Xi : \Omega \to \mathcal{M}$, called the *problem* function.

The set Ω is the collection of objects that give rise to our computational problems $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$. It can be a family of matrices (infinite or finite), a collection of polynomials, a family of Schrödinger (or Dirac) operators with a certain potential, etc. The problem function $\Xi : \Omega \to \mathcal{M}$ is what we are interested in computing. It could be the set of eigenvalues of an $n \times n$ matrix, the spectrum of a Hilbert (or Banach) space operator, a polynomial's root(s), etc. Finally, the set Λ is the collection of functions that provide us with the information we can read, such as matrix elements, polynomial coefficients, or pointwise values of a potential function of a Schrödinger operator. In most cases, it is convenient to consider a metric space $\mathcal{M}$; however, in the case of polynomials, it may be more useful to use a pseudo metric space (see Example 7.1 (III)). To explain this rather abstract setup in (7.1) we commence with the following examples:

Example 7.1.

(I) **(Spectral problems)** Let $\Omega = \mathcal{B}(\mathcal{H})$, the set of all bounded linear operators on a separable Hilbert space $\mathcal{H}$, and the problem function Ξ be the mapping $A \mapsto \mathrm{sp}(A)$ (the spectrum of A). Here $(\mathcal{M}, d)$ is the set of all non-empty compact subsets of $\mathbb{C}$ provided with the Hausdorff metric $d = d_{\mathrm{H}}$ (defined precisely in (7.3)). The evaluation functions in Λ could, for example, consist of the family of all functions $f_{i,j} : A \mapsto \langle Ae_j, e_i \rangle$, $i, j \in \mathbb{N}$, which provide the entries of the matrix representation of A w.r.t. an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. Of course, Ω could be a strict subset of $\mathcal{B}(\mathcal{H})$, for example, the set of self-adjoint or normal operators, and Ξ could have represented the pseudospectrum, the essential spectrum or any other interesting information about the operator.

(II) **(Inverse problems)** Let $\Omega = \mathcal{B}_{\mathrm{inv}}(\mathcal{H}) \times \mathcal{H}$, where $\mathcal{B}_{\mathrm{inv}}(\mathcal{H})$ denotes the set of all bounded invertible operators on $\mathcal{H}$, and let the problem function Ξ be the mapping $(A, b) \mapsto A^{-1}b$, which assigns to a linear problem $Ax = b$ its solution x. The metric space $\mathcal{M}$ would simply be $\mathcal{H}$ and Λ the collection of mappings $\{f_{i,j}\}_{i\in\mathbb{N}, j\in\mathbb{Z}_+}$ where $f_{i,j} : (A, b) \mapsto \langle Ae_j, e_i\rangle$ for $j \in \mathbb{N}$ and $f_{i,0} : (A, b) \mapsto \langle b, e_i\rangle$. Also, here Ω could consist of operators with specific properties (off-diagonal decay, self-adjointness, isometric properties).

(III) **(Polynomial root finding)** Let $\Omega = \mathbb{P}_s$, the set of polynomials of degree $\leq s$ over $\mathbb{C}$ and let the problem function Ξ be the mapping $p \mapsto \{\alpha \in \mathbb{C} \mid p(\alpha) = 0\}$ (the roots of p). Let $(\mathcal{M}, d)$ denote the collection of finite sets of points in $\mathbb{C}$ equipped with the pseudo metric $d : \mathcal{M} \times \mathcal{M} \to [0, \infty]$, defined by $d(x, y) = \min_{1\leq i\leq n, 1\leq j\leq m} |x_j - y_i|$, where $x = \{x_1, \ldots, x_n\}$, $y = \{y_1, \ldots, y_m\} \in \mathcal{M}$. The pseudo metric is because the techniques of Doyle and McMullen that we will consider are based on computing a single root of a polynomial (as Newton's method does). In this case Λ is the finite set of functions $\{f_j\}_{j=1}^s$ where $f_j : p \mapsto \alpha_j$ for $p(t) = \sum_{k=1}^s \alpha_k t^k$.

(IV) **(Computational quantum mechanics)** Let $\Omega = \mathrm{L}^\infty(\mathbb{R}^d) \cap \mathrm{C}(\mathbb{R}^d)$ and let $\Xi : V \mapsto \mathrm{sp}(-\Delta + V)$, where the domain $\mathcal{D}(-\Delta + V) = \mathrm{W}^{2,2}(\mathbb{R}^d)$ (the standard Sobolev space) and $-\Delta + V$ is the usual Schrödinger operator. Given that the spectra are unbounded, we cannot use the Hausdorff metric anymore but will let $(\mathcal{M}, d_{\mathrm{AW}})$ denote the set of non-empty closed subsets of $\mathbb{C}$ equipped with the *Attouch–Wets* metric (see (7.4)). In this case, a natural choice of Λ would be the set of all evaluations $f_x : V \mapsto V(x)$, $x \in \mathbb{Q}^d$.

(V) **(Decision making)** Let Ω denote the set of infinite matrices with values in $\{0, 1\}$ and $\Xi : \Omega \to \mathcal{M} = \{\textit{Yes}, \textit{No}\}$ where $\mathcal{M}$ is equipped with the discrete metric d_{disc}. The evaluation functions would naturally be $f_{i,j} : A \mapsto A_{i,j}$, $i, j \in \mathbb{N}$, the (i, j)th matrix coordinate of A. A typical example of Ξ could be: $\Xi(\{A_{i,j}\})$: Does $\{A_{i,j}\}$ have a column containing infinitely many non-zero entries? Naturally, Ω can be replaced with the natural numbers including zero $\mathbb{Z}_+$, and Ξ could be a question about membership in a certain set, as in classical recursion theory. In this case the evaluation set would be $\Lambda = \{\lambda\}$ consisting of the function $\lambda : \mathbb{Z}_+ \to \mathbb{C}$, $x \mapsto x$.

Given this setup and motivation, we can now define what we mean by a computational problem.

Definition 7.2 (Computational problem)**.** Given a domain Ω; an evaluation set Λ, such that for $A_1, A_2 \in \Omega$ then $A_1 = A_2$ if and only if $f(A_1) = f(A_2)$ for all $f \in \Lambda$; a metric space $\mathcal{M}$; and a problem function $\Xi : \Omega \to \mathcal{M}$, we call the collection $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ a computational problem.

We aim to find and study families of functions (that we will sometimes refer to as algorithms) that permit us to approximate the function Ξ. The central pillar of our framework is the concept of a tower of algorithms. However, before that, we will define a general algorithm.

Definition 7.3 (General Algorithm)**.** Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, a *general algorithm* is a mapping $\Gamma : \Omega \to \mathcal{M}$ such that for each $A \in \Omega$:

(i) there exists a finite subset of evaluations $\Lambda_\Gamma(A) \subset \Lambda$,
(ii) the action of Γ on A is uniquely determined by $\{A_f\}_{f\in\Lambda_\Gamma(A)}$ where $A_f := f(A)$,
(iii) for every $B \in \Omega$ such that $B_f = A_f$ for every $f \in \Lambda_\Gamma(A)$, it holds that $\Lambda_\Gamma(B) = \Lambda_\Gamma(A)$.

We will sometimes write $\Gamma(\{A_f\}_{f\in\Lambda_\Gamma(A)})$, in order to emphasise that $\Gamma(A)$ only depends on the results $\{A_f\}_{f\in\Lambda_\Gamma(A)}$ of finitely many evaluations.

Note that there are no restrictions on the operations allowed for a general algorithm. The only restriction is that it can only take a finite amount of information, though it *is* allowed to *adaptively* choose the finite amount of information it reads depending on the input (which may very well be infinite, say an infinite matrix, or a function). Condition (iii) ensures that the algorithm is well-defined and consistent since, put in simple words, changing the input A shall not affect the algorithm's action as long as the change does not affect the output of the relevant evaluations in $\Lambda_\Gamma(A)$.

Remark 7.4 (The purpose of a general algorithm)**.** The purpose of a general algorithm is to have a definition that will encompass any model of computation and allow lower bounds and impossibility results to become universal. Given that there are several non-equivalent models of computation, lower bounds will be shown with a general definition of an algorithm. Upper bounds will always be done with more structure on the algorithms, for example, using a Turing machine or a Blum–Shub–Smale (BSS) machine.

However, more than a general algorithm is needed to describe the universe of computational problems. We need the concept of *towers of algorithms* for that.

Definition 7.5 (Tower of algorithms)**.** Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, a *tower of algorithms of height k for* $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ is a collection of sequences of functions

$$\Gamma_{n_k} : \Omega \to \mathcal{M}, \quad \Gamma_{n_k,n_{k-1}} : \Omega \to \mathcal{M}, \, \dots \, , \Gamma_{n_k,\dots,n_1} : \Omega \to \mathcal{M},$$

where $n_k, \dots, n_1 \in \mathbb{N}$ and the functions $\Gamma_{n_k,\dots,n_1}$ at the lowest level in the tower are general algorithms in the sense of Definition 7.3. Moreover, for every $A \in \Omega$,

$$\begin{aligned} \Xi(A) &= \lim_{n_k\to\infty} \Gamma_{n_k}(A), \\ \Gamma_{n_k}(A) &= \lim_{n_{k-1}\to\infty} \Gamma_{n_k,n_{k-1}}(A), \\ &\;\;\vdots \\ \Gamma_{n_k,\dots,n_2}(A) &= \lim_{n_1\to\infty} \Gamma_{n_k,\dots,n_1}(A), \end{aligned} \tag{7.2}$$

where $S = \lim_{n\to\infty} S_n$ means convergence $S_n \to S$ in the (pseudo) metric space $\mathcal{M}$. For simplicity, and with a slight abuse of notation, we will often refer to $\{\Gamma_{n_k,\dots,n_1}\}$ as a tower of algorithms, implicitly meaning the whole collection as described above.

In this paper, we will discuss several types of towers: *General towers*, when there is no extra structure on the functions at the lowest level in the tower; *Doyle–McMullen towers*, that are used for Smale's problem on polynomial root finding (see §10); *Arithmetic towers*, that restricts the algorithm to arithmetic operations and comparisons; *Radical towers*, that also allows the operation of $\sqrt{\cdot}$ of a real number. A General tower will refer to the very general definition in Definition 7.5 specifying that there are no further restrictions as will be the case for the other towers. When we specify the tower type, we specify requirements on the functions $\Gamma_{n_k,\dots,n_1}$, in particular, what kind of operations may be allowed. We can now define an *arithmetic tower of algorithms* and a *radical tower of algorithms*.

Definition 7.6 (Arithmetic towers)**.** Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, where Λ is countable, we define the following: An *Arithmetic tower of algorithms* of height k for $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ is a tower of algorithms where the lowest functions $\Gamma = \Gamma_{n_k,\ldots,n_1} : \Omega \to \mathcal{M}$ satisfy the following: For each $A \in \Omega$ the mapping $(n_k, \ldots, n_1, \{A_f\}_{f\in\Lambda})) \mapsto \Gamma_{n_k,\ldots,n_1}(A) = \Gamma_{n_k,\ldots,n_1}(\{A_f\}_{f\in\Lambda})$ is recursive, and $\Gamma_{n_k,\ldots,n_1}(A)$ is a finite string of complex numbers that can be identified with an element in $\mathcal{M}$. For arithmetic towers, we let $\alpha = A$

Remark 7.7 (Recursiveness)**.** By recursive we mean the following. If $f(A) \in \mathbb{Q}$ for all $f \in \Lambda$, $A \in \Omega$, and Λ is countable, then $\Gamma_{n_k,\ldots,n_1}(\{A_f\}_{f\in\Lambda})$ can be executed by a Turing machine [116], that takes $(n_k, \ldots, n_1, \{A_f\}_{f\in\Lambda}))$ as input, and that has an oracle tape consisting of $\{A_f\}_{f\in\Lambda}$. If $f(A) \in \mathbb{R}$ (or $\mathbb{C}$) for all $f \in \Lambda$, then $\Gamma_{n_k,\ldots,n_1}(\{A_f\}_{f\in\Lambda})$ can be executed by a Blum-Shub-Smale (BSS) machine [19] that takes $(n_k, \ldots, n_1, \{A_f\}_{f\in\Lambda}))$, as input, and that has an oracle that can access any A_f for $f \in \Lambda$.

Remark 7.8 (Radical towers and beyond - the SCI and the insolvability of the quintic)**.** Similarly to the definition of an arithmetic tower, one could define a radical tower, $\alpha = R$, by allowing, in addition to arithmetic operations and comparisons, the operation $\sqrt{\cdot}$ on real numbers. In that case, the recursiveness requirement above would mean recursive in the sense of a BSS machine with an oracle for computing $\sqrt{\cdot}$. In this case, the insolvability of the quintic becomes a question of the SCI with respect to a radical tower of algorithms. Similarly, one could define other towers by allowing other operations.

Given the definition of a tower of algorithms, we can now define the main concept of this paper: the Solvability Complexity Index (SCI). The SCI was first discussed in [71] for a specific spectral problem. However, this definition extends to include general problems in computations.

Definition 7.9 (Solvability Complexity Index)**.** Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, it is said to have *Solvability Complexity Index* $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}, \Lambda)_\alpha = k$ with respect to a tower of algorithms of type α if k is the smallest integer for which there exists a tower of algorithms of type α of height k. If no such tower exists then $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}, \Lambda)_\alpha = \infty$. If there exists a tower $\{\Gamma_n\}_{n\in\mathbb{N}}$ of type α and height one such that $\Xi = \Gamma_{n_1}$ for some $n_1 < \infty$, then we define $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}, \Lambda)_\alpha = 0$.

With the definition of the SCI, we can define the SCI hierarchy, for which any computational problem can be classified. Without any extra structure on the metric space $\mathcal{M}$, the Δ_k^α classes are the finest refinement we can obtain regarding the SCI. However, as described below, the hierarchy becomes much richer when more structure is present.

Definition 7.10 (The Solvability Complexity Index hierarchy)**.** Consider a collection $\mathcal{C}$ of computational problems and let $\mathcal{T}$ be the collection of all towers of algorithms of type α for the computational problems in $\mathcal{C}$. Define

$$\Delta_0^\alpha := \{\{\Xi, \Omega\} \in \mathcal{C} \mid \mathrm{SCI}(\Xi, \Omega)_\alpha = 0\}$$
$$\Delta_{m+1}^\alpha := \{\{\Xi, \Omega\} \in \mathcal{C} \mid \mathrm{SCI}(\Xi, \Omega)_\alpha \le m\}, \qquad m \in \mathbb{N},$$

as well as

$$\Delta_1^\alpha := \{\{\Xi, \Omega\} \in \mathcal{C} \mid \exists\, \{\Gamma_n\} \in \mathcal{T} \text{ s.t. } \forall A \in \Omega\ d(\Gamma_n(A), \Xi(A)) \le 2^{-n}\}.$$

7.1. **Extending the hierarchy for totally ordered $\mathcal{M}$.** When there is extra structure on the metric space $\mathcal{M}$, say $\mathcal{M} = \mathbb{R}$ or $\mathcal{M} = \{0, 1\}$ with the standard metric, one may be able to define convergence of functions from above or below. This is an extra form of structure that allows for a type of error control. As we argue below, this is important, for example, in computer-assisted proofs and crucial in scientific computing.

Definition 7.11 (The SCI Hierarchy (totally ordered set))**.** Given the setup in Definition 7.10, suppose in addition that $\mathcal{M}$ is a totally ordered set. Define

$$
\begin{aligned}
\Sigma_0^\alpha &= \Pi_0^\alpha = \Delta_0^\alpha, \\
\Sigma_1^\alpha &= \{\{\Xi,\Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T} \text{ s.t. } \Gamma_n(A) \nearrow \Xi(A) \ \forall A \in \Omega\}, \\
\Pi_1^\alpha &= \{\{\Xi,\Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T} \text{ s.t. } \Gamma_n(A) \searrow \Xi(A) \ \forall A \in \Omega\},
\end{aligned}
$$

where $\nearrow$ and $\searrow$ denotes convergence from below and above respectively, as well as, for $m \in \mathbb{N}$,

$$
\begin{aligned}
\Sigma_{m+1}^\alpha &= \{\{\Xi,\Omega\} \in \Delta_{m+2}^\alpha \mid \exists \{\Gamma_{n_{m+1},\dots,n_1}\} \in \mathcal{T} \text{ s.t. } \Gamma_{n_{m+1}}(A) \nearrow \Xi(A) \ \forall A \in \Omega\}, \\
\Pi_{m+1}^\alpha &= \{\{\Xi,\Omega\} \in \Delta_{m+2}^\alpha \mid \exists \{\Gamma_{n_{m+1},\dots,n_1}\} \in \mathcal{T} \text{ s.t. } \Gamma_{n_{m+1}}(A) \searrow \Xi(A) \ \forall A \in \Omega\}.
\end{aligned}
$$

If the metric space $\mathcal{M} = \{0,1\}$, it is a totally ordered set, and hence, from Definition 7.11, we get the SCI hierarchy for arbitrary decision problems.

7.2. **Extending the hierarchy for spectral problems.** In the case where $\mathcal{M}$ is the collection of non-empty closed subsets of another metric space $(\mathcal{M}', d')$ it is custom to equip $\mathcal{M}$ with the Hausdorff metric (bounded case)

$$
d_{\mathrm{H}}(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d'(x,y), \sup_{y\in Y}\inf_{x\in X} d'(x,y)\right\}, \tag{7.3}
$$

or the Attouch–Wets metric (unbounded case)

$$
d_{\mathrm{AW}}(A,B) = \sum_{m=1}^{\infty} 2^{-m}\min\left\{1, \sup_{d'(x,x_0)<m} |\mathrm{dist}(x,A) - \mathrm{dist}(x,B)|\right\}, \tag{7.4}
$$

where A and B are non-empty closed subsets of $\mathcal{M}'$, and where $\mathrm{dist}(x,A)$ denotes the distance between the point $x \in \mathcal{M}'$ and $A \subset \mathcal{M}'$, and where $x_0 \in \mathcal{M}'$ can be chosen arbitrarily.

Definition 7.12 (The SCI Hierarchy (Attouch–Wets/Hausdorff metric))**.** Given the set-up in Definition 7.10, and suppose in addition that $(\mathcal{M}, d)$ has the Attouch–Wets or the Hausdorff metric induced by another metric space $(\mathcal{M}', d')$, define, for $m \in \mathbb{N}$,

$$
\begin{aligned}
\Sigma_0^\alpha &= \Pi_0^\alpha = \Delta_0^\alpha, \\
\Sigma_1^\alpha &= \{\{\Xi,\Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T},\ \{X_n(A)\} \subset \mathcal{M} \text{ s.t. } \Gamma_n(A) \underset{\mathcal{M}'}{\subset} X_n(A), \\
&\qquad \lim_{n\to\infty} \Gamma_n(A) = \Xi(A),\ d(X_n(A), \Xi(A)) \leq 2^{-n} \ \forall A \in \Omega\}, \\
\Pi_1^\alpha &= \{\{\Xi,\Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T},\ \{X_n(A)\} \subset \mathcal{M} \text{ s.t. } \Xi(A) \underset{\mathcal{M}'}{\subset} X_n(A), \\
&\qquad \lim_{n\to\infty} \Gamma_n(A) = \Xi(A),\ d(X_n(A), \Gamma_n(A)) \leq 2^{-n} \ \forall A \in \Omega\},
\end{aligned}
$$

where $\subset_{\mathcal{M}'}$ means inclusion in the metric space $\mathcal{M}'$, and $\{X_n(A)\}$ is a sequence where $X_n(A) \in \mathcal{M}$ depends on A. Moreover,

$$
\begin{aligned}
\Sigma_{m+1}^\alpha &= \{\{\Xi,\Omega\} \in \Delta_{m+2}^\alpha \mid \exists \{\Gamma_{n_{m+1},\dots,n_1}\} \in \mathcal{T},\ \{X_{n_{m+1}}(A)\} \subset \mathcal{M} \text{ s.t. } \Gamma_{n_{m+1}}(A) \underset{\mathcal{M}'}{\subset} X_{n_{m+1}}(A), \\
&\qquad \lim_{n_{m+1}\to\infty} \Gamma_{n_{m+1}}(A) = \Xi(A),\ d(X_{n_{m+1}}(A), \Xi(A)) \leq 2^{-n_{m+1}} \ \forall A \in \Omega\}, \\
\Pi_{m+1}^\alpha &= \{\{\Xi,\Omega\} \in \Delta_{m+2}^\alpha \mid \exists \{\Gamma_{n_{m+1},\dots,n_1}\} \in \mathcal{T},\ \{X_{n_{m+1}}(A)\} \subset \mathcal{M} \text{ s.t. } \Xi(A) \underset{\mathcal{M}'}{\subset} X_{n_{m+1}}(A), \\
&\qquad \lim_{n_{m+1}\to\infty} \Gamma_{n_{m+1}}(A) = \Xi(A),\ d(X_{n_{m+1}}(A), \Gamma_{n_{m+1}}(A)) \leq 2^{-n_{m+1}} \ \forall A \in \Omega\},
\end{aligned}
$$

where d can be either d_{H} or d_{AW}.

Remark 7.13 (Convergence from below and above)**.** Intuitively, Definition 7.12 captures convergence from below or above, respectively, up to a small error parameter 2^{-n}. Indeed, in the case of the Hausdorff metric case, it is easy to see that the above Definition 7.12 yields

$$\Sigma_1^\alpha = \{\{\Xi,\Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T} \text{ s.t. } \Gamma_n(A) \subset \bar{\mathcal{N}}_{2^{-n}}(\Xi(A)) \ \forall A \in \Omega\},$$
$$\Pi_1^\alpha = \{\{\Xi,\Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T} \text{ s.t. } \bar{\mathcal{N}}_{2^{-n}}(\Gamma_n(A)) \supset \Xi(A) \ \forall A \in \Omega\},$$

where $\bar{\mathcal{N}}_\delta(\omega)$ denotes the closed δ-neighbourhood of $\omega \subset \mathcal{M}'$, and similar definitions for Σ_{m+1}^α and Π_{m+1}^α.

Note that to build a Σ_1 algorithm, it is enough, by taking subsequences, to construct $\Gamma_n(A)$ such that $\Gamma_n(A) \subset \bar{\mathcal{N}}_{E_n(A)}(\Xi(A))$ with some computable $E_n(A)$ that converges to zero.

Definition 7.14. Given a totally ordered metric space $(\mathcal{M}, d)$, we say that the metric is order respecting if for any $a, b, c \in \mathcal{M}$ with $a \leq b \leq c$ we have $d(a, b) \leq d(a, c)$.

Proposition 7.15 (Properties of the SCI hierarchy)**.** *Given the setup, let $(\mathcal{M}, d)$ be either the Hausdorff or Attouch–Wets metric or a totally ordered metric space with order respecting metric. Let $k = 1, 2$ or 3, then we have the following.*

- *(i) $\Delta_k^G = \Sigma_k^G \cap \Pi_k^G$. In particular, if for a problem $\Xi : \Omega \to \mathcal{M}$ we have $\Delta_k^G \not\ni \{\Xi,\Omega\} \in X_k^\alpha$, where $X = \Sigma$ or Π and α denotes any type of tower, then $\{\Xi,\Omega\} \notin Y_k^\alpha$, where $Y = \Pi$ or Σ respectively.*
- *(ii) Suppose for a computational problem $\Xi : \Omega \to \mathcal{M}$ we have a corresponding convergent Σ_k^A tower $\Gamma^1_{n_k,\ldots,n_1}$ and a corresponding convergent Π_k^A tower $\Gamma^2_{n_k,\ldots,n_1}$. Suppose also that we can compute for every $A \in \Omega$ the distance $d(\Gamma^1_{n_k,\ldots,n_1}(A), \Gamma^2_{n_k,\ldots,n_1}(A))$ to arbitrary precision using finitely many arithmetic operations and comparisons. Then $\{\Xi,\Omega\} \in \Delta_k^A$.*

Finally, we also have the following property:

- *(iii) When $\mathcal{M} = \{0, 1\}$, $\Delta_k^\alpha = \Sigma_k^\alpha \cap \Pi_k^\alpha$ for all $k \in \mathbb{N}$ and $\alpha = G, A$.*

The proof of Proposition 7.15 can be found in §A.

Remark 7.16. Part (i) of Proposition 7.15 shows that the classifications obtained in this paper are sharp in the SCI hierarchy.

7.2.1. *Computing approximate eigenvectors.* Let $\mathcal{C}$ denote the collection of computation spectral problems $\{\Xi,\Omega,\mathcal{M},\Lambda\}$ where Ω is a collection of normal operators on some Hilbert space $\mathcal{H}$ and $\Xi(A) = \mathrm{sp}(A)$. If we consider bounded operators, $\mathcal{M}$ is the collection of compact subsets of $\mathbb{C}$ equipped with the Hausdorff, and in the unbounded case $\mathcal{M}$ is the collection of closed subsets of $\mathbb{C}$ with the Attouch–Wets metric.

$$\begin{aligned}\Sigma_1^{\alpha,\mathrm{eigv}} = \{\{\Xi,\Omega\} \in \Sigma_1^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T} \text{ s.t. } &\Gamma_n(A) = \{(\lambda_{1,n},\xi_{1,n}),\ldots,(\lambda_{K,n},\xi_{K,n})\},\\ &K = K(n) \in \mathbb{N},\ \lambda_{j,n} \in \bar{\mathcal{N}}_{2^{-n}}(\mathrm{sp}(A)),\ \|A\xi_{j,n} - \lambda_{j,n}\xi_{j,n}\| \leq 2^{-n},\\ &\|\xi_{j,n}\| = 1 + a_n,\ |a_n| \leq 2^{-n}\,\forall j,\ \cup_{j=1}^K \lambda_{j,n} \to \mathrm{sp}(A), n \to \infty,\ \forall A \in \Omega\}.\end{aligned}$$

In words $\Sigma_1^{\alpha,\mathrm{eigv}}$ can be described as follows.

> $\Sigma_1^{\alpha,\mathrm{eigv}}$ *is the collection of computational spectral problems concerning normal operators that are in Σ_1^α, where there exists an algorithm that can also compute approximate eigenvectors.*

7.3. **Inexact input.** Suppose we are given a computational problem $\{\Xi,\Omega,\mathcal{M},\Lambda\}$, and that $\Lambda = \{f_j\}_{j\in\beta}$, where β is some index set that can be finite or infinite. However, obtaining f_j may be a computational task on its own, which is exactly the problem in most areas of computational mathematics. In particular, for $A \in \Omega$, $f_j(A)$ could be the number $e^{\frac{\pi}{j}i}$ for example. Hence, we cannot access $f_j(A)$, but rather $f_{j,n}(A)$ where $f_{j,n}(A) \to f_j(A)$ as $n \to \infty$. Or, just as for problems that are high up in the SCI hierarchy, it could be that we need several limits, in particular, one may need mappings $f_{j,n_m,\ldots,n_1} : \Omega \to \mathbb{D} + i\mathbb{D}$, where $\mathbb{D}$ denotes the dyadic rational numbers, such that

$$\lim_{n_m\to\infty} \cdots \lim_{n_1\to\infty} \|\{f_{j,n_m,\ldots,n_1}(A)\}_{j\in\beta} - \{f_j(A)\}_{j\in\beta}\|_\infty = 0 \quad \forall A \in \Omega. \tag{7.5}$$

In particular, we may view the problem of obtaining $f_j(A)$ as a problem in the SCI hierarchy, where Δ_1 classification would correspond to the existence of mappings $f_{j,n} : \Omega \to \mathbb{D} + i\mathbb{D}$ such that

$$\|\{f_{j,n}(A)\}_{j\in\beta} - \{f_j(A)\}_{j\in\beta}\|_\infty \leq 2^{-n} \quad \forall A \in \Omega. \tag{7.6}$$

This idea is formalized in the following definition.

Definition 7.17 (Δ_m-information)**.** Let $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ be a computational problem. For $m \in \mathbb{N}$ we say that Λ has Δ_{m+1}-information if each $f_j \in \Lambda$ is not available, however, there are mappings $f_{j,n_m,\ldots,n_1} : \Omega \to \mathbb{D}+i\mathbb{D}$ such that (7.5) holds. Similarly, for $m = 0$ there are mappings $f_{j,n} : \Omega \to \mathbb{D} + i\mathbb{D}$ such that (7.6) holds. Finally, if $k \in \mathbb{N}$ and $\hat{\Lambda}$ is a collection of such functions described above such that Λ has Δ_k-information, we say that $\hat{\Lambda}$ provides Δ_k information for Λ. Moreover, we denote the family of all such $\hat{\Lambda}$ by $\mathcal{L}^k(\Lambda)$.

Note that we want to have algorithms that can handle all computational problems $\{\Xi, \Omega, \mathcal{M}, \hat{\Lambda}\}$ when $\hat{\Lambda} \in \mathcal{L}^m(\Lambda)$. In order to formalize this, we define what we mean by a computational problem with Δ_m information.

Definition 7.18 (Computational problem with Δ_m information)**.** Given $m \in \mathbb{N}$, a computational problem where Λ has Δ_m-information is denoted by $\{\Xi, \Omega, \mathcal{M}, \Lambda\}^{\Delta_m} := \{\tilde{\Xi}, \tilde{\Omega}, \mathcal{M}, \tilde{\Lambda}\}$, where

$$\tilde{\Omega} = \left\{\tilde{A} = \{f_{j,n_m,\ldots,n_1}(A)\}_{j,n_m,\ldots,n_1\in\beta\times\mathbb{N}^m} \mid A \in \Omega, \{f_j\}_{j\in\beta} = \Lambda, f_{j,n_m,\ldots,n_1} \text{ satisfy (*)}\right\},$$

and (*) denotes (7.5) if $m > 1$ and (*) denotes (7.6) if $m = 1$. Moreover, $\tilde{\Xi}(\tilde{A}) = \Xi(A)$, and we have $\tilde{\Lambda} = \{\tilde{f}_{j,n_m,\ldots,n_1}\}_{j,n_m,\ldots,n_1\in\beta\times\mathbb{N}^m}$ where $\tilde{f}_{j,n_m,\ldots,n_1}(\tilde{A}) = f_{j,n_m,\ldots,n_1}(A)$. Note that $\tilde{\Xi}$ is well defined by Definition 7.2 of a computational problem.

The SCI and the SCI hierarchy, given Δ_m-information, are then defined in the standard obvious way. We will use the notation

$$\{\Xi, \Omega, \mathcal{M}, \Lambda\}^{\Delta_m} \in \Delta_k^\alpha$$

to denote that the computational problem is in Δ_k^α given Δ_m-information. When $\mathcal{M}$ and Λ are obvious then we will write $\{\Xi, \Omega\}^{\Delta_m} \in \Delta_k^\alpha$ for short.

8. Proof of Theorem 4.4

We start the sections on the proofs of our main results with a simple but fundamental observation on the smallest singular values $\sigma_{\inf}(B)$ of finite matrices $B \in \mathbb{C}^{m\times n}$, which constitutes one of the cornerstones for most of the general algorithms we will construct in the subsequent proofs. Note that when dealing with infinite-dimensional operators, we will also use the notation $\sigma_{\inf}$ to denote the injection modulus defined for $A \in \mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, as

$$\sigma_{\inf}(A) := \inf_{\|x\|=1} \|Ax\|.$$

Proposition 8.1. *Given a matrix $B \in \mathbb{C}^{m\times n}$ and a number $\epsilon > 0$, one can test with finitely many arithmetic operations of the entries of B whether the smallest singular value $\sigma_{\inf}(B)$ of B is greater than ϵ.*

Proof. The matrix B^*B is self-adjoint and positive semi-definite; hence, it has its eigenvalues in $[0, \infty)$. The singular values of B are the square roots of these eigenvalues of B^*B. The smallest singular value is greater than ϵ if and only if the smallest eigenvalue of B^*B is greater than ϵ^2, which is the case if and only if $C := B^*B - \epsilon^2 I$ is positive definite. It is well known that C is positive definite if and only if the pivots left after Gaussian elimination (without row exchange) are all positive. Thus, if C is positive definite, Gaussian elimination leads to pivots that are all positive, and this requires finitely many arithmetic operations. If C is not positive definite, then at some point, a pivot is zero or negative; at this point the algorithm aborts. An alternative is the Cholesky decomposition. Although forming the lower triangular $L \in \mathbb{C}^{n\times n}$ (if it exists) such that $C = LL^*$ requires the use of radicals, the existence of L can be determined using finitely many arithmetic operations. This follows from the standard Cholesky algorithm, and we omit the details. □

Proposition 8.2. *Given a matrix $B \in \mathbb{C}^{m\times n}$ with Δ_1-information for the matrix entries of B, and $\eta > 0$, we can compute $\sigma_{\mathrm{inf}}(B)$ to accuracy η using finitely many arithmetic operations and comparisons over $\mathbb{Q}$.*

Proof. Without loss of generality, we can assume that $\eta \in \mathbb{Q}$. Let $\hat{B}$ be a rational approximation of B, obtained using Δ_1-information, such that $\|B - \hat{B}\| \leq \eta/2$. Note that we can bound the operator norm by the Frobenius norm and hence can guarantee $\|B - \hat{B}\| \leq \eta/2$ if each matrix entry of $\hat{B}$ is accurate to $\eta(2\sqrt{mn})^{-1}$ (we can choose a smaller rational accuracy parameter). It then follows that $|\sigma_{\mathrm{inf}}(B) - \sigma_{\mathrm{inf}}(\hat{B})| \leq \|B - \hat{B}\| \leq \frac{\eta}{2}$. The proposition follows if we can compute $\sigma_{\mathrm{inf}}(\hat{B})$ to accuracy $\eta/2$. To do this, let $M \in \mathbb{N}$ be such that $M^{-1} < \eta/2$. Using Proposition 8.1 (note that this only requires arithmetic operations and comparisons over $\mathbb{Q}$) and applying successive tests to $\epsilon = 1/M, 2/M, ...$, we can compute the smallest $k \in \mathbb{N}$ such that $\sigma_{\mathrm{inf}}(\hat{B}) \leq k/M$. Our approximation is then given by k/M. □

Remark 8.3 (Proofs of $\{\Xi, \Omega\}^{\Delta_1} \in \Delta/\Pi/\Sigma$)**.** All our theorems are valid regardless of inexact input (Δ_1 information), and the main reason is Proposition 8.2. Only minor alterations need to be done in the proofs to deal with inexact input, and there will be guidelines specifying where the changes are needed. Note that there are much more numerically efficient procedures than in the proof of Proposition 8.2. However, the purpose of Proposition 8.2 is to show that the algorithms we construct in this paper can be made to work in a 100% rigorous manner on a Turing machine with inexact Δ_1-information.

We will split the proof of Theorem 4.4 into several parts, and a brief roadmap for the proof is as follows. We first deal with computing the spectra and pseudospectra of compact operators since the constructive parts of the proof use a different (most likely more familiar) method, the finite section method, than the proof for the other classes of operators. Step I of this part also contains one of the arguments used to prove lower bounds throughout this paper and is written out in detail for the reader's convenience. We then move onto pseudospectra, where variants on the method of uneven sections are used to approximate the relevant resolvent norms. In some cases, these towers are used directly to provide (with an additional limit) towers of algorithms for the spectra. The proof that $\{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma_2^A$ uses a very different method to those usually found in the literature, a local estimation of the resolvent norm (using similar ideas to §8.2) together with the function g gives rise to upper bounds on the distance of a point to the spectrum. This is then used in a local search routine to compute the spectrum. The proof that $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \notin \Delta_3^G$ relies on reducing a decision problem, known to require three limits, to $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\}$. Proof that the decision problem requires three limits is provided in §8.6 via a Baire category argument. The constructive proofs for essential spectra build on the towers of algorithms for computing spectra but are more involved. We end with the problem Ξ_{sp}^z where the proof of lower bounds uses similar arguments for the other problem functions, and the construction of towers of algorithms uses the towers constructed in §8.3 for the spectrum.

8.1. **Spectra of compact operators.**

Proof of Theorem 4.4 for compact operators. **Step I**: $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{C}}\} \notin \Sigma_1^G$. We argue by contradiction and suppose that there is a sequence $\{\Gamma_n\}$ of general algorithms such that, for every $A \in \Omega_{\mathrm{C}}$, $\Gamma_n(A) \to \mathrm{sp}(A)$ with $\Gamma_n(A) \subset \mathrm{sp}(A) + B_{2^{-n}}(0)$, and in particular each $\Lambda_{\Gamma_n}(A)$ is finite. Thus, we define $N(A, n) :=$

$\max\{i,j \mid f_{i,j} \in \Lambda_{\Gamma_n}(A)\}$. We consider an operator of the type

$$A := A_k \oplus \mathrm{diag}\{0,0,...\} \quad \text{with } A_k := \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{k\times k},$$

where we will choose the specific value of k later. Let $C = \mathrm{diag}\{1,0,0,...\}$ then $\mathrm{sp}(C) = \{0,1\}$ and clearly A is compact with $\mathrm{sp}(A) = \{0,2\}$. We choose k to gain a contradiction as follows. There exists n such that $\Gamma_n(C) \cap B_{1/4}(1) \neq \emptyset$. Let $k > N(C,n)$. By this construction, it follows that $\Gamma_n(C) = \Gamma_n(A)$. Indeed, since any evaluation function $f_{i,j} \in \Lambda$ just provides the (i,j)-th matrix element, it follows by the choice of k that for any evaluation functions $f_{i,j} \in \Lambda_{\Gamma_n}(A)$ we have that that $f_{i,j}(A) = f_{i,j}(C)$. Thus, by assumption (iii) in the definition of a general algorithm (Definition 7.3), we get that $\Lambda_{\Gamma_n}(A) = \Lambda_{\Gamma_n}(C)$ which, by assumption (ii) in Definition 7.3, yields $\Gamma_n(C) = \Gamma_n(A)$. But then $\Gamma_n(A) \cap B_{1/4}(1) \neq \emptyset$, which is impossible since $\Gamma_n(A) \subset \{0,2\} + B_{2^{-n}}(0)$, a contradiction.

Step II: $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{C}}\} \notin \Pi_1^G$. This is essentially the same argument. Assume that there exists Γ_n such that $\mathrm{sp}(A) \subset \Gamma_n(A) + B_{2^{-n}}(0)$. Let A and C be as before. But now we know that there exists n such that $\Gamma_n(C) \cap B_{3/4}(2) = \emptyset$. We argue as before, choosing $k > N(C,n)$, to get $\Gamma_n(C) = \Gamma_n(A)$. But we must have $2 \in \Gamma_n(A) + B_{2^{-n}}(0)$, a contradiction.

Step III: $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{\mathrm{C}}\} \notin \Pi_1^G \cup \Sigma_1^G$. For sufficiently small ϵ, we have the required separation such that the above argument works for $\Xi^N_{\mathrm{sp},\epsilon}$. For larger ϵ, we appropriately rescale the operators in the argument.

Step IV: $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{C}}\} \in \Delta_2^A$. For $n \in \mathbb{N}$, let $G_n = \frac{1}{n}(\mathbb{Z} + i\mathbb{Z}) \cap B_n(0)$. For $A \in \Omega_{\mathrm{C}}$ let $\Gamma_n(A) = \{z \in G_n : \sigma_{\inf}(P_n(A - zI)P_n) \leq 1/n\}$, where P_n denotes the orthogonal projection onto the linear span of the first n basis vectors. By Proposition 8.1, it is clear that this can be computed in finitely many arithmetical operations and comparisons. Hence we are done if we can prove convergence, the proof of which will make clear that we can make $\Gamma_n(A)$ non-empty by replacing $\Gamma_n(A)$ with $\Gamma_{m(n)}(A)$ such that $m(n) \geq n$ is minimal with $\Gamma_{m(n)}(A) \neq \emptyset$. Let $\epsilon > 0$, then choose $N > 2/\epsilon$. If $n \geq N$ and $z \in \Gamma_n(A)$ then we must have $\sigma_{\inf}(P_n(A - zI)P_n) \leq \epsilon/2$. Hence there exists $x_n \in l^2(\mathbb{N})$ of norm 1 and with $x_n = P_n x_n$ such that $\|(P_n A - zI)x_n\| \leq \epsilon/2$. A is compact and hence we can choose N large if necessary to ensure that $\|(I - P_n)A\| \leq \epsilon/2$. It follows that $\|(A - zI)x_n\| \leq \epsilon$ and hence z is in $\mathrm{sp}_\epsilon(A)$. Note that N does not depend on the point z so for large n we have $\Gamma_n(A) \subset \mathrm{sp}_\epsilon(A)$.

Conversely, let $z \in \mathrm{sp}(A)$. The method of finite section converges for compact operators, and hence there exists $z_n \in \mathrm{sp}(P_n A P_n)$ with $z_n \to z$. Let $w_n \in G_n$ be of minimal distance to z_n then for large n we must have $|w_n - z_n| \leq 1/(\sqrt{2}n)$ and hence $\sigma_{\inf}(P_n(A - w_n I)P_n) \leq 1/(\sqrt{2}n) < 1/n$. It follows that $w_n \in \Gamma_n(A)$. Let $\epsilon > 0$, then we can choose a finite set $S_\epsilon \subset \mathrm{sp}(A)$ with $d_{\mathrm{H}}(S_\epsilon, \mathrm{sp}(A)) < \epsilon/2$. Applying the above argument to all points in S_ϵ implies that for large n we must have that $\mathrm{sp}(A) \subset \Gamma_n(A) + B_\epsilon(0)$. Hence, since $\epsilon > 0$ was arbitrary, the fact that $\Gamma_n(A) \subset \mathrm{sp}_\epsilon(A)$ implies the required convergence.

Step V: $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{\mathrm{C}}\} \in \Delta_2^A$. This will follow from the classification of $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_f\}$ since we can use a dispersion bounding function $f(n) = n + 1$. Note that we do not necessarily know the dispersion bound (in the form of the null sequence $\{c_n\}$) and hence (see Remark 4.5) this provides a Δ_2^A tower (however not the Σ_1^A classification). □

Remark 8.4. To deal with Δ_1-information in the above construction (Step IV), we can replace $\sigma_{\inf}(P_n(A - zI)P_n)$ by a rational approximation accurate to $1/n^2$ (see Proposition 8.2) and the proof follows through with minor changes.

8.2. N**-pseudospectrum.** Since $\Omega_{\mathrm{SA}} \subset \Omega_{\mathrm{N}} \subset \Omega_g \subset \Omega_{\mathrm{B}}$, $\Omega_{fg} \subset \Omega_f$ and we have already dealt with compact operators, we only need to show that $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{\mathrm{B}}\} \in \Sigma^A_2$, $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_f\} \in \Sigma^A_1$, $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{\mathrm{SA}}\} \notin \Delta^G_2$ and $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{fg}\} \notin \Delta^G_1$.

Proof of Theorem 4.4 for the pseudospectrum. **Step I:** $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{\mathrm{B}}\} \in \Sigma^A_2$. Let $A \in \Omega_{\mathrm{B}}$, and $\epsilon > 0$. We introduce the following continuous functions $\gamma^N : \mathbb{C} \to \mathbb{R}_+$, $\gamma^N_m : \mathbb{C} \to \mathbb{R}_+$ and $\gamma^N_{m,n} : \mathbb{C} \to \mathbb{R}_+$,

$$\gamma^N(z) := \left(\min\left\{\sigma_{\inf}\left((A - zI)^{2^N}\right), \sigma_{\inf}\left((A^* - \bar{z}I)^{2^N}\right)\right\}\right)^{2^{-N}} = \left\|(A - zI)^{-2^N}\right\|^{-2^{-N}}$$

$$\gamma^N_m(z) := \left(\min\left\{\sigma_{\inf}\left((A - zI)^{2^N} P_m\right), \sigma_{\inf}\left((A^* - \bar{z}I)^{2^N} P_m\right)\right\}\right)^{2^{-N}}$$

$$\gamma^N_{m,n}(z) := \left(\min\left\{\sigma_{\inf}\left((P_n(A - zI)P_n)^{2^N} P_m\right), \sigma_{\inf}\left((P_n(A^* - \bar{z}I)P_n)^{2^N} P_m\right)\right\}\right)^{2^{-N}},$$

where $\sigma_{\inf}(B)$ denotes the injection modulus of B, and in the terms such as $\sigma_{\inf}(P_n B P_m)$ the operator $P_n B P_m$ is regarded as element of $\mathcal{B}(\mathrm{Ran}(P_m), \mathrm{Ran}(P_n))$. For the proof that $\gamma^N(z) = \|(A-zI)^{-2^N}\|^{-2^{-N}}$ see [71]. We define initial approximations $\hat{\Gamma}_{m,n}(A)$ for $\mathrm{sp}_{N,\epsilon}(A)$ by $\hat{\Gamma}_{m,n}(A) := \{z \in G_n : \gamma^N_{m,n}(z) \le \epsilon\}$, where $G_j := (j^{-1}(\mathbb{Z} + \mathrm{i}\mathbb{Z})) \cap B_j(0)$. Writing $\gamma^N_{m,n}(z) \le \epsilon$ as $(\gamma^N_{m,n}(z))^{2^N} \le \epsilon^{2^N}$ and due to Proposition 8.1 it is clear that the computation of $\hat{\Gamma}_{m,n}(A)$ requires only finitely many arithmetic operations on finitely many evaluations $\{\langle Ae_j, e_i\rangle : i, j = 1, \dots, n\}$ of A. The problem with this tower is that it might produce the empty set. To get round this and construct our Σ^A_2 arithmetical tower, there are several facts we will state that can be found in [71]. First, $\gamma^N_{m,n}$ converges uniformly to γ^N_m on compact subsets of $\mathbb{C}$ as $n \to \infty$. Second, γ^N_m is non-increasing in m and converges uniformly to γ^N on compact subsets of $\mathbb{C}$ as $m \to \infty$. Finally, we have

$$\mathrm{cl}\{z \in \mathbb{C} : \gamma^N_m(z) < \epsilon\} = \{z \in \mathbb{C} : \gamma^N_m(A) \le \epsilon\} \tag{8.1}$$

for all $\epsilon > 0$. Now it is straightforward to show via a Neumann series argument (see the proof that $\{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma^A_2$ below) that there exists a compact ball K such that if $z \notin K$ then $\gamma^N_{m,n}(z) > 2\epsilon$ for all m, n. In particular, by considering the minimum of $\gamma^N_m(\cdot)$, this, together with the above closure property, shows that the minimum is zero and $\{z \in \mathbb{C} : \gamma^N_m(A) \le \epsilon\} \neq \emptyset$.

Now let $z_0 \in \{z \in \mathbb{C} : \gamma^N_m(z) < \epsilon\}$. On the compact set K, and for any m, the functions $\gamma^N_{m,n}$ and γ^N_m are Lipschitz continuous with a uniform Lipschitz constant. Using this and (8.1), it follows that for large enough n, there exists $z_n \in \hat{\Gamma}_{m,n}(A)$ with $z_n \to z_0$. Furthermore, if $z_n \in \hat{\Gamma}_{m,n}(A)$ and we select a subsequence such that $z_{n_j} \to z$ as $n_j \to \infty$, we see that $\gamma^N_m(z) \le \epsilon$. These observations together imply that

$$\lim_{n\to\infty} \hat{\Gamma}_{m,n}(A) = \{z \in \mathbb{C} : \gamma^N_m(A) \le \epsilon\} \subset \mathrm{sp}_{N,\epsilon}(A).$$

Since γ^N_m converges to γ^N uniformly on compact sets and are uniformly Lipschitz, it is easy to show that $\lim_{m\to\infty}\{z \in \mathbb{C} : \gamma^N_m(A) \le \epsilon\} = \mathrm{sp}_{N,\epsilon}(A)$. Hence in order to construct our Σ^A_2 arithmetical tower we define $\Gamma_{m,n}(A) = \hat{\Gamma}_{m,j(m,n)}(A)$, where $j(m,n) \ge n$ is minimal such that $\hat{\Gamma}_{m,j(m,n)}(A) \neq \emptyset$. Such a $j(m,n)$ is guaranteed to exist and can be found by successively computing finitely many of the $\hat{\Gamma}_{m,k}(A)$'s.

Step II: $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_f\} \in \Sigma^A_1$. Let A be such that f is a bound for its dispersion, and $\epsilon > 0$. Recall that $f(n) \ge n+1$ for every n. Define the composition $F^N := f \circ \dots \circ f$ of 2^N copies of f. Besides the already defined functions γ^N, γ^N_m and $\gamma^N_{m,n}$ we additionally introduce $\psi^N_m := \gamma^N_{m,F^N(m)}$, i.e.

$$\psi^N_m(z) := \left(\min\left\{\sigma_{\inf}\left((P_{F^N(m)}(A - zI)P_{F^N(m)})^{2^N} P_m\right), \sigma_{\inf}\left((P_{F^N(m)}(A^* - \bar{z}I)P_{F^N(m)})^{2^N} P_m\right)\right\}\right)^{2^{-N}},$$

and we define the desired approximations $\hat{\Gamma}_m(A)$ for $\mathrm{sp}_{N,\epsilon}(A)$ by $\hat{\Gamma}_m(A) := \{z \in G_m : \psi^N_m(z) \le \epsilon\}$. Writing $\psi^N_m(z) \le \epsilon$ as $(\psi^N_m(z))^{2^N} \le \epsilon^{2^N}$ and using Proposition 8.1, we see that again the computation of $\hat{\Gamma}_m(A)$ requires only finitely many arithmetic operations on finitely many evaluations $\{\langle Ae_j, e_i\rangle : i, j = 1, \dots, F^N(m)\}$ of A.

Again, there exists a compact ball $K \subset \mathbb{C}$ such that $\gamma_m^N(z) > 2\epsilon$ and $\psi_m^N(z) > 2\epsilon$ for all $z \in \mathbb{C} \setminus K$ and all m. Further note that ψ_m^N converges to γ_m^N uniformly on K. Indeed, since all $z \mapsto (P_{F^N(m)}(A - zI)P_{F^N(m)})^{2^N} P_m$ and $z \mapsto (A - zI)^{2^N} P_m$ are operator-valued polynomials of the same degree whose coefficients converge in the norm due to the choice of the function F^N, we can take into account that $|\sigma_{\inf}(B+C) - \sigma_{\inf}(B)| \leq \|C\|$ holds for arbitrary bounded operators B, C, and we arrive at the conclusion that $|\gamma_m^N(z) - \psi_m^N(z)| \to 0$ as $m \to \infty$ uniformly with respect to $z \in K$. To construct a Σ_1^A tower we bound this difference using the sequence $\{c_n\}$ and the constant $\|A\|$ (for the case $N > 0$ as follows).

If $N = 0$, then clearly we have $\|P_{f(m)}(A - zI)P_m - (A - zI)P_m\| \leq c_m$ by definition of the $\{c_n\}$. Suppose that we have a bound

$$\|(P_{F^N(m)}(A - zI)P_{F^N(m)})^{2^N} P_m - (A - zI)^{2^N} P_m\| \leq \alpha(N, m, z), \tag{8.2}$$

for some function $\alpha(N, m, z)$. We can write

$$\begin{aligned}
&(P_{F^{N+1}(m)}(A - zI)P_{F^{N+1}(m)})^{2^{N+1}} P_m - (A - zI)^{2^{N+1}} P_m \\
&\quad= \big((P_{F^{N+1}(m)}(A - zI)P_{F^{N+1}(m)})^{2^N} - (A - zI)^{2^N}\big)(P_{F^{N+1}(m)}(A - zI)P_{F^{N+1}(m)})^{2^N} P_m \\
&\quad\quad - (A - zI)^{2^N}\big((A - zI)^{2^N} - (P_{F^{N+1}(m)}(A - zI)P_{F^{N+1}(m)})^{2^N}\big)P_m.
\end{aligned}$$

Using the fact that $F^{N+1}(m) = F^N(F^N(m))$ and $P_{F^N(m)}P_{F^{N+1}(m)} = P_{F^{N+1}(m)}$, we can bound the first of the above terms in norm by $\alpha(N, F^N(m), z)(\|A\| + |z|)^{2^N}$. Similarly, we can bound the second term in norm by the same quantity. It follows that we can choose

$$\alpha(N, m, z) = 2\alpha(N - 1, F^{N-1}(m), z)(\|A\| + |z|)^{2^{N-1}}$$

and iterating this N times we can take

$$\alpha(N, m, z) = 2^N c_n\,(\|A\| + |z|)^{2^N - 1}, \quad n = F^{\frac{N(N-1)}{2}}(m),$$

such that (8.2) holds. Note that this estimate can be computed with finitely many arithmetic operations and comparisons from the given data.

In order to simplify the notation, we choose a sequence (δ_m) which converges monotonically to zero such that

$$\gamma_m^N(z) + \delta_m \geq \psi_m^N(z) \geq \gamma_m^N(z) - \delta_m \text{ for every } m \text{ and every } z \in K.$$

Moreover, we point out that each of the functions $z \mapsto \psi_m^N(z)$ is continuous on the compact set K, hence even uniformly continuous, and we can assume without loss of generality that, for every m,

$$|\psi_m^N(z) - \psi_m^N(y)| < \delta_m \text{ for arbitrary } z, y \in K,\ |z - y| < 1/m. \tag{8.3}$$

Now let $\zeta_\epsilon(A) := \{z \in \mathbb{C} : \gamma^N(z) \leq \epsilon\}$, $\zeta_{\epsilon,m}(A) := \{z \in \mathbb{C} : \gamma_m^N(z) \leq \epsilon\}$, and $\Psi_{\epsilon,m}(A) := \{z \in \mathbb{C} : \psi_m^N(z) \leq \epsilon\}$. By the discussion above, we conclude for all $m \geq k$ that

$$\zeta_{\epsilon+\delta_k,m}(A) \supset \zeta_{\epsilon+\delta_m,m}(A) \supset \Psi_{\epsilon,m}(A) \supset \zeta_{\epsilon-\delta_m,m}(A) \supset \zeta_{\epsilon-\delta_k,m}(A). \tag{8.4}$$

Since, $P_m \leq P_{m+1}$ and $P_m \to I$ strongly, $\gamma_m^N \to \gamma^N$ monotonically from above pointwise (and hence locally uniformly by Dini's Theorem). Thus, by [71], $\zeta_{\epsilon+\delta_k,m}(A) \to \zeta_{\epsilon+\delta_k}(A) = \mathrm{sp}_{N,\epsilon+\delta_k}(A)$ and $\zeta_{\epsilon-\delta_k,m}(A) \to \zeta_{\epsilon-\delta_k}(A) = \mathrm{sp}_{N,\epsilon-\delta_k}(A)$ as $m \to \infty$. Hence, since $\mathrm{sp}_{N,\epsilon\pm\delta_k}(A) \to \mathrm{sp}_{N,\epsilon}(A)$ as $k \to \infty$, (8.4) yields $\lim_{m\to\infty} \Psi_{\epsilon,m}(A) = \mathrm{sp}_{N,\epsilon}(A)$. To finish the convergence proof, we observe that it is clear that on the one hand, $\Psi_{\epsilon,m}(A) \supset \hat{\Gamma}_m(A)$. On the other hand, for sufficiently large m, it holds that for every point $x \in \Psi_{\epsilon-\delta_m,m}(A)$ there is a point $y_x \in G_m$ with $|x - y_x| < 1/m$ and, by (8.3) we get $|\psi_m^N(y_x) - \psi_m^N(x)| < \delta_m$ that is y_x even belongs to $\hat{\Gamma}_m(A)$. Thus, $\hat{\Gamma}_m(A) + B_{1/m}(0) \supset \Psi_{\epsilon-\delta_m,m}(A)$ for sufficiently large m. Combining this, we arrive at

$$\Psi_{\epsilon,m}(A) + B_{1/k}(0) \supset \hat{\Gamma}_m(A) + B_{1/m}(0) \supset \Psi_{\epsilon-\delta_m,m}(A) \supset \Psi_{\epsilon-\delta_k,m}(A),$$

for $m \geq k$ large. By the above, the sets on the left converge to $\mathrm{sp}_{N,\epsilon}(A) + B_{1/k}(0)$ as $m \to \infty$, and the sets on the right converge to $\mathrm{sp}_{N,\epsilon-\delta_k}(A)$ for every k. Since both of these sets converge to $\mathrm{sp}_{N,\epsilon}(A)$ as $k \to \infty$ this provides $\lim_{m\to\infty} \hat{\Gamma}_m(A) = \mathrm{sp}_{N,\epsilon}(A)$. This shows that (upon altering as in Step I to avoid the empty set), we can gain convergence in one limit without knowing $\{c_n\}$ and $\|A\|$.

Now we have that $|(\psi_m^N(z))^{2^N} - (\gamma_m^N(z))^{2^N}| \leq \alpha(N,m,z)$. Hence we define

$$\tilde{\Gamma}_m(A) := \{z \in G_m : (\psi_m^N(z))^{2^N} \leq \epsilon^{2^N} - \alpha(N,m,z), \epsilon^{2^N} - \alpha(N,m,z) > 0\},$$

which can be computed in finitely many arithmetic operations and comparisons. Of course, this may be empty, but it has the property that $\tilde{\Gamma}_m(A) \subset \mathrm{sp}_{N,\epsilon}(A)$. Suppose for a contradiction that we do not have convergence to $\mathrm{sp}_{N,\epsilon}(A)$. Without loss of generality, by taking a subsequence if necessary, there exists $z_m \in \mathrm{sp}_{N,\epsilon}(A)$, $z \in \mathrm{sp}_{N,\epsilon}(A)$ and $\delta > 0$ such that $\gamma^N(z) < \epsilon$, $z_m \to z$ but $\mathrm{dist}(z_m, \tilde{\Gamma}_m(A)) \geq \delta$. Let $\hat{z}_m \in G_m$ with $\hat{z}_m \to z$. Then for large m we must have $\gamma^N(\hat{z}_m) < \epsilon$. But $\alpha(N,m,\hat{z}_m) \to 0$ and hence $\hat{z}_m \in \tilde{\Gamma}_m(A)$ for large m, the required contradiction. To finish, we define $\Gamma_m(A) = \hat{\Gamma}_{j(m)}(A)$, where $j(m) \geq m$ is minimal such that $\hat{\Gamma}_{j(m)}(A) \neq \emptyset$. Such a $j(m)$ must exist, and hence, we avoid the empty set. Finally, the fact that $\tilde{\Gamma}_m(A) \subset \mathrm{sp}_{N,\epsilon}(A)$ ensures we have Σ_1^A convergence.

Step III: $\{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{SA}}\} \notin \Delta_2^G$. Assume for a contradiction that there is a sequence $\{\Gamma_k\}$ of general algorithms such that $\Gamma_k(A) \to \mathrm{sp}_{N,\epsilon}(A)$ for all $A \in \Omega_{\mathrm{SA}}$, and consider operators of the type

$$A := \bigoplus_{r=1}^{\infty} A_{l_r} \qquad \text{with } \{l_r\} \subset \mathbb{N} \text{ and } A_n := \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{n\times n}. \tag{8.5}$$

Then $\mathrm{sp}(A_n) = \{0,2\}$, hence A is bounded, self-adjoint, and $\mathrm{sp}(A) = \{0,2\}$ as well. For sufficiently small ϵ the (N,ϵ)-pseudospectrum is a certain neighbourhood of $\{0,2\}$ disjoint from $B_{\frac{1}{2}}(1)$, independently of the choice of $\{l_r\}$. In order to find a counterexample, we construct an appropriate sequence $\{l_r\} \subset \mathbb{N}$ by induction: For $C := \mathrm{diag}\{1,0,0,0,\ldots\}$ one has $\mathrm{sp}(C) = \{0,1\}$. Choose $k_0 := 1$ and $l_1 > N(C,k_0)$, where $N(C,n) = \max\{i,j \,|\, f_{i,j} \in \Lambda_{\Gamma_n}(C)\}$ for $n \in \mathbb{N}$. Suppose that $l_1,\ldots,l_n$ are already chosen. Then we have that $\mathrm{sp}\,(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C) = \{0,1,2\}$, hence, there exists a k_n such that $\Gamma_k\,(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C) \cap B_{\frac{1}{n}}(1) \neq \emptyset$ for every $k \geq k_n$, where $B_{\frac{1}{n}}(1)$ denotes the closed ball of radius $1/n$ and centre 1. Now, choose

$$l_{n+1} > N(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C, k_n) - l_1 - l_2 - \ldots - l_n. \tag{8.6}$$

By construction, it follows that

$$\Gamma_{k_n}(\oplus_{r=1}^{\infty} A_{l_r}) \cap B_{\frac{1}{n}}(1) = \Gamma_{k_n}(A_{l_1} \oplus \ldots \oplus A_{l_n} \oplus C) \cap B_{\frac{1}{n}}(1) \neq \emptyset \quad \forall\, n \in \mathbb{N}. \tag{8.7}$$

Indeed, since any evaluation function $f_{i,j} \in \Lambda$ just provides the (i,j)-th matrix element, it follows by (8.6) that for any evaluation functions $f_{i,j} \in \Lambda_{\Gamma_{k_n}}(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C)$ we have that $f_{i,j}(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C) = f_{i,j}(\oplus_{r=1}^{\infty} A_{l_r})$. Thus, by assumption (iii) in the definition of a General algorithm (Definition 7.3), we get that $\Lambda_{\Gamma_{k_n}}(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C) = \Lambda_{\Gamma_{k_n}}(\oplus_{r=1}^{\infty} A_{l_r})$ which, by assumption (ii) in Definition 7.3, yields (8.7). So, from (8.7), we see that the point 1 is contained in the partial limiting set of the sequence $\{\Gamma_k(\oplus_{r=1}^{\infty} A_{l_r})\}_{k=1}^{\infty}$ which approximates $\mathrm{sp}_{N,\epsilon}(A)$, a contradiction. For general N and ϵ, we apply the above argument after appropriate re-scaling.

Step IV: $\{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{fg}\} \notin \Delta_1^G$. This is clear by considering diagonal operators. The point is that given any general Δ_1^G tower, Γ_n, and any n, $\Gamma_n(A)$ uses only finitely many matrix evaluations $\{f_{i,j}(A) : i,j \leq N_0(n,A)\}$. We can choose m large such that $m > N_0(1,0)$ and set $f_{m,m}(A) = 2\epsilon + 2$. Then $\Gamma_1(A) = \Gamma_1(0) \subset B_{1/2+\epsilon}(0)$, a contradiction since $2\epsilon + 2 \in \mathrm{sp}_{N,\epsilon}(A)$. ☐

Remark 8.5. To deal with Δ_1-information in Step I of the above proof, we can replace $(\gamma_{m,n}^N(z))^{2^N}$ by a suitable rational approximation accurate to $1/n$ (see Proposition 8.2). For Step II, we can replace $(\psi_m^N(z))^{2^N}$ and $\alpha(N,m,z)$ by rational approximations from above accurate to $1/m$. If ϵ is not rational, we approximate with a rational from below accurate to $1/n^2$ in Step I and $1/m^2$ in Step II.

8.3. **Spectrum.** Again, using the inclusions $\Omega_{\mathrm{SA}} \subset \Omega_{\mathrm{N}} \subset \Omega_g$,when considering the spectrum we only need to show that $\{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Sigma_1^A$, $\{\Xi_{\mathrm{sp}}, \Omega_f\} \in \Pi_2^A$, $\{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma_2^A$, $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$, $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{SA}}\} \notin \Delta_2^G$, $\{\Xi_{\mathrm{sp}}, \Omega_f\} \notin \Delta_2^G$ and $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \notin \Delta_3^G$ (the fact that $\{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \notin \Delta_1^G$ is clear by considering diagonal operators). We then prove that $\{\Xi_{\mathrm{sp}}, \Omega_f \cap \Omega_{\mathrm{N}}\} \in \Sigma_1^{A,\mathrm{eigv}}$ separately since the argument easily extends to the Schrödinger case in §9.1. The proof that $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \notin \Delta_3^G$ relies on some results from decision-making problems which we shall prove in Section 8.6.

Proof of Theorem 4.4 for the spectrum. **Step I:** We begin with the easy cases that $\{\Xi_{\mathrm{sp}}, \Omega_f\} \in \Pi_2^A$ and $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$. To prove that $\{\Xi_{\mathrm{sp}}, \Omega_f\} \in \Pi_2^A$, let $\epsilon > 0$ and let Γ_n^ϵ denote the height one arithmetic tower to compute the (classical) pseudospectrum of operators in Ω_f. Using the fact that $\mathrm{sp}_{\mathrm{N},\epsilon}(A)$ are continuous with respect to the parameter $\epsilon > 0$, and converge to $\mathrm{sp}(A)$ as $\epsilon \to 0$ for every A, we simply set $\Gamma_{m,n}(A) = \Gamma_n^{1/m}(A)$. This is a Π_2^A tower since $\mathrm{sp}_{0,1/m}(A)$ contains $\mathrm{sp}(A)$. $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ is similar and just requires the additional first limit.

Step II: $\{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma_2^A$. Let $g : [0,\infty) \to [0,\infty)$ be as in Definition 4.4, in particular, continuous, vanishing only at $x = 0$ and diverging to ∞ as $x \to \infty$. Note that $g(x) \le x$ for all x, and without loss of generality, we can also assume that g is strictly increasing. Then the inverse function $h(y) := g^{-1}(y) : [0,\infty) \to [0,\infty)$ is well defined, continuous, strictly increasing, $h(y) \ge y$ for every y, and $\lim_{y\to 0} h(y) = 0$.

Let $K \subset \mathbb{C}$ be a compact set and $\delta > 0$. We introduce a δ-grid for K by $G^\delta(K) := (K + B_\delta(0)) \cap (\delta(\mathbb{Z} + \mathrm{i}\mathbb{Z}))$, where $B_\delta(0)$ denotes the closed ball of radius δ centred at 0. Without loss of generality, we may assume that δ^{-1} is an integer, and obviously, $G^\delta(K)$ is finite. Moreover, introduce $h_\delta(y) := \min\{k\delta : k \in \mathbb{N}, g(k\delta) > y\}$ and observe that for each y, evaluating $h_\delta(y)$ requires only finitely many evaluations of g. Also, notice that $h(y) \le h_\delta(y) \le h(y) + \delta$. For a given function $\zeta : \mathbb{C} \to [0,\infty)$ we define sets $\Upsilon_K^\delta(\zeta)$ as follows: For each $z \in G^\delta(K)$ let $I_z := B_{h_\delta(\zeta(z))}(z) \cap (\delta(\mathbb{Z} + \mathrm{i}\mathbb{Z}))$. Further

- If $\zeta(z) \le 1$ then introduce the set M_z of all $w \in I_z$ for which $\zeta(w) \le \zeta(v)$ holds for all $v \in I_z$.
- Otherwise, if $\zeta(z) > 1$, just set $M_z := \emptyset$.

Now define

$$\Upsilon_K^\delta(\zeta) := \bigcup_{z \in G^\delta(K)} M_z. \tag{8.8}$$

Notice that for the computation of $\Upsilon_K^\delta(\zeta)$, only finitely many evaluations of ζ and g are required.

Claim: Let K be a compact set containing the spectrum of A and $0 < \delta < \epsilon < 1/2$. Further assume that ζ is a function with $\|\zeta - \gamma\|_{\infty,\hat{K}} := \|(\zeta - \gamma)\chi_{\hat{K}}\|_\infty < \epsilon$ on $\hat{K} := (K + B_{h(\mathrm{diam}(K)+2\epsilon)+\epsilon}(0))$, where $\chi_{\hat{K}}$ denotes the characteristic function of $\hat{K}$. Finally, let

$$u(\xi) := \max\{h(3\xi + h(t+\xi) - h(t)) + \xi : t \in [0,1]\}. \tag{8.9}$$

Then we have that $d_{\mathrm{H}}(\Upsilon_K^\delta(\zeta), \mathrm{sp}(A)) \le u(\epsilon)$ and $\lim_{\xi\to 0} u(\xi) = 0$.

Proof of claim: To prove the claim, let $z \in G^\delta(K)$ and notice that $I_z \subset \hat{K}$ since, for every $v \in I_z$,

$$\begin{aligned} |z - v| &\le h_\delta(\zeta(z)) \le h_\delta(\gamma(z) + \epsilon) \le h(\mathrm{dist}(z, \mathrm{sp}(A)) + \epsilon) + \delta \\ &\le h(\mathrm{diam}(K) + \delta + \epsilon) + \delta. \end{aligned} \tag{8.10}$$

Suppose that $M_z \neq \emptyset$. Note that by (4.4), the monotonicity of h, and the compactness of $\mathrm{sp}(A)$ there is a $y \in \mathrm{sp}(A)$ of minimal distance to z with $|z - y| \le h(\gamma(z))$. Since $\|\zeta - \gamma\|_{\infty,\hat{K}} < \epsilon$ we get $|z - y| \le h(\zeta(z) + \epsilon)$. Hence, at least one of the $v \in I_z$, let's say v_0, satisfies $|v_0 - y| < h(\zeta(z) + \epsilon) - h(\zeta(z)) + \delta$.

Noting again that $\gamma(v_0) \leq \mathrm{dist}(v_0, \mathrm{sp}(A))$, we get $\zeta(v_0) < \gamma(v_0) + \epsilon < h(\zeta(z) + \epsilon) - h(\zeta(z)) + 2\epsilon$. By the definition of M_z this estimate now holds for all points $w \in M_z$ and we conclude that, for all $w \in M_z$,

$$\begin{aligned} \mathrm{dist}(w, \mathrm{sp}(A)) &= h(g(\mathrm{dist}(w, \mathrm{sp}(A)))) \leq h(\gamma(w)) \\ &\leq h(\zeta(w) + \epsilon) \leq h(h(\zeta(z) + \epsilon) - h(\zeta(z)) + 3\epsilon). \end{aligned} \tag{8.11}$$

This observation holds for every $z \in G^{\delta}(K)$ and all $w \in M_z$, hence all points in $\Upsilon_K^{\delta}(\zeta)$ are closer to $\mathrm{sp}(A)$ than $u(\epsilon)$.

Conversely, take any $y \in \mathrm{sp}(A) \subset K$. Then there is a point $z \in G^{\delta}(K)$ with $|z - y| < \delta < \epsilon$, hence $\zeta(z) < \gamma(z) + \epsilon \leq \mathrm{dist}(z, \mathrm{sp}(A)) + \epsilon < 2\epsilon < 1$. Thus, M_z is not empty and contains a point that is closer to y than $h(\zeta(z)) + \epsilon \leq h(2\epsilon) + \epsilon \leq u(\epsilon)$. Finally, notice that the mapping

$$(t, \xi) \mapsto h(h(t + \xi) - h(t) + 3\xi) + \xi$$

is continuous on the compact set $[0, 1] \times [0, 1]$, hence uniformly continuous. Moreover, for every fixed t it tends to 0 as $\xi \to 0$, thus we can conclude $u(\xi) \to 0$, and we have proved the claim. □

Define the function $\gamma_{m,n}(z, A) := \min\{\sigma_{\inf}(P_n(A - zI)P_m), \sigma_{\inf}(P_n(A^* - \bar{z}I)P_m)\}$, and note that we can compute an approximation to $\gamma_{m,n}(z, A)$ from *above* to within an accuracy of $1/m$ in finitely many arithmetic operations and comparisons using Proposition 8.2 (this also includes the case of Δ_1-information). Call this approximation function $\zeta_{m,n}(z, A)$ and we can assume that it takes values in $\frac{1}{2m}\mathbb{N}$. As $n \to \infty$, $\gamma_{m,n}(\cdot, A)$ converges to $\gamma_m(z, A) := \min\{\sigma_{\inf}((A - zI)P_m), \sigma_{\inf}((A^* - \bar{z}I)P_m)\}$ monotonically from below. By taking successive maxima over n and then minima over m if necessary: $\min_{1 \leq j \leq m} \max_{1 \leq k \leq n} \zeta_{j,k}(z, A)$, we can assume that $\zeta_{m,n}(\cdot, A)$ is non-decreasing in n and non-increasing in m. Since $\gamma_{m,n}$ obeys these monotonicity relations, this preserves the error bound of $1/m$. It follows that $\zeta_{m,n}(\cdot, A)$ converges to $\zeta_m(\cdot, A)$ which takes values in the set $\frac{1}{2m}\mathbb{N}$ (i.e. $\zeta_{m,n}(z, A)$ is eventually constant for a given z) and such that $\gamma_m(z, A) \leq \zeta_m(z, A) \leq \gamma_m(z, A) + 1/m$.

Now let

$$\hat{\Gamma}_{m,n}(A) = \Upsilon_{B_m(0)}^{1/2^m}(\zeta_{m,n}).$$

To show that this provides an arithmetic tower of algorithms, note that the computation of $\Upsilon_{B_m(0)}^{1/2^m}(\zeta_{m,n})$ requires only finitely many evaluations of $\zeta_{m,n}$, and the finite number of constants $g(k/m) \leq 1$, $k = 1, 2, \ldots$. Since $G^{1/2^m}(B_m(0))$ is finite and we restricted values of $\zeta_{m,n}$ to $\frac{1}{2m}\mathbb{N}$, we must have that for large n, $\hat{\Gamma}_{m,n}(A)$ is constant and equal to $\Upsilon_{B_m(0)}^{1/2^m}(\zeta_m)$. Denote this eventually constant set by $\hat{\Gamma}_m(A)$. We must now adapt $\hat{\Gamma}_{m,n}$ such that the output is non-empty and such that we gain the desired convergence in the Hausdorff topology yielding the Σ_2^A classification. For any $\hat{\Gamma}_{m,n}(A)$ let $S(m, n, A) := \max_{z \in \hat{\Gamma}_{m,n}(A)} \zeta_{m,n}(z, A)$, where we take the maximum over the empty set to be $+\infty$. Note that $\hat{\Gamma}_{m,n}(A)$ is empty if and only if $\zeta_{m,n}(z, A) > 1$ for all $z \in G^{1/2^m}(B_m(0))$ and note also that $S(m, n, A)$ can be computed using finitely many arithmetic operations and comparisons from the given data.

For given m, n, if $n < m$ then set $\Gamma_{m,n}(A) = \{0\}$. Otherwise, compute $S(k, n, A)$ for $m \leq k \leq n$. If there exists such a k with $S(k, n, A) \leq g(2^{-m})$, then choose a minimal such k and set $\Gamma_{m,n}(A) = \hat{\Gamma}_{k,n}(A)$ (which must be non-empty by the definition of $S(m, n, A)$), otherwise set $\Gamma_{m,n}(A) = \{0\}$. This defines an arithmetic algorithm mapping into the appropriate metric space (in particular, it outputs a non-empty compact set). Since $\zeta_{m,n}$ increases to ζ_m and g is continuous, if $\hat{\Gamma}_l(A) \neq \emptyset$ then $S(l, n, A)$ is finite for all $n \in \mathbb{N}$. For such an l, we must have $S(l, n, A)$ non-decreasing in n, convergent to $S_l(A) := \max_{z \in \hat{\Gamma}_l(A)} g(\zeta_l(z, A))$. On the other hand if $\hat{\Gamma}_l(A) = \emptyset$ then $\zeta_l(z, A) > 1$ for all $z \in G^{1/2^l}(B_l(0))$ and the fact that $\zeta_{m,n}$ increases to ζ_m shows that $S(l, n, A) = +\infty$ for large n.

Define the function $\gamma(z) := \min\{\sigma_{\inf}(A - zI), \sigma_{\inf}(A^* - \bar{z}I)\} = \|(A - zI)^{-1}\|^{-1}$. To see why $\min\{\sigma_{\inf}(A - zI), \sigma_{\inf}(A^* - \bar{z}I)\} = \|(A - zI)^{-1}\|^{-1}$ see for example [71]. Now

$$\sigma_{\inf}(P_n(A - zI)P_m) = \inf\{\|P_n(A - zI)P_m\xi\| : \xi \in \mathrm{Ran}(P_m), \|\xi\| = 1\}$$

and $\sigma_{\inf}((A-zI)P_m) = \inf\{\|(A-zI)P_m\xi\| : \xi \in \mathrm{Ran}(P_m), \|\xi\| = 1\}$. Thus, since $P_m \to I$ strongly and $P_{m+1} \geq P_m$, then $\gamma_m \to \gamma$ pointwise and monotonically from above, and by Dini's Theorem, the convergence is uniform on every compact set, in particular on the ball $K := B_{m_0}(0)$, with a fixed $m_0 > 2\|A\| + 4$. Also, $\gamma_{m,n} \to \gamma_m$ pointwise monotonically from below as $n \to \infty$, hence again by Dini's Theorem it follows that the convergence is uniform on the ball $K = B_{m_0}(0)$. Outside this ball we have, for $n > m$, by a Neumann argument

$$\begin{aligned}\gamma_{m,n}(z) &= \min\{\sigma_{\inf}(P_n(A-zI)P_nP_m), \sigma_{\inf}(P_n(A^*-\bar{z}I)P_nP_m)\}\\ &\geq \min\{\sigma_{\inf}(P_n(A-zI)P_n), \sigma_{\inf}(P_n(A^*-\bar{z}I)P_n)\}\\ &= \|(P_n(A-zI)P_n)^{-1}\|^{-1} = |z|\|(P_n - z^{-1}P_nAP_n)^{-1}\|^{-1} \geq 2.\end{aligned}$$

For all $n > m > m_0$, the points in the finite set $G^{1/2^m}(B_m(0)) \setminus K$ lead to function values of $\zeta_{m,n}$ being larger than 1 (since $\zeta_{m,n}$ approximates $\gamma_{m,n}$ to within $1/m$), hence $\hat{\Gamma}_{m,n}(A) = \Upsilon_K^{1/2^m}(\zeta_{m,n})$. Fix $\epsilon \in (0, 1/2)$. Then there is an $m_1 > m_0$ with $m_1 > 3/\epsilon$ such that $\|\gamma - \zeta_m\|_{\infty,\hat{K}} < \epsilon/3$ on $\hat{K} := B_{h(\mathrm{diam}(K)+2\epsilon)+\epsilon}(0)$ for all $m > m_1$. Moreover, for every m there is an $n_1(m)$ such that $\|\gamma_m - \gamma_{m,n}\|_{\infty,\hat{K}} < \epsilon/3$ for all $n > n_1(m)$. This yields

$$\begin{aligned}\|\gamma - \zeta_{m,n}\|_{\infty,\hat{K}} &\leq \|\gamma - \gamma_m\|_{\infty,\hat{K}} + \|\gamma_m - \gamma_{m,n}\|_{\infty,\hat{K}} + \|\gamma_{m,n} - \zeta_{m,n}\|_{\infty,\hat{K}}\\ &\leq \epsilon/3 + \epsilon/3 + 1/m < \epsilon\end{aligned} \tag{8.12}$$

whenever $m > m_1$ and $n > n_1(m)$. Hence, by the above claim, we must have that $d_{\mathrm{H}}(\hat{\Gamma}_{m,n}(A), \mathrm{sp}(A)) \leq u(\epsilon)$ whenever $m > m_1$ and $n > n_1(m)$. Since this bound tends to zero as $\epsilon \to 0$, it is proved that

$$\lim_{m\to\infty} \limsup_{n\to\infty} d_{\mathrm{H}}(\hat{\Gamma}_{m,n}(A), \mathrm{sp}(A)) = 0.$$

It follows that there exists $N_0 \in \mathbb{N}$ minimal such that $S_{N_0}(A) < +\infty$, equivalently such that $\hat{\Gamma}_{N_0}(A) \neq \emptyset$. Monotonicity of ζ_m in m and the fact that the grid refines itself now ensures that if $m \geq N_0$ then $S_m(A) < +\infty$. Furthermore, the above claim (as well as continuity in g) shows that $\lim_{m\to\infty} S_m(A) = 0$. Let $N_1(m) \geq m$ be minimal such that $S_{N_1(m)} \leq g(2^{-m})$. It follows that we must have $\lim_{n\to\infty} \Gamma_{m,n}(A) = \hat{\Gamma}_{N_1(m)}(A)$. We must also have $\lim_{m\to\infty} \Gamma_m(A) = \mathrm{sp}(A)$. Furthermore,

$$\max_{z\in\Gamma_m(A)} g(\mathrm{dist}(z, \mathrm{sp}(A))) \leq \max_{z\in\Gamma_m(A)} \gamma(z, A) \leq S_{N_1(m)}(A) \leq g(2^{-m}). \tag{8.13}$$

But g is strictly increasing so that we must have $\Gamma_m(A) \subset \mathrm{sp}(A) + B_{2^{-m}}(0)$ and hence Σ_2^A convergence.

Step III: $\{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Sigma_1^A$. This is very similar to Step II, but now we use the function f to collapse the first limit. We can approximate

$$F_n(z, A) := \min\{\sigma_{\inf}(P_{f(n)}(A-zI)P_n), \sigma_{\inf}(P_{f(n)}(A^*-\bar{z}I)P_n)\} + c_n,$$

from *above* to within an accuracy $1/n$ in finitely many arithmetic operations and comparisons using Proposition 8.2 (this also includes the case of Δ_1-information). Call this approximation function $\tilde{F}_n(z, A)$ and assume that $\tilde{F}_n(z, A) \in \mathbb{Q}$. Note that by definition of $D_{f,n}$ and the fact that $D_{f,n}(A) \leq c_n$, we must have $\tilde{F}_n(z, A) \geq \gamma_n(z, A)$ and without loss of generality (take successive minima if necessary) we can assume that $\tilde{F}_n$ converges locally uniformly to γ monotonically from above. Now let $\Gamma_n(A) = \Upsilon_{B_n(0)}^{1/2^n}(\tilde{F}_n)$. Arguing as before, we see that this provides an arithmetic tower of algorithms, is non-empty for large n (so we can assume this holds for all n without loss of generality), and has $\lim_{n\to\infty} \Gamma_n(A) = \mathrm{sp}(A)$. Hence, we only need to argue for the Σ_1^A error control. Define

$$E_n(A) = \sup_{z\in\Gamma_n(A)} h_{2^{-n}}(\tilde{F}_n(z, A)), \tag{8.14}$$

then since $h_{2^{-n}} \geq h$, we must have $E_n(A) \geq \sup_{z\in\Gamma_n(A)} \mathrm{dist}(z, \mathrm{sp}(A))$. Moreover, $\sup_{z\in\Gamma_n(A)} \tilde{F}_n(z, A)$ converges to 0 as $n \to \infty$. Since $h_{2^{-n}} \leq h + 2^{-n}$, it follows that $E_n(A) \to 0$ and hence (by the usual argument of taking subsequences if necessary) we have $\{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Sigma_1^A$.

Step IV: $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{SA}}\} \notin \Delta_2^G$. This is almost the same argument as the pseudospectrum case. Assume that there is a sequence $\{\Gamma_k\}$ of general algorithms such that $\Gamma_k(A) \to \mathrm{sp}(A)$ for all $A \in \Omega_{\mathrm{SA}}$, and consider operators of the type (8.5). The spectrum is $\{0, 2\}$ disjoint to $B_{\frac{1}{2}}(1)$, independently of the choice of $\{l_r\}$. By the same procedure as before, one obtains again that 1 belongs to the partial limiting set of $\Gamma_k(A)$ for a certain A, hence a contradiction.

Step V: $\{\Xi_{\mathrm{sp}}, \Omega_f\} \notin \Delta_2^G$. Recall that Ω_f denotes the set of bounded operators on $l^2(\mathbb{N})$ whose dispersion is bounded by f. Thus, to show the claim, it suffices to show that for any height one general tower of algorithms $\{\Gamma_n\}_{n\in\mathbb{N}}$ for Ξ_{sp}, there exists a weighted shift S, with $(Su)_1 = 0$ for all $u \in l^\infty(\mathbb{N})$ and $Se_n = \alpha_n e_{n+1}$ where $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} \in l^\infty(\mathbb{N})$, such that $\Gamma_m(S) \nrightarrow \mathrm{sp}(S)$ when $m \to \infty$, Obviously $S \in \Omega_f$ (recall $f(n) \geq n+1$). To construct such an S, we let

$$\alpha = \{0, 0, \ldots, 0, 1, 0, 0, \ldots, 0, 1, 1, 0, 0, \ldots, 0, 1, 1, 1, 0, \ldots\}, \qquad \alpha_{l_j+1}, \alpha_{l_j+2}, \ldots, \alpha_{l_j+j} = 1,$$

for some sequence $\{l_j\}_{j\in\mathbb{N}}$ where $l_{j+1} > l_j + 2j$ that we will determine. Observe that regardless of the choice of $\{l_j\}_{j\in\mathbb{N}}$ we have that $\mathrm{sp}(S) = B_1(0)$ (the closed disc centered at zero with radius one). Indeed, on the one hand $\|S\| = 1$, hence $\mathrm{sp}(S) \subset B_1(0)$. On the other hand, one can define the elementary shift operator $V : e_n \mapsto e_{n+1}$, $n \in \mathbb{N}$, and its left inverse $V^- : e_{n+1} \mapsto e_n$, $n \in \mathbb{N}$, $e_1 \mapsto 0$. Then the shifted copies $(V^-)^{l_j} S V^{l_j}$ converge strongly to the limit operator V whose spectrum $\mathrm{sp}(V) = B_1(0)$ is necessarily contained in the essential spectrum of S (cf. [100] or [88]).

To construct S we will inductively choose $\{l_j\}_{j\in\mathbb{N}}$ with the help of another sequence $\{m_j\}_{j\in\mathbb{Z}_+}$ that will also be chosen inductively. Before we start, define, for any $A \in \Omega_f$ and $m \in \mathbb{N}$, $N(A, m)$ to be the smallest integer so that $\Lambda_{\Gamma_m}(A)$ only includes matrix entries $A_{ij} = \langle Ae_j, e_i\rangle$ with $i, j \leq N(A, m)$. Now let $m_0 = 1$ and choose $l_1 > N(0, m_1)$. Suppose that $l_1, \ldots, l_n$ and $m_0, \ldots, m_{n-1}$ are already chosen. Note that $\mathrm{sp}(P_r S) = \{0\}$, since $P_r S = P_r S P_r$ can be regarded as a $r \times r$-triangular matrix with zero-diagonal. Thus, since by assumption $\{\Gamma_m\}_{m\in\mathbb{N}}$ is a General tower of algorithms for Ξ_1, there is an m_n such that $\Gamma_m(P_{l_n+n+1}S) \subset B_{\frac{1}{2}}(0)$, for all $m \geq m_n$. Let

$$l_{n+1} > N(P_{l_n+n+1}S, m_n) \text{ such that also } l_{n+1} > l_n + 2n. \tag{8.15}$$

Then, it follows that $\Gamma_{m_n}(S) = \Gamma_{m_n}(P_{l_{n+1}}S) = \Gamma_{m_n}(P_{l_n+n+1}S)$. Indeed, since any evaluation function $f_{i,j} \in \Lambda$ just provides the (i,j)-th matrix element, it follows by (8.15) that for any evaluation functions $f_{i,j} \in \Lambda_{\Gamma_{m_n}}(S)$ we have that $f_{i,j}(S) = f_{i,j}(P_{l_{n+1}}S) = f_{i,j}(P_{l_n+n+1}S)$. Thus, by assumption (iii) in the definition of a General algorithm (Definition 7.3), we get that $\Lambda_{\Gamma_{m_n}}(S) = \Lambda_{\Gamma_{m_n}}(P_{l_{n+1}}S) = \Lambda_{\Gamma_{m_n}}(P_{l_n+n+1}S)$ which, by assumption (ii) in Definition 7.3 implies the assertion. Thus, by the choice of the sequences $\{l_j\}_{j\in\mathbb{N}}$ and $\{m_j\}_{j\in\mathbb{Z}_+}$, it follows that $\Gamma_{m_n}(S) = \Gamma_{m_n}(P_{l_n+n+1}S) \subset B_{\frac{1}{2}}(0)$ for every n. Since $\mathrm{sp(S)} = B_1(0)$ we observe that $\Gamma_m(S) \nrightarrow \mathrm{sp}(S)$.

Step VI: $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \notin \Delta_3^G$. To prove this, we need one of the results from §8.6. Namely, if we define Ω' to be the collection of all infinite matrices $\{a_{i,j}\}_{i,j\in\mathbb{Z}}$ with entries $a_{i,j} \in \{0, 1\}$ and consider

$$\Xi' : \Omega' \ni \{a_{i,j}\}_{i,j\in\mathbb{Z}} \mapsto \left(\exists D \forall j \left(\left(\forall i \sum_{k=-i}^{i} a_{k,j} < D\right) \vee \left(\forall R \exists i \sum_{k=0}^{i} a_{k,j} > R \wedge \sum_{k=-i}^{0} a_{k,j} > R\right)\right)\right)$$

("there is a bound D such that every column has either less than D 1s or is two-sided infinite")

(where we map into the discrete space $\{\textit{Yes}, \textit{No}\}$), then $\mathrm{SCI}(\Xi', \Omega')_{\mathrm{G}} = 3$.

We may identify $\Omega_{\mathrm{B}} = \mathcal{B}(l^2(\mathbb{N}))$ with $\Omega = \mathcal{B}(X)$, where $X = \oplus_{n=-\infty}^{\infty} X_n$ in the l^2-sense and where $X_n = l^2(\mathbb{Z})$. Consider sequences $a = \{a_i\}_{i\in\mathbb{Z}}$ over $\mathbb{Z}$ with $a_i \in \{0, 1\}$, and define respective operators $B_a \in \mathcal{B}(l^2(\mathbb{Z}))$ with matrix representation $B_a = \{b_{k,i}\}$ by

$$b_{k,i} := \begin{cases} 1: & k = i \text{ and } a_k = 0 \\ 1: & k < i \text{ and } a_k = a_i = 1 \text{ and } a_j = 0 \text{ for all } k < j < i \\ 0: & \text{otherwise.} \end{cases}$$

Then B_a is again a shift on a certain subset of basis elements and the identity on the other basis elements, hence we have the following possible spectra:

- $\mathrm{sp}(B_a) \subset \{0, 1\}$ if $\{a_i\}$ has finitely many 1s.
- $\mathrm{sp}(B_a) = \mathbb{T}$, the unit circle, if there are infinitely many $i > 0$ with $a_i = 1$ and infinitely many $i < 0$ with $a_i = 1$ (we say $\{a_i\}$ is two-sided infinite).
- $\mathrm{sp}(B_a) = \mathbb{D}$, the unit disc, if $\{a_i\}$ has infinitely many 1s, but only finitely many for $i < 0$ or finitely many for $i > 0$ (we say $\{a_i\}$ is one-sided infinite in that case).

Next for a matrix $\{a_{i,j}\}_{i,j\in\mathbb{Z}}$ we define the operator

$$C := \bigoplus_{k=-\infty}^{\infty} B_k \tag{8.16}$$

on X, where $B_k = B_{\{a_{i,k}\}_{i\in\mathbb{Z}}}$ corresponds to the column $\{a_{i,k}\}_{i\in\mathbb{Z}}$ in the above sense. Concerning its spectrum, we have $\bigcup_{k\in\mathbb{Z}} \mathrm{sp}(B_k) \subset \mathrm{sp}(C) \subset \mathbb{D}$ since $\|C\| = 1$. Clearly, if one of the columns is one-sided infinite, then $\mathrm{sp}(C) = \mathbb{D}$. The same holds if for every $k \in \mathbb{N}$ there is a finite column with at least k 1s. Otherwise (that is if there is a number D such that for every column it holds that it either has less than D 1s or is two-sided infinite) the spectrum $\mathrm{sp}(C)$ is a subset of $\{0\} \cup \mathbb{T}$.

Suppose for a contradiction that there exists a height two tower, Γ_{n_2,n_1} solving $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\}$. Consider the intervals $J_1 = [0, 1/8]$, and $J_2 = [3/8, \infty)$. Set $\alpha_{n_2,n_1} = \mathrm{dist}(1/2, \Gamma_{n_2,n_1}(A))$. Let $k(n_2, n_1) \leq n_1$ be maximal such that $\alpha_{n_2,k}(A) \in J_1 \cup J_2$. If no such k exists or $\alpha_{n_2,k}(A) \in J_1$ then set $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) =$ *No*. Otherwise set $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) =$ *Yes*. It is clear from the construction of the matrix C from $\{a_{i,k}\}_{i\in\mathbb{Z}}$ that this defines a generalized algorithm. In particular, given N we can evaluate $\{f_{k,l}(C) : k, l \leq N\}$ using only finitely many evaluations of $\{a_{i,j}\}$, where we can use a bijection between the canonical bases to view C as acting on $l^2(\mathbb{N})$. The point of the intervals J_1, J_2 is that we can show $\lim_{n_1\to\infty} \tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \tilde{\Gamma}_{n_2}(\{a_{i,j}\})$ exists (the distance to the point $1/2$ cannot oscillate infinitely often between J_1 and J_2). If $\Xi'(\{a_{i,j}\}) =$ *No* then for large n_2 we have $\lim_{n_1\to\infty} \alpha_{n_2,n_1}(A) < 1/8$ and hence $\lim_{n_2\to\infty} \tilde{\Gamma}_{n_2}(\{a_{i,j}\}) =$ *No*. Similarly, if $\Xi'(\{a_{i,j}\}) =$ *Yes* then for large n_2 we have $\lim_{n_1\to\infty} \alpha_{n_2,n_1}(A) > 3/8$ and hence $\lim_{n_2\to\infty} \tilde{\Gamma}_{n_2}(\{a_{i,j}\}) =$ *Yes*. Hence $\tilde{\Gamma}_{n_2,n_1}$ is a height two tower of general algorithms solving $\{\Xi', \Omega'\}$, a contradiction. □

Remark 8.6. In the case of self-adjoint bounded operators, the spectrum $\mathrm{sp}(A)$ is real, and the function g can be chosen as $x \mapsto x$. Thus, in the definition of $\Upsilon_K^\delta(\zeta)$ it suffices to consider compact $K \subset \mathbb{R}$, the real grid $G^\delta(K) := (K + [-\delta, \delta]) \cap (\delta\mathbb{Z})$, and for all $z \in G^\delta(K)$ only the two points $z_{1/2} := z \pm \zeta(z)$ in I_z. Also, in the case of normal operators, where $g : x \mapsto x$ does the job again, the construction simplifies. In particular, for a given function $\zeta : \mathbb{C} \to [0, \infty)$ we may define sets $\Upsilon_K^\delta(\zeta)$ as follows: For $z \in G^\delta(K)$ consider $I_z := \{z + \zeta(z)e^{j\delta\mathrm{i}} : j = 0, 1, \ldots, \lceil 2\pi\delta^{-1}\rceil\}$ and define $\Upsilon_K^\delta(\zeta)$ again as in (8.8). The proof is then the same, up to some obvious adaptations.

Proof that $\{\Xi_{\mathrm{sp}}, \Omega_f \cap \Omega_{\mathrm{N}}\} \in \Sigma_1^{A,\mathrm{eigv}}$. Since $\Omega_f \cap \Omega_{\mathrm{N}} \subset \Omega_{fg}$, the only part left of the proof is the result concerning approximate eigenvectors. Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ denote the sequence of arithmetic algorithms defined in Step III of the above proof. By the now standard argument of taking subsequences, it is enough to show that given $z \in \Gamma_n(A)$ and $\delta \in \mathbb{Q}_{>0}$ with $\delta < 1$, we can compute in finitely many arithmetic operations and comparisons a vector ψ_n such that $\max\{\|A\psi_n - z\psi_n\|, |1 - \|\psi_n\||\} \leq E_n(A) + 2c_n + 2\delta$, where $E_n(A)$ is defined in (8.14). Without loss of generality, we can assume that $z = 0$ by an appropriate shift of the operator A. By construction of the algorithm, we must have that $\sigma_{\inf}(P_{f(n)}\tilde{A}P_n) + c_n < E_n(A) + \delta$, where $\tilde{A}$ is the approximation of the matrix A used when computing $\Gamma_n(A)$ (recall we deal with Δ_1 information). We assume, without loss of generality, that $\|AP_n - P_{f(n)}\tilde{A}P_n\| \leq c_n + \delta/2$. Let $\epsilon = (E_n(A) + \delta)^2$ and consider the matrix

$$B = \left(P_{f(n)}\tilde{A}P_n\right)^* \left(P_{f(n)}\tilde{A}P_n\right) - \epsilon I \in \mathbb{C}^{n\times n}$$

which is Hermitian but not positive definite. It follows that B can be put into the form $PBP^T = LDL^*$, where L is lower triangular with 1's along its diagonal, D is block diagonal with block sizes 1 or 2 and P is a permutation matrix. This can be computed in finitely many arithmetic operations. Without loss of generality, we assume that $P = I$. Let x be an eigenvector of B with non-positive eigenvalue then set $y = L^*x$. Such an x exists by assumption. Note that, since $\delta > 0$,

$$\langle y, Dy\rangle = \langle L^*x, DL^*x\rangle = \langle x, Bx\rangle < 0. \tag{8.17}$$

It follows that there exists a non-zero vector y_n with $\langle y_n, Dy_n\rangle \leq 0$. Since the inequality in (8.17) is strict, such a vector is easy to compute using arithmetic operations by considering determinants and traces of 1 blocks or 2 blocks in the block diagonal matrix D. L^* is invertible and upper triangular so we can solve for $\tilde{\psi}_n = (L^*)^{-1}y_n$. We can then approximately normalize $\tilde{\psi}_n$ to ψ_n using finitely many arithmetic operations (e.g., by approximating the norm of $\tilde{\psi}_n$) so that $1 - \delta < \|\psi_n\| \leq 1$. Note also that

$$\|P_{f(n)}\tilde{A}P_n\psi_n\|^2 = \langle\psi_n, B\psi_n\rangle + \|\psi_n\|^2\epsilon = \frac{\|\psi_n\|^2}{\|\tilde{\psi}_n\|^2}\langle y_n, Dy_n\rangle + \|\psi_n\|^2\epsilon \leq \epsilon.$$

It follows that (using ψ_n to also denote the zero padding of ψ_n to form a vector in $l^2(\mathbb{N})$)

$$\begin{aligned}\|AP_n\psi_n\| &\leq E_n(A) + \delta + \|AP_n - P_{f(n)}\tilde{A}P_n\|\|\psi_n\|\\ &\leq E_n(A) + \delta + (c_n + \delta/2)(1+\delta) \leq E_n(A) + 2c_n + 2\delta\end{aligned}$$

since $\delta < 1$. The result now follows. □

8.4. **Essential Spectrum.** In this section, we prove the results for the essential spectrum. Since $\Omega_{\mathrm{D}} \subset \Omega_{fg} \subset \Omega_f$ and $\Omega_{\mathrm{SA}} \subset \Omega_{\mathrm{N}} \subset \Omega_g \subset \Omega_{\mathrm{B}}$, we only need to prove that $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{D}}\} \notin \Delta_2^G$, $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{SA}}\} \notin \Delta_3^G$, $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ and $\{\Xi_{\text{e-sp}}, \Omega_f\} \in \Pi_2^A$.

Proof of Theorem 4.4 for the essential spectrum. **Step I:** $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{D}}\} \notin \Delta_2^G$. To see this, suppose for a contradiction that a height one tower Γ_n solves the computational problem. For the contradiction we will construct $A \in \Omega_{\mathrm{D}}$ with diagonal entries in $\{0,1\}$ such that $\Gamma_n(A)$ does not converge. Let $A_n = \mathrm{diag}(0,0,...,0) \in \mathbb{C}^{n\times n}$ and $B_n = \mathrm{diag}(1,1,...,1) \in \mathbb{C}^{n\times n}$ (we let A_∞ and B_∞ be the obvious infinite analogues). We will construct

$$A = \bigoplus_{n\in\mathbb{N}} A_{a_n} \oplus B_{b_n},$$

for $a_n, b_n \in \mathbb{N}$ inductively. Suppose that $a_1, b_1, a_2, b_2, ..., a_m, b_m$ have been chosen. Then the operator

$$C_m := \Big(\bigoplus_{n=1}^{m} A_{a_n} \oplus B_{b_n}\Big) \oplus A_\infty$$

has essential spectrum $\{0\}$. Hence there exists $n_m \geq m$ such that $\Gamma_{n_m}(C_m) \subset B_{1/4}(0)$. However, by the definition of a general tower, there must exist some $N(m)$ such that $\Gamma_{n_m}(C_m)$ only uses the evaluations of matrix elements $f_{i,j}(C_m)$ with $i,j \leq N(m)$. Now choose $a_{m+1} \geq \max\{N_m - (a_1+b_1+...+a_m+b_m), 1\}$ then we must have $\Gamma_{n_m}(A) = \Gamma_{n_m}(C_m)$. Similarly, if $a_1, b_1, a_2, b_2, ..., b_m, a_{m+1}$ have been chosen then we consider

$$D_m := \Big(\bigoplus_{n=1}^{m} A_{a_n} \oplus B_{b_n}\Big) \oplus A_{m+1} \oplus B_\infty$$

and choose b_{m+1} large so that $\Gamma_{\hat{n}_m}(A) = \Gamma_{\hat{n}_m}(D_m) \subset B_{1/4}(1)$ for some $\hat{n}_m \geq n_m$. This gives the required contradiction since the sequence $\Gamma_n(A)$ does not converge.

Step II: $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{SA}}\} \notin \Delta_3^G$. Suppose for a contradiction that Γ_{n_2,n_1} is a height two tower solving this problem. Let $(\mathcal{M}, d)$ be the discrete space $\{\textit{Yes}, \textit{No}\}$, let Ω' denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries $a_{i,j} \in \{0,1\}$ and consider the problem function

$$\Xi'(\{a_{i,j}\}): \text{ Does } \{a_{i,j}\} \text{ have (only) finitely many columns with (only) finitely many 1s?}$$

In Section 8.6 we prove that $\mathrm{SCI}(\Xi', \Omega')_G = 3$. We will gain a contradiction by using the supposed height two tower for $\{\Xi_{\text{e-sp}}, \Omega_{\text{SA}}\}$, Γ_{n_2,n_1}, to solve $\{\Xi', \Omega'\}$.

Without loss of generality, identify Ω_{SA} with self adjoint operators in $\mathcal{B}(X)$ where $X = \bigoplus_{j=1}^{\infty} X_j$ in the l^2-sense with $X_j = l^2(\mathbb{N})$. Now let $\{a_{i,j}\} \in \Omega'$ and for the jth column define $B_j \in \mathcal{B}(X_j)$ with the following matrix representation:

$$B_j = \bigoplus_{r=1}^{M_j} A_{l_r^j}, \quad A_m := \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{m \times m},$$

where if M_j is finite then $l_{M_j}^j = \infty$ with $A_\infty = \mathrm{diag}(1, 0, 0, ...)$. The l_r^j are defined by the relation

$$\sum_{r=1}^{\sum_{i=1}^{m} a_{i,j}} l_r^j = m + \sum_{i=1}^{m} a_{i,j}, \tag{8.18}$$

and measure the lengths $(+1)$ of successive gaps between 1's in the jth column. Define the self-adjoint operator $A = \bigoplus_{j=1}^{\infty} B_j$. We then have that

$$\mathrm{sp}_{\text{ess}}(A) = \begin{cases} \{0, 1, 2\}, & \text{if } \Xi'(\{a_{i,j}\}) = \textit{No} \\ \{0, 2\}, & \text{otherwise .} \end{cases}$$

Consider the intervals $J_1 = [0, 1/2]$, and $J_2 = [3/4, \infty)$. Set $\alpha_{n_2,n_1} = \mathrm{dist}(1, \Gamma_{n_2,n_1}(A))$. Let $k(n_2, n_1) \leq n_1$ be maximal such that $\alpha_{n_2,k}(A) \in J_1 \cup J_2$. If no such k exists or $\alpha_{n_2,k}(A) \in J_1$ then set $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \textit{No}$. Otherwise set $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \textit{Yes}$. It is clear from (8.18) and the definition of the A_m that this defines a generalized algorithm. In particular, given N, we can evaluate $\{A_{k,l} : k, l \leq N\}$ using only finitely many evaluations of $\{a_{i,j}\}$, where we can use a bijection between the canonical bases to view A as acting on $l^2(\mathbb{N})$. Again, the point of the intervals J_1, J_2 is that we can show $\lim_{n_1\to\infty} \tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \tilde{\Gamma}_{n_2}(\{a_{i,j}\})$ exists. If $\Xi'(\{a_{i,j}\}) = \textit{No}$ then for large n_2 we have $\lim_{n_1\to\infty} \alpha_{n_2,k}(A) < 1/2$ and hence $\lim_{n_2\to\infty} \tilde{\Gamma}_{n_2}(\{a_{i,j}\}) = \textit{No}$. Similarly, if $\Xi'(\{a_{i,j}\}) = \textit{Yes}$ then for large n_2 we have $\lim_{n_1\to\infty} \alpha_{n_2,k}(A) > 3/4$ and hence $\lim_{n_2\to\infty} \tilde{\Gamma}_{n_2}(\{a_{i,j}\}) = \textit{Yes}$. Hence $\tilde{\Gamma}_{n_2,n_1}$ is a height two tower of general algorithms solving $\{\Xi', \Omega'\}$, a contradiction.

Step III: $\{\Xi_{\text{e-sp}}, \Omega_{\text{B}}\} \in \Pi_3^A$. We start by defining the following functions on $\mathbb{C}$, where $Q_n := I - P_n$,

$$\begin{aligned} \mu_{m,n,k} &: z \mapsto \min\{\sigma_{\inf}(P_k(A - zI)Q_m P_n), \sigma_{\inf}(P_k(A - zI)^* Q_m P_n)\} \\ \mu_{m,n} &: z \mapsto \min\{\sigma_{\inf}((A - zI)Q_m P_n), \sigma_{\inf}((A - zI)^* Q_m P_n)\} \\ \mu_m &: z \mapsto \min\{\sigma_{\inf}((A - zI)Q_m), \sigma_{\inf}((A - zI)^* Q_m)\}. \end{aligned}$$

Here $P_k(A - zI)Q_m P_n$ is considered as operator on $\mathrm{Ran}(Q_m P_n)$, etc. as usual. Recall from the previous proofs that, for every n, m, $\mu_{m,n,k} \to \mu_{m,n}$ pointwise and monotonically from below as $k \to \infty$ and for every m $\mu_{m,n} \to \mu_m$ pointwise and monotonically from above as $n \to \infty$. Furthermore, $\{\mu_m\}_{m\in\mathbb{N}}$ is pointwise increasing and bounded, hence converges as well. By Proposition 8.2 we can compute with finitely many arithmetic operations and comparisons, for any given z, an approximation $\tilde{\mu}_{m,n,k}(z) \in \mathbb{Q}$ with $|\mu_{m,n,k}(z) - \tilde{\mu}_{m,n,k}(z)| \leq 1/k$. Furthermore, we can approximate from below and assume without loss of generality (by taking successive maxima) that $\tilde{\mu}_{m,n,k}(z)$ converges to $\mu_{m,n}$ pointwise and monotonically from below (again, this also includes the case of Δ_1-information).

Next, we define the finite grids

$$G_n := \left\{ \frac{s + \mathrm{i}t}{2^n} : s, t \in \{-2^{2n}, \dots, 2^{2n}\} \right\},$$

and, for $A \in \mathcal{B}(l^2(\mathbb{N}))$,

$$\hat{\Gamma}_{m,n,k}(A) := \left\{ z \in G_n : \tilde{\mu}_{m,n,k}(z) \leq \frac{1}{m} \right\}$$

$$\hat{\Gamma}_{m,n}(A) := \bigcap_{k \in \mathbb{N}} \hat{\Gamma}_{m,n,k}(A) = \lim_{k \to \infty} \hat{\Gamma}_{m,n,k}(A), \tag{8.19}$$

$$\hat{\Gamma}_{m}(A) := \overline{\bigcup_{n \in \mathbb{N}} \hat{\Gamma}_{m,n}(A)} = \lim_{n \to \infty} \hat{\Gamma}_{m,n}(A), \tag{8.20}$$

$$\hat{\Gamma}(A) := \bigcap_{m \in \mathbb{N}} \hat{\Gamma}_{m}(A) = \lim_{m \to \infty} \hat{\Gamma}_{m}(A). \tag{8.21}$$

It follows again that all $\hat{\Gamma}_{m,n,k}$ are general algorithms in the sense of Definition 7.3 that require only finitely many arithmetic operations. We shall show that for large enough n, the above sets are non-empty and establish the limits in (8.19), (8.20) and (8.21) and that $\hat{\Gamma}(A)$ equals $\mathrm{sp}_{\mathrm{ess}}(A)$. We will show that it is possible to choose subsequences of n such that this holds (each output and any limits must never empty since we require convergence in the Hausdorff metric), allowing us to construct a height three arithmetic tower. The final limit will be from above and hence the Π_3^A classification.

To do that we abbreviate $\mathcal{H} := l^2(\mathbb{N})$ and first show that

$$\mu(z) := \lim_{m \to \infty} \mu_m(z) \quad \text{equals} \quad \|(A - zI + \mathcal{K}(\mathcal{H}))^{-1}\|^{-1} \quad \text{for all} \quad z \in \mathbb{C}, \tag{8.22}$$

where $A - zI + \mathcal{K}(\mathcal{H})$ denotes the element in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and where we use the convention $\|b^{-1}\|^{-1} := 0$ if the element b is not invertible. Clearly, it suffices to consider $z = 0$. The estimate "$\leq$" is trivial in case $\mu(0) = 0$. So, let $\mu(0) > \epsilon > 0$. Choose $m \in \mathbb{N}$ such that $\mu_m(0) \geq \mu(0) - \epsilon$. The operator $A_0 := AQ_m : \mathrm{Ran}Q_m \to \mathrm{Ran}(AQ_m)$ is invertible, hence the kernel of $A = AQ_m + AP_m$ has finite dimension. $\sigma_{\inf}(A^*Q_m) > 0$ yields that $\mathrm{Ran}A$ has finite codimension, hence both A and AQ_m are Fredholm. Let R be the orthogonal projection onto $\mathrm{Ran}AQ_m$, B_0 the inverse of A_0 and $B := B_0R$. Then

$$BA - I = (BA - I)P_m + (BA - I)Q_m = (BA - I)P_m \quad \text{and}$$
$$AB - I = (AB - I)(I - R) + (AB - I)R = (AB - I)(I - R)$$

are compact, i.e. B is a regulariser for A. Now

$$\begin{aligned}\|(A + \mathcal{K}(\mathcal{H}))^{-1}\|^{-1} &\geq \|B\|^{-1} = \|B_0R\|^{-1} \\ &\geq (\|B_0\|\|R\|)^{-1} = \|B_0\|^{-1} = \sigma_{\inf}(AQ_m) \geq \mu(0) - \epsilon\end{aligned}$$

gives the estimate "$\leq$" since ϵ is arbitrary.

Conversely, there is nothing to prove if A is not Fredholm, so let $\epsilon > 0$ and $B \in (A + \mathcal{K}(\mathcal{H}))^{-1}$ be a regulariser with $\|B\| \leq \|(A + \mathcal{K}(\mathcal{H}))^{-1}\| + \epsilon$. Since the operator $K := BA - I$ is compact we get for all sufficiently large m that $\|Q_mBAQ_m - Q_m\| = \|Q_mKQ_m\|$ is so small such that $Q_m + Q_mKQ_m$ is invertible in $\mathcal{B}(\mathrm{Ran}(Q_m))$,

$$\underbrace{Q_m(Q_m + Q_mKQ_m)^{-1}Q_mB}_{=:\, B_1 \in \mathcal{B}(\mathcal{H})} AQ_m = Q_m \quad \text{and} \quad \|Q_mB - B_1\| < \epsilon.$$

We get that $\sigma_{\inf}(AQ_m) > 0$, hence the compression $AQ_m : \mathrm{Ran}(Q_m) \to \mathrm{Ran}(AQ_m)$ is invertible and the compression $B_1|_{\mathrm{Ran}(AQ_m)} : \mathrm{Ran}(AQ_m) \to \mathrm{Ran}(Q_m)$ is its (unique) inverse. Thus, we have $\|B_1\| \geq \|B_1|_{\mathrm{Ran}(AQ_m)}\| = \sigma_{\inf}(AQ_m)^{-1}$ and further $\|B\| \geq \|Q_mB\| \geq \|B_1\| - \|Q_mB - B_1\| \geq \sigma_{\inf}(AQ_m)^{-1} - \epsilon$. We conclude that for sufficiently large m that $\sigma_{\inf}(AQ_m)^{-1} \leq \|B\| + \epsilon \leq \|(A + \mathcal{K}(\mathcal{H}))^{-1}\| + 2\epsilon$. Since $\epsilon > 0$ is arbitrary we arrive at $\lim_{m \to \infty} \sigma_{\inf}(AQ_m) \geq \|(A + \mathcal{K}(\mathcal{H}))^{-1}\|^{-1}$. Applying this observation to A^* we also find

$$\lim_{m \to \infty} \sigma_{\inf}(A^*Q_m) \geq \|(A^* + \mathcal{K}(\mathcal{H}^*))^{-1}\|^{-1} = \|(A + \mathcal{K}(\mathcal{H}))^{-1}\|^{-1},$$

which finishes the proof of (8.22). In particular, we now can apply that all of the above functions $\mu_{m,n,k}$, $\mu_{m,n}$, μ_m, μ are continuous with respect to z, and together with the already discussed pointwise monotone convergence results, Dini's Theorem gives that the convergences are even locally uniform.

We can now establish the limits in (8.19), (8.20) and (8.21) for large enough n. Obviously, $\{\hat{\Gamma}_{m,n,k}(A)\}_k$ is decreasing. If $\hat{\Gamma}_{m,n}(A) = \emptyset$ then there must exist some finite k with $\hat{\Gamma}_{m,n,k}(A) = \emptyset$ since the sets are nested, closed and uniformly bounded. Furthermore, $\{\hat{\Gamma}_{m,n}(A)\}_n$ is increasing since, for every k, $\hat{\Gamma}_{m,n}(A) \subset \hat{\Gamma}_{m,n,k}(A) \subset \hat{\Gamma}_{m,n+1,k}(A)$. Let $z \in \mathrm{sp}_{\mathrm{ess}}(A)$. For $m \in \mathbb{N}$, $\mu_m(z) = 0$ and furthermore, there is an $n_0(m)$ and a $z_m \in G_{n_0(m)}$ with $|z - z_m| < 1/m$, $\mu_m(z_m) < 1/(2m)$ and $\mu_{m,n}(z_m) < 1/m$ for every $n \geq n_0(m)$. Then, for every k, $\hat{\mu}_{m,n,k}(z_m) < 1/m$ as well. Since the essential spectrum of a bounded linear operator is non-empty, it follows that there exists a minimal $N(m)$ such that if $n \geq N(m)$ then $\hat{\Gamma}_{m,n}(A) \neq \emptyset$.

We now alter $\hat{\Gamma}_{m,n,k}$ as follows. For a given m, n and k we successively compute $\hat{\Gamma}_{m,n,k}(A), \hat{\Gamma}_{m,n+1,k}(A), \ldots$ and choose $N(m,n,k) \geq n$ minimal such that $\hat{\Gamma}_{m,N(m,n,k),k}(A) \neq \emptyset$. By the above remarks, it follows that this process must terminate. We also have that $\Gamma_{m,n}(A) := \lim_{k\to\infty} \Gamma_{m,n,k}(A)$ exists (in fact $\Gamma_{m,n,k}(A)$ is eventually constant as we increase k since $\hat{\mu}_{m,n,k}$ is increasing) and also that $\Gamma_{m,n}(A) = \hat{\Gamma}_{m,\max\{n,N(m)\}}(A)$. Since $\Gamma_{m,n}(A)$ are increasing in n, it then follows that

$$\Gamma_m(A) := \lim_{n\to\infty} \Gamma_{m,n}(A) = \overline{\bigcup_{n\in\mathbb{N}} \Gamma_{m,n}(A)} = \overline{\bigcup_{n\in\mathbb{N}} \hat{\Gamma}_{m,n}(A)}.$$

Finally, $\{\Gamma_m(A)\}_m$ is decreasing. To see this, choose $z \in \Gamma_m(A)$ and a sequence (z_n) with $z_n \to z$ and $z_n \in \hat{\Gamma}_{m,n}(A)$ (for large n), respectively. The functions $\mu_{m,n}$ are non-decreasing in m and hence we have

$$\hat{\Gamma}_{m,n}(A) = \left\{ z \in G_n : \mu_{m,n}(z) \leq \frac{1}{m} \right\} \subset \hat{\Gamma}_{m-1,n}(A)$$

from which we conclude $z_n \in \hat{\Gamma}_{m-1,n}(A)$, hence $z \in \Gamma_{m-1}(A)$. It follows that the limit $\Gamma(A) := \lim_{m\to\infty} \Gamma_m(A)$ exists.

We are left with proving that $\Gamma(A) = \mathrm{sp}_{\mathrm{ess}}(A)$. Let $z \in \mathrm{sp}_{\mathrm{ess}}(A)$. Arguing as before, for $m \in \mathbb{N}$, $\mu_m(z) = 0$ and furthermore, there is an $n_0(m)$ and a $z_m \in G_{n_0(m)}$ with $|z - z_m| < 1/m$, $\mu_m(z_m) < 1/(2m)$ and $\mu_{m,n}(z_m) < 1/m$ for every $n \geq n_0(m)$. Then for every k $\mu_{m,n,k}(z_m) < 1/m$ as well. We conclude that $z_m \in \Gamma_m(A) \subset \Gamma_l(A)$, $l = 1, \ldots, m$. Thus the limit z of the sequence $\{z_m\}$ belongs to all $\Gamma_l(A)$ and hence $\mathrm{sp}_{\mathrm{ess}}(A) \subset \Gamma(A)$. Conversely, let $z \notin \mathrm{sp}_{\mathrm{ess}}(A)$. Then $\mu(z) > \epsilon > 0$ for a certain $\epsilon > 0$ and for all z in a certain neighbourhood U of z. Moreover there is an $m_0 > 3/\epsilon$ such that $\mu_m(z) > \epsilon/2$ for all $m \geq m_0$ and $z \in U$, hence $\mu_{m,n}(z) > \epsilon/2$ for all $m \geq m_0$, all n and all $z \in U$. Further, for every $m > m_0$ and n there is a $k_0(m,n)$ such that $\mu_{m,n,k}(z) > \epsilon/3 > 1/m_0 > 1/m$ for all $k \geq k_0(m,n)$ and $z \in U$. Thus, the intersection of U and $\Gamma(A)$ is empty, in particular $z \notin \Gamma(A)$.

Step IV: $\{\Xi_{\text{e-sp}}, \Omega_f\} \in \Pi_2^A$. Knowing a bound f on the dispersion of A suggests to plug it into the previously defined algorithms and define

$$\kappa_{m,n} : z \mapsto \min\{\sigma_{\inf}(P_{f(n)}(A - zI)Q_m P_n), \sigma_{\inf}(P_{f(n)}(A - zI)^* Q_m P_n)\}$$
$$\tilde{\Gamma}_{m,n}(A) := \left\{ z \in G_n : \hat{\kappa}_{m,n}(z) \leq \frac{1}{m} \right\}.$$

Where, as usual, we will approximate $\kappa_{m,n}$ to within $1/n$ by a function $\hat{\kappa}_{m,n}$ taking rational values that can be computed (using Proposition 8.2 to cope with Δ_1-information if needed) at any point using finitely many arithmetic operations and comparisons. Unfortunately, all we know about the functions $\kappa_{m,n}$, μ_m is that they are Lipschitz continuous with Lipschitz constant 1 and that $\kappa_{m,n}$ converge pointwise to μ_m, but not, whether or when this convergence is monotone. Therefore, we have to make a modification in order to guarantee the existence of the desired limiting sets. The following idea is similar to using the intervals J_1 and J_2 in Step II and avoids possible oscillations at the boundary.

Let V_m denote the square $V_m := \{z \in \mathbb{C} : |\Re(z)|, |\Im(z)| \leq 2^{-(m+1)}\}$ and $V_m(z) := z + V_m$ the respective shifted copies. Moreover, set $Z_m := \{\frac{s+it}{2^m} : s, t \in \mathbb{Z}\}$ and

$$S_{m,n}(z) := \{i = m+1, \dots, n : \exists z \in V_m(z) \cap G_i : \hat{\kappa}_{m,i}(z) \leq 1/m\}$$
$$T_{m,n}(z) := \{i = m+1, \dots, n : \exists z \in V_m(z) \cap G_i : \hat{\kappa}_{m,i}(z) \leq 1/(m+1)\},$$

as well as

$$E_{m,n}(z) := |S_{m,n}(z)| + |T_{m,n}(z)| - n$$
$$I_{m,n} := \{z \in Z_m : E_{m,n}(z) > 0 \text{ and } |z| \leq n\}$$
$$\hat{\Gamma}_{m,n}(A) := \bigcup_{z \in I_{m,n}} V_m(z).$$

Roughly speaking, $\hat{\Gamma}_{m,n}(A)$ is the union of a family of squares $V_m(z)$ with $E_{m,n}(z)$ being positive, which is the case if "most of the $\hat{\kappa}_{m,i}$ are small on $V_m(z)$".

To make this precise, we first notice that all $\hat{\kappa}_{m,i}(z)$, $i \geq m+1$, with z outside the compact ball $K := B_{2\|A\|+2}(0)$ are larger than one, $I_{m,n}$ are finite, and all $\hat{\Gamma}_{m,n}(A)$ are contained in K, due to a simple Neumann series argument. Furthermore, $\hat{\kappa}_{m,n} \to \mu_m$ uniformly on K due to the Lipschitz continuity (uniform in n) of $\hat{\kappa}_{m,n}$ and μ_m.

We now show that for each $m \geq 5$, the sign of $E_{m,n}(z)$ are eventually constant with respect to n for every $z \in Z_m \cap K$, if n is sufficiently large. That is, for every z there is an $n(z)$ such that either $E_{m,n}(z) \leq 0$ or $E_{m,n}(z) > 0$ for all $n \geq n(z)$. For fixed z and $m \geq 5$, we have to consider three possible cases: The first one is $\mu_m(w) > 1/m$ for all $w \in V_m(z)$. Then there exists an n_0 such that $\hat{\kappa}_{m,n}(w) > 1/m$ for all $n \geq n_0$ and all $w \in V_m(z)$ (take into account that $V_m(z)$ is compact and $\hat{\kappa}_{m,n} \to \mu_m$ locally uniformly), hence $|S_{m,n}(z)| + |T_{m,n}(z)|$ is constant and $E_{m,n}(z)$ is monotonically decreasing. Secondly, assume that $\mu_m(w) < 1/m$ for all $w \in V_m(z)$. Then there exists an n_0 such that $\hat{\kappa}_{m,n}(w) < 1/m$ for all $n \geq n_0$ and all $w \in V_m(z)$, hence $|S_{m,n}(z)| = n - c$ with a certain constant c, and $E_{m,n}(z) = |T_{m,n}(z)| - c$ is monotonically increasing. Finally, assume that $1/m$ belongs to the interval

$$[\min\{\mu_m(w) : w \in V_m(z)\}, \max\{\mu_m(w) : w \in V_m(z)\}]$$

and notice that the length of that interval is at most 2^{-m}, which is less than $1/m - 1/(m+1)$ for $m \geq 5$. Then there exists an n_0 such that $\hat{\kappa}_{m,n}(w) > 1/(m+1)$ for all $n \geq n_0$ and all $w \in V_m(z)$, hence $\{|T_{m,n}(z)|\}_{n \geq n_0}$ is constant, and

$$E_{m,n}(z) = (|S_{m,n}(z)| - n) + |T_{m,n}(z)|$$

is monotonically decreasing.

Taking the maximum N of the finite set $\{n(z) : z \in Z_m \cap K\}$ then yields that the $\hat{\Gamma}_{m,n}(A)$, $n \geq N$, are constant, hence converge (if this constant set is non-empty) as $n \to \infty$. If $z_0 \in \mathrm{sp}_{\mathrm{ess}}(A)$ then $\mu(z_0) = 0$, hence $\mu_m(z_0) = 0$ for all m. So, for fixed m, we have $\hat{\kappa}_{m,n}(z) < 1/(m+1)$ for all sufficiently large n and all z in the neighbourhood $U_{1/(2m)}(z_0)$. Choose $z \in Z_m$ such that $z_0 \in V_m(z) \subset U_{1/(2m)}(z_0)$. This is possible since $m \geq 5$. Then it is immediate from the definitions that $E_{m,n}(z) = n - c$ with a constant c for all sufficiently large n, hence $z_0 \in \Gamma_{m,n}(A)$ for n large. Now given m, n, successively compute $\hat{\Gamma}_{m+5,n}(A), \hat{\Gamma}_{m+5,n+1}(A), \ldots$ and let $N(m,n) \geq n$ be minimal such that $\hat{\Gamma}_{m+5,N(m,n)}(A) \neq \emptyset$. Define $\Gamma_{m,n}(A) = \hat{\Gamma}_{m+4,N(m,n)}(A)$. The above arguments, in particular the fact that $\mathrm{sp}_{\mathrm{ess}}(A) \neq \emptyset$, demonstrate that this sequence of computations halts and $\Gamma_{m,n}$ is an arithmetical algorithm. Note also that $\Gamma_m(A) := \lim_{n\to\infty} \Gamma_{m,n}(A)$ exists, and the above argument shows that it contains the essential spectrum. Note also that $\Gamma_{m,n}(A)$ is in fact equal to $\Gamma_m(A)$ for large n.

We claim that $\{\Gamma_m(A)\}_m$ is a decreasing nested sequence, hence converges as well. Indeed, let $z \in \Gamma_{m+1}(A)$, then $z \in \hat{\Gamma}_{m+5,n}(A)$ for large n, i.e. $z \in V_{m+5}(w)$ for a $w \in I_{m+5,n}$, i.e. $w \in Z_{m+5}$ and $E_{m+5,n}(w) > 0$. Clearly, (for large enough n) there exists a $w_0 \in Z_{m+4}$ with $V_{m+5}(w) \subset V_{m+4}(w_0)$, and further (since we can assume without loss of generality by computing maxima over successive m that $\hat{\kappa}_{m+4,i}(z) \leq \hat{\kappa}_{m+5,i}(z)$ holds whenever $n > m+5$)

$$\begin{aligned} S_{m+5,n}(w) &= \{i = m+6, \dots, n : \exists z \in V_{m+5}(w) \cap G_i : \hat{\kappa}_{m+5,i}(z) \leq 1/(m+5)\} \\ &\subset \{i = m+5, \dots, n : \exists z \in V_{m+4}(w_0) \cap G_i : \hat{\kappa}_{m+4,i}(z) \leq 1/(m+4)\} = S_{m+4,n}(w_0) \end{aligned}$$

and analogously $T_{m+5,n}(w) \subset T_{m+4,n}(w_0)$. Therefore $E_{m+5,n}(w) \leq E_{m+4,n}(w_0)$, which shows that $w_0 \in I_{m+4,n}$ and thus $z \in \Gamma_m(A)$.

It remains to prove that the final limiting set $\lim_{m\to\infty} \Gamma_m(A)$ coincides with the essential spectrum. We have already proven that it must contain the essential spectrum. Conversely, let $z_0 \notin \mathrm{sp}_{\mathrm{ess}}(A)$, i.e. $\mu(z_0) > 0$. Then, for large m_0, there exists an $\epsilon > 3/m_0$ such that $\mu_m(z_0) > \epsilon$ and $\hat{\kappa}_{m,n}(z_0) > \epsilon/2$ for $m \geq m_0$ and large n, and then also $\hat{\kappa}_{m,n}(z) > \epsilon/3 > 1/m_0$ for all z in a certain neighbourhood U of z_0. For all sufficiently large $m \geq m_0$ all $V_m(z)$ which contain z_0 are subsets of U, hence $E_{m,n}(z) = d - n$ with a constant d for large n, that is $\lim_{n\to\infty} \hat{\Gamma}_{m,n}(A)$ and $\{z_0\}$ are separated. But since the $\{\Gamma_m(A)\}_m$ are nested, it follows z_0 is not in the limiting set $\lim_{m\to\infty} \Gamma_m(A)$. This finishes the proof. □

8.5. **Determining if a point z lies in $\mathrm{sp}(A)$.** Recall that for this problem, we restrict to $z \in \mathbb{R}$ when considering Ω_{D} or Ω_{SA}. We also restrict to $z \neq 0$ when considering Ω_{C}. Since $\Omega_{\mathrm{D}} \subset \Omega_{fg} \subset \Omega_f$, $\Omega_{\mathrm{C}} \subset \Omega_f$ and $\Omega_{\mathrm{SA}} \subset \Omega_{\mathrm{N}} \subset \Omega_g \subset \Omega_B$ it is enough to prove that $\{\Xi^z_{\mathrm{sp}}, \Omega_{\mathrm{SA}}\} \notin \Delta^G_3$, $\{\Xi^z_{\mathrm{sp}}, \Omega_B\} \in \Pi^A_3$, $\{\Xi^z_{\mathrm{sp}}, \Omega_f\} \in \Pi^A_2$, $\{\Xi^z_{\mathrm{sp}}, \Omega_{\mathrm{D}}\} \notin \Delta^G_2$ and $\{\Xi^z_{\mathrm{sp}}, \Omega_{\mathrm{C}}\} \notin \Delta^G_2$.

Proof of Theorem 4.4 for determining if a point lies in the spectrum. **Step I:** $\{\Xi^z_{\mathrm{sp}}, \Omega_{\mathrm{SA}}\} \notin \Delta^G_3$. By considering the shift $A - zI$, we can, without loss of generality, assume that $z = 0$. Suppose for a contradiction that Γ_{n_2,n_1} is a height two tower solving $\{\Xi^0_{\mathrm{sp}}, \Omega_{\mathrm{SA}}\}$. Let $(\mathcal{M}, d)$ be the discrete space $\{\textit{Yes}, \textit{No}\}$, let Ω' denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries $a_{i,j} \in \{0,1\}$ and consider the problem function

$$\Xi'(\{a_{i,j}\}): \text{ Does } \{a_{i,j}\} \text{ have (only) finitely many columns with (only) finitely many 1s?}$$

In Section 8.6 we prove that $\mathrm{SCI}(\Xi', \Omega')_G = 3$. Our strategy will be the same as the proof that $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \notin \Delta^G_3$ - we will gain a contradiction by using the supposed height two tower Γ_{n_2,n_1} to solve $\{\Xi', \Omega'\}$.

First, we need a certain periodic semi-infinite Jacobi matrix, which gives rise to spectral pollution when applying the finite section method. Define

$$A_\infty := \begin{pmatrix} 0 & 3 & & & & \\ 3 & 0 & 1 & & & \\ & 1 & 0 & 3 & & \\ & & 3 & 0 & 1 & \\ & & & 1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

It is well known that $\mathrm{sp}(A_\infty) = [-4,-2] \cup [2,4]$ (see for instance [42]). However, an easy check shows that 0 is an eigenvalue of the finite truncated matrix $P_n A_\infty P_n$ whenever n is odd. With an abuse of notation, we also define $A_n := P_n A_\infty P_n \oplus C_\infty \in \mathcal{B}(l^2(\mathbb{N}))$, where C_n denotes the $n \times n$ diagonal matrix with diagonal entries equal to -4.

Without loss of generality, we identify Ω_{SA} with self adjoint operators in $\mathcal{B}(X)$ where $X = \bigoplus_{j=1}^\infty X_j$ in the l^2-sense with $X_j = l^2(\mathbb{N})$. Now let $\{a_{i,j}\} \in \Omega'$ and for the jth column define $B_j \in \mathcal{B}(X_j)$ as follows. Let $I_j = \{i \in \mathbb{N} : a_{i,j} = 1\}$ and $J_j = \{i \in \mathbb{N} : a_{i,j} = 0\}$. We partition $\mathbb{N}$ into two sets:

$$N_1(j) = \{1\} \cup \{2k, 2k+1 : k \in I_j\}, \quad N_2(j) = \{2k, 2k+1 : k \in J_j\}.$$

On $\mathrm{span}\{e_k : k \in N_1(j)\}$ we let B_j act as $A_{|N_1(j)|}$, whereas on $\mathrm{span}\{e_k : k \in N_2(j)\}$ we let B_j act as $C_{|N_2(j)|}$ (both with respect to the natural bases and ordering). It is clear that B_j is unitarily equivalent to

$A_{|N_1(j)|} \oplus C_{|N_2(j)|}$. Hence $\mathrm{sp}(B_j)$ is equal to $[-4,-2] \cup [2,4] \cup K_j$, where $K = \{0\}$ if $\sum_i a_{i,j} < \infty$ and $K_j = \emptyset$ otherwise.

Next, we define the operator

$$C := \bigoplus_{j=1}^{\infty} \left(B_j + \frac{1}{2j} I \right)$$

on X. Concerning its spectrum, we note that any non-zero point of $\mathrm{sp}(C)$ inside the interval $[-1,1]$ is equal to $1/(2j)$ corresponding to precisely when the column $\{a_{i,j}\}_{i\in\mathbb{N}}$ has finitely many 1's. It is also clear that $0 \in \mathrm{sp}(C)$ precisely when this happens infinitely many times (0 is a limit point of a descending sequence in the spectrum). Hence $\Xi^0_{\mathrm{sp}}(C) = \textit{Yes}$ if and only if $\Xi'(\{a_{i,j}\}) = \textit{No}$.

We then define $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \textit{Yes}$ if $\Gamma_{n_2,n_1}(C) = \textit{No}$ and $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \textit{No}$ if $\Gamma_{n_2,n_1}(C) = \textit{Yes}$. Given N, we can evaluate $\{f_{k,l}(C) : k,l \leq N\}$ using only finitely many evaluations of $\{a_{i,j}\}$, where we can use a bijection between the canonical bases to view C as acting on $l^2(\mathbb{N})$. This follows since given any finite i, we can compute the sets $\{1,...,i\} \cap N_1(j)$ and $\{1,...,i\} \cap N_2(j)$. Hence $\tilde{\Gamma}_{n_2,n_1}$ defines a generalized algorithm and provides a height two tower of general algorithms solving $\{\Xi', \Omega'\}$, a contradiction.

Step II: $\{\Xi^z_{\mathrm{sp}}, \Omega_B\} \in \Pi^A_3$. By considering the shift $A - zI$, we can, without loss of generality, assume that $z = 0$ (note also that only having Δ_1-information regarding z is captured by only having Δ_1-information on matrix entries after this shift). Define the numbers

$$\gamma := \min\{\sigma_{\inf}(A), \sigma_{\inf}(A^*)\}, \quad \gamma_m := \min\{\sigma_{\inf}(AP_m), \sigma_{\inf}(A^*P_m)\},$$

$$\gamma_{m,n} := \min\{\sigma_{\inf}(P_nAP_m), \sigma_{\inf}(P_nA^*P_m)\}$$
$$\delta_{m,n} := \min\{2^{-m}k : k \in \mathbb{N}, 2^{-m}k \geq \sigma_{\inf}(P_nAP_m) \text{ or } 2^{-m}k \geq \sigma_{\inf}(P_nA^*P_m)\}.$$

As pointed out before, A is invertible if and only if $\gamma > 0$. Furthermore, note that $\gamma_m \downarrow_m \gamma$, and that $\gamma_{m,n} \uparrow_n \gamma_m$ for every fixed m. The sequences $\{\delta_{m,n}\}_n$ are bounded and monotonically non-decreasing, and $\gamma_{m,n} \leq \delta_{m,n} \leq \gamma_{m,n} + 2^{-m} \leq \gamma_m + 2^{-m}$. Thus, for $\epsilon > 0$ there is an m_0, and for every $m \geq m_0$ there is an $n_0 = n_0(m)$ such that

$$|\gamma - \delta_{m,n}| \leq |\gamma - \gamma_m| + |\gamma_m - \gamma_{m,n}| + |\gamma_{m,n} - \delta_{m,n}| \leq \epsilon/3 + \epsilon/3 + 2^{-m} \leq \epsilon \tag{8.23}$$

whenever $m \geq m_0$ and $n \geq n_0(m)$. So we see that the numbers $\delta_{m,n}$ converge monotonically from below for every m as $n \to \infty$, and the respective limits form a non-increasing sequence with respect to m, tending to γ. Moreover, each $\delta_{m,n}$ can be computed with finitely many arithmetic operations by Proposition 8.1. Thus, if we define $\Gamma_{k,m,n}(A) := (\delta_{m,n} < k^{-1})$, the monotonicity ensure that $\Gamma_k(A) := \lim_{m\to\infty} \lim_{n\to\infty} \Gamma_{k,m,n}(A)$ exists. Moreover, if $\gamma < k^{-1}$ then $\Gamma_k(A) = \textit{Yes}$. If $\Gamma_k(A) = \textit{No}$ then we must have that $\gamma \geq k^{-1}$ and hence $\Xi^0_{\mathrm{sp}}(A) = \textit{No}$. Finally, if $\gamma > k^{-1}$ then $\Gamma_k(A) = \textit{No}$. Hence $\Gamma_{k,m,n}$ provides a Π^A_3 tower.

Step III: $\{\Xi^z_{\mathrm{sp}}, \Omega_f\} \in \Pi^A_2$. Again, by considering the shift $A - zI$, we can, without loss of generality, assume that $z = 0$. If one considers operators for which a bound f on their dispersion is known, then choosing $n = f(m)$ turns (8.23) into

$$|\gamma - \delta_{m,f(m)}| \leq |\gamma - \gamma_m| + |\gamma_m - \gamma_{m,f(m)}| + |\gamma_{m,f(m)} - \delta_{m,f(m)}| \leq \epsilon/3 + \epsilon/3 + 2^{-m} \leq \epsilon \tag{8.24}$$

for large m taking $|\sigma_{\inf}(BP_m) - \sigma_{\inf}(P_{f(m)}BP_m)| \leq \|(I - P_{f(m)})BP_m\|$ into account. Therefore, a natural first guess for our general algorithms could be $\tilde{\Gamma}_{k,m}(A) := (\delta_{m,f(m)} < k^{-1})$. Unfortunately, although $\delta_{m,f(m)}$ converges to γ as $m \to \infty$ by (8.24), this is not monotone in general. Hence, it might be the case that $\gamma = k^{-1}$, but $\delta_{m,f(m)}$ oscillates around k^{-1} such that $\{\tilde{\Gamma}_{k,m}(A)\}_m$ may not converge. To overcome this drawback, we can use the same interval trick as before. Define $J^1_k = [0, k^{-1}]$ and $J^2_k = [2k^{-1}, \infty)$. For any given m, let $j(m) \leq m$ be maximal such that $\delta_{j,f(j)} \in J^1_k \cup J^2_k$. If no such j exists or $\delta_{j,f(j)} \in J^2_k$ then set $\Gamma_{k,m}(A) = \textit{No}$, otherwise set $\Gamma_{k,m}(A) = \textit{Yes}$. By our now standard argument, this converges as $m \to \infty$. If $\gamma > 0$, then for large enough k (such that $2k^{-1} < \gamma$), $\Gamma_{k,m}(A) = \textit{No}$ for large m. Conversely, if $\gamma = 0$ then for any k, $\delta_{m,f(m)} \in J_k^1$ for large m and hence $\Gamma_{k,m}(A) = \textit{Yes}$ for large m. This gives Π_2^A convergence.

Step IV: $\{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{D}}\} \notin \Delta_2^G$. Again, by considering the shift $A - zI$, we can, without loss of generality, assume that $z = 0$. If we assume that there is a general height-one-tower of algorithms $\{\Gamma_n\}$ over Ω_{D} then we can again construct counterexamples very easily: For a decreasing sequence $\{a_i\}$ of positive numbers we consider the diagonal operator $A := \mathrm{diag}\{a_i\}$. Clearly, 0 belongs to the spectrum of A if and only if the a_is tend to zero. As a start, set $\{a_i^1\} := \{1, 1, \ldots\}$, choose n_1 such that $\Gamma_n(\mathrm{diag}\{a_i^1\}) = \textit{No}$ for all $n \geq n_1$, and i_1 such that $\max\{i, j \,|\, f_{i,j} \in \Lambda_{\Gamma_{n_1}}(\mathrm{diag}\{a_i^1\})\} < i_1$. This is possible by (iii) in Definition 7.3 and our now standard argument. Then set $\{a_i^2\} := \{1, 1, \ldots, 1, 1/2, 1/2, \ldots\}$ with $1/2$s starting at the i_1th position. If $n_1, \ldots, n_{k-1}$ and $i_1, \ldots, i_{k-1}$ are already chosen then pick n_k such that $\Gamma_n(\mathrm{diag}\{a_i^k\}) = \textit{No}$ for all $n \geq n_k$, and i_k such that $\max\{i, j \,|\, f_{i,j} \in \Lambda_{\Gamma_{n_k}}(\mathrm{diag}\{a_i^k\})\} < i_k$, and modify $\{a_i^k\}$ to $\{a_i^{k+1}\} := \{1, \ldots, 2^{-k}, 2^{-k}, \ldots\}$ with 2^{-k}s starting at the i_kth position. Now, the contradiction is as in the previous proofs, and we see that $\{\Xi^0, \Omega_{\mathrm{D}}\} \notin \Delta_2^G$.

Step V: $\{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{C}}\} \notin \Delta_2^G$. Recall in this case that $z \neq 0$. By scaling any $A \in \Omega_{\mathrm{C}}$ by the factor $3/(2z)$, we can assume without loss of generality that $z = 3/2$. Suppose for a contradiction that a general height-one-tower of algorithms $\{\Gamma_n\}$ solves $\{\Xi_{\mathrm{sp}}^{\frac{3}{2}}, \Omega_{\mathrm{C}}\}$. Consider the arrowhead matrix:

$$A_n(\epsilon) := \begin{pmatrix} 1 & \epsilon & \epsilon^2 & \cdots & \epsilon^n \\ \epsilon & 0 & & & \\ \epsilon^2 & & \ddots & & \\ \vdots & & & 0 & \\ \epsilon^n & & & & 0 \end{pmatrix},$$

where $\epsilon \in (0, 1)$. A simple calculation yields that the eigenvalues of $A_n(\epsilon)$ are $\{0, 1/2 \pm \sqrt{1 + 4a_n(\epsilon)}/2\}$, where $a_n(\epsilon) = \frac{\epsilon^2(1-\epsilon^{2n})}{1-\epsilon^2}$. In particular, we choose $\epsilon = \sqrt{3/7}$ for which the only positive eigenvalue is $b_n := \frac{1+\sqrt{1+3(1-\frac{3^n}{7^n})}}{2}$. We now choose an increasing sequence of integers (greater than 1) $r_1, r_2, \ldots$ inductively, and define $A \in \Omega_{\mathrm{C}}$ such that when projected onto the span of the basis vectors $\{e_1, e_{r_1}, \ldots, e_{r_n}\}$ (with the natural order), with projection denoted by Q_n, $Q_n A Q_n$ has matrix $A_n(\sqrt{3/7})$. We also enforce that if $j \notin \{r_n\}_{n \in \mathbb{N}} \cup \{1\}$, then the $j$the column and row of A are zero. In other words, $A_{1,r_n} = A_{r_n,1} = (\sqrt{3/7})^n$, $A_{1,1} = 1$ and all other entries are 0. It follows that $\mathrm{sp}(A) = \{0, 1/2 \pm 1\}$ and hence $\Xi_{\mathrm{sp}}^{\frac{3}{2}}(A) = \textit{Yes}$. However, we choose $\{r_n\}$ such that there is an increasing sequence $\{c_n\}$ with $\Gamma_{c_n}(A) = \textit{No}$, yielding the contradiction.

Suppose that $r_1, \ldots, r_n$ have been chosen. Then let B_n be the infinite matrix with $Q_n B_n Q_n$ having matrix $A_n(\sqrt{3/7})$ and zeros elsewhere. Clearly the only positive eigenvalue of B_n is $b_n < 3/2$ and hence $\Xi_{\mathrm{sp}}^{\frac{3}{2}}(B_n) = \textit{No}$. So there exists $c_n > r_n$ with $\Gamma_{c_n}(B_n) = \textit{No}$. But by our now standard argument using the Definition 7.3 of a general algorithm, we can choose $r_{n+1} > r_n$ large such that $\Gamma_{c_n}(A) = \Gamma_{c_n}(B_n)$. □

Remark 8.7. To deal with Δ_1-information in Step II of the above proof, we can approximate $\delta_{m,n}$ from below to accuracy $1/n$ (taking rational values) and take successive maxima to preserve monotonicity as $n \to \infty$. In Step III, we approximate $\delta_{m,f(m)}$ to accuracy $1/m$ (taking rational values). In both cases, we use Proposition 8.2.

8.6. **Techniques for proving lower bounds.** Here, we collect two results concerning decision-making problems, which are used to show lower bounds for two of our spectral problems. Within this section, we exclusively deal with problems (functions)

$$\Xi : \Omega \to \mathcal{M} := \{\textit{Yes}, \textit{No}\},$$

where $\mathcal{M}$ is equipped with the discrete metric. This means that for such problems, we search for General algorithms $\Gamma_{n_k,\ldots,n_1} : \Omega \to \mathcal{M}$ which, for a given input $\omega \in \Omega$, answer *Yes* or *No*. We will refer to such problems as decision-making problems. Clearly, a sequence $\{m_i\} \subset \mathcal{M}$ of such "answers" converges to $m \in \mathcal{M}$ if and only if finitely many m_i are different from m. Let Ω_1 denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries $a_{i,j} \in \{0,1\}$ and let Ω_2 denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j\in\mathbb{Z}}$ with entries $a_{i,j} \in \{0,1\}$. Consider the following two problems:

$$\Xi_1 : \Omega_1 \ni \{a_{i,j}\}_{i,j\in\mathbb{N}} \mapsto \text{ Does } \{a_{i,j}\} \text{ have (only) finitely many columns with (only) finitely many 1s?}$$

$$\Xi_2 : \Omega_2 \ni \{a_{i,j}\}_{i,j\in\mathbb{Z}} \mapsto \left(\exists D \forall j \left(\left(\forall i \sum_{k=-i}^{i} a_{k,j} < D\right) \vee \left(\forall R \exists i \sum_{k=0}^{i} a_{k,j} > R \wedge \sum_{k=-i}^{0} a_{k,j} > R\right)\right)\right)$$

("there is a bound D such that every column has either less than D 1s or is two-sided infinite")

Theorem 8.8 (Decision making problems)**.** *Given the setup above, we have*

$$\mathrm{SCI}(\Xi_1, \Omega_1)_{\mathrm{G}} = \mathrm{SCI}(\Xi_1, \Omega_1)_{\mathrm{A}} = 3,$$
$$\mathrm{SCI}(\Xi_2, \Omega_2)_{\mathrm{G}} = \mathrm{SCI}(\Xi_2, \Omega_2)_{\mathrm{A}} = 3.$$

Remark 8.9. Note that the SCI of the decision problems above are considered with respect to general and arithmetic towers. These towers do not assume any computability model but only a model on the mathematical tools allowed (arithmetic operations in the case of an arithmetic tower) and how the algorithm can read the available information (only a finite amount of input). However, the SCI framework with towers of algorithms fit naturally into the classical theory of computability and the Arithmetical Hierarchy.

To prove Theorem 8.8, we must introduce some helpful background. Equip the set of all sequences $\{x_i\}_{i\in\mathbb{N}} \subset \{0,1\}$ with the following metric:

$$d_{\mathrm{B}}(\{x_i\}, \{y_i\}) := \sum_{n\in\mathbb{N}} 3^{-n} |x_n - y_n|. \tag{8.25}$$

The resulting metric space is known as the Cantor space. By the usual enumeration of the elements of $\mathbb{N}^2$ this metric translates to a metric on the set Ω_1 of all matrices $A = \{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries in $\{0,1\}$. Similarly, we do this for the set Ω_2 of all matrices $A = \{a_{i,j}\}_{i,j\in\mathbb{Z}}$ with entries in $\{0,1\}$. In each case, this gives a complete metric space, hence a so-called Baire space, i.e., it is of second category (in itself). To make this precise, we recall the following definitions:

Definition 8.10 (Meager set)**.** A set $S \subset \Omega$ in a metric space Ω is nowhere dense if every open set $U \subset \Omega$ has an open subset $V \subset U$ such that $V \cap S = \emptyset$, i.e. if the interior of the closure of S is empty. A set $S \subset \Omega$ is meager (or of the first category) if it is an at most countable union of nowhere dense sets. Otherwise, S is non-meager (or of the second category).

Notice that every subset of a meager set is meager, as is every countable union of meager sets. By the Baire category theorem, every (non-empty) complete metric space is non-meager.

Definition 8.11 (Initial segment)**.** We call a finite matrix $\sigma \in \mathbb{C}^{n\times m}$ an initial segment for an infinite matrix $A \in \Omega_1$ and say that A is an extension of σ if σ is in the upper left corner of A. In particular, $\sigma = P_n A P_m$ for some $n, m \in \mathbb{N}$, where we, with slight abuse of notation, consider $P_n A P_m \in \mathbb{C}^{n\times m}$. P_n is as usual the projection onto $\mathrm{span}\{e_j\}_{j=1}^n$, where $\{e_j\}_{j\in\mathbb{N}}$ is the canonical basis for $l^2(\mathbb{N})$.

Similarly, a finite matrix $\sigma \in \mathbb{C}^{(2n+1)\times(2m+1)}$ is an initial segment for an infinite matrix $B \in \Omega_2$ if σ is in the centre of B i.e. $\sigma = \tilde{P}_n B \tilde{P}_m$ where $\tilde{P}_n$ is the projection onto $\mathrm{span}\{e_j\}_{j=-n}^n$, where $\{e_j\}_{j\in\mathbb{Z}}$ is the canonical basis for $l^2(\mathbb{Z})$. We denote that A is an extension of σ by $\sigma \subset A$, and the set of all extensions of σ by $E(\sigma)$. The notion of extension extends in an obvious way to finite matrices.

Notice that the set $E(\sigma)$ of all extensions of σ is a non-empty open and closed neighborhood for every extension of σ.

Lemma 8.12. *Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a sequence of General algorithms mapping $\Omega_1 \to \mathcal{M}$, $T \subset \Omega_1$ be a non-empty closed set, and $S \subset T$ be a non-meager set (in T) such that $\xi = \lim_{n\to\infty} \Gamma_n(A)$ exists and is the same for all $A \in S$. Then there exists an initial segment σ and a number n_0 such that $E^T(\sigma) := T \cap E(\sigma)$ is not empty, and such that $\Gamma_n(A) = \xi$ for all $A \in E^T(\sigma)$ and all $n \geq n_0$. The same statement is true if we consider Ω_2 instead of Ω_1.*

Proof. We are in a complete metric space T. Since $S = \bigcup_{k\in\mathbb{N}} S_k$ with $S_k := \{A \in S : \Gamma_n(A) = \xi\ \forall n \geq k\}$ and S is non-meager, not all of the S_k can be meager, hence there is a non-meager S_k, and we set $n_0 := k$. Now, let A be in the closure $\overline{S_{n_0}}$, i.e. there is a sequence $\{A_j\} \subset S_{n_0}$ converging to A. Note that by assumption (i) in Definition 7.3 and the fact that Γ_n are General algorithms, we have that, for every fixed $n \geq n_0$, $|\Lambda_{\Gamma_n}(A)| < \infty$. Thus, by (ii) in Definition 7.3, the General algorithm Γ_n only depends on a finite part of A, in particular $\{A_f\}_{f\in\Lambda_{\Gamma_n}(A)}$ where $A_f = f(A)$. Since each $f \in \Lambda_{\Gamma_n}(A)$ represents a coordinate evaluation of A and by the definition of the metric d_B in (8.25), it follows that for all sufficiently large j, $f(A) = f(A_j)$ for all $f \in \Lambda_{\Gamma_n}(A)$. By assumption (iii) in Definition 7.3, it then follows that $\Lambda_{\Gamma_n}(A_j) = \Lambda_{\Gamma_n}(A)$ for all sufficiently large j. Hence, by assumption (ii) in Definition 7.3, we have that $\Gamma_n(A) = \Gamma_n(A_j) = \xi$ for all sufficiently large j. Thus, $\Gamma_n(A) = \xi$ for all $n \geq n_0$ and all $A \in \overline{S_{n_0}}$. Since S_{n_0} is not nowhere dense, we can choose a point $\tilde{A}$ in the interior of $\overline{S_{n_0}}$ and fix a sufficiently large initial segment σ of $\tilde{A}$ such that $E^T(\sigma)$ is a subset of $\overline{S_{n_0}}$. The assertion of the lemma now follows. The extension of the proof to Ω_2 is clear. □

Roughly speaking, this shows that there is a nice open and closed non-meager subspace of T for which $\lim_{n\to\infty} \Gamma_n(A)$ exists even in a uniform manner. Note that this result particularly applies to the case $T = \Omega$.

Proof of Theorem 8.8. **Step I:** $\mathrm{SCI}(\Xi_1, \Omega_2)_{\mathrm{G}} \geq 3$. We argue by contradiction and assume that there is a height two tower $\{\Gamma_r\}$, $\{\Gamma_{r,s}\}$ for Ξ_1, where Γ_r denote, as usual, the pointwise limits $\lim_{s\to\infty} \Gamma_{r,s}$. We will inductively construct initial segments $\{\sigma_n\}$ with $\sigma_{n+1} \supset \sigma_n$ yielding an infinite matrix $A \supset \sigma_n$ for all $n \in \mathbb{N}$, such that $\lim_{r\to\infty} \Gamma_r(A)$ does not exist. We construct $\{\sigma_n\}$ with the help of two sequences of subsets $\{T_n\}$ and $\{S_n\}$ of Ω, with the properties that $T_{n+1} \subset T_n$, each T_n is closed, and either $T_n = \Omega_1$ or there is an initial segment $\sigma \in \mathbb{C}^{m\times m}$ where $m \geq n$ such that T_n is the set of all extensions of σ with all the remaining entries in the first n columns being zero.

Suppose that we have chosen T_n. Note that the subset of all matrices in T_n with one particular entry being fixed is closed in T_n. Hence, the set of all matrices with one particular column being fixed is closed (as an intersection of closed sets). The latter set has no interior points in T_n; hence, it is nowhere dense in T_n. This provides that the set of all matrices in T_n for which a particular column has only finitely many 1s is a countable union of nowhere dense sets in T_n, hence is meager in T_n. Hence, the set of all matrices in $A \in T_n$ with $\Xi_1(A) =$ *No* (i.e., matrices with infinitely many "finite columns") is meager in T_n as well. Let R be its complement in T_n, i.e., the non-meager set of all matrices $A \in T_n$ with $\Xi_1(A) =$ *Yes*.

Clearly, $R = \bigcup_{r\in\mathbb{N}} R_r$ with $R_r := \{A \in R : \Gamma_k(A) = \textit{Yes}\ \forall k \geq r\}$, and there is an r_n such that $S_n := R_{r_n}$ is non-meager in T_n. Note that $\Gamma_{r_n,s}$ are General algorithms and $\Gamma_{r_n}(A) = \lim_{s\to\infty} \Gamma_{r_n,s}(A) =$ *Yes* for all $A \in S_n$. Thus, Lemma 8.12 applies and yields an initial segment σ_n, such that

$$E^{T_n}(\sigma_n) \neq \emptyset \quad \text{and } \Gamma_{r_n}(A) = \textit{Yes} \text{ for all } A \in E^{T_n}(\sigma_n). \tag{8.26}$$

Now, let $T_{n+1} \subset T_n$ be the (closed) set of all matrices in $E^{T_n}(\sigma_n)$ with all remaining [1] entries in the first $n+1$ columns being zero. Letting $T_0 = \Omega_1$ we have completed the construction.

The nested initial segments $\sigma_{n+1} \supset \sigma_n$ yield a matrix $A \in \cap_{n=0}^{\infty} T_n$ and this A has only finitely many 1s in each of its columns. However, by the construction of $\{T_n\}$, we have that $A \in E^{T_n}(\sigma_n)$ for all $n \in \mathbb{N}$. Thus, $\Xi_1(A) =$ *No*, but by (8.26), $\Gamma_k(A) =$ *Yes* for infinitely many k.

[1] I.e. outside the initial segment σ_n.

Step II: $\mathrm{SCI}(\Xi_2, \Omega_2)_{\mathrm{G}} \geq 3$. The proof is very similar to the proof of Step I. In particular, we argue by contradiction and assume that there is a height two tower $\{\Gamma_r\}$, $\{\Gamma_{r,s}\}$ for Ξ_2. As above, we inductively construct initial segments $\{\sigma_n\}$ with $\sigma_{n+1} \supset \sigma_n$ yielding an infinite matrix $A \supset \sigma_n$ for all $n \in \mathbb{N}$, such that $\lim_{r\to\infty} \Gamma_r(A)$ does not exist. We construct $\{\sigma_n\}$ with the help of two sequences of subsets $\{T_n\}$ and $\{S_n\}$ of Ω_2, with the properties that $T_{n+1} \subset T_n$, each T_n is closed, and either $T_n = \Omega_2$ or there is an initial segment $\sigma \in \mathbb{C}^{(2m+1)\times(2m+1)}$ where $m \geq n$ such that T_n is the set of all extensions of σ with all $\pm n$th semi-columns being filled by n additional 1s and infinitely many 0s, and and all the other kth columns, $|k| \leq n-1$, are being filled with zeros. In particular, if $\{a_{i,j}\}_{i,j\in\mathbb{Z}} \in T_n$ then

$$\begin{aligned} &\{a_{i,\pm n}\}_{i\in\mathbb{Z}} = \{\ldots, 0, \underbrace{1,\ldots,1}_{n\text{ times}}, \sigma_{-m,\pm n}, \ldots, \sigma_{m,\pm n}, \underbrace{1,\ldots,1}_{n\text{ times}}, 0, \ldots\}^T, \\ &\{a_{i,k}\}_{i\in\mathbb{Z}} = \{\ldots, 0, \sigma_{-m,k}, \ldots, \sigma_{m,k}, 0, \ldots\}^T, \quad k \in \mathbb{Z}_+,\ |k| \leq n-1. \end{aligned} \tag{8.27}$$

Suppose that we have chosen T_n. We argue as in Step I and deduce that for $k \in \mathbb{Z}$ the set of all matrices in T_n with one of the two kth semi-columns being fixed is nowhere dense in T_n, hence the set of all matrices in T_n with (one of the two) kth semi-columns having finitely many 1s is meager in T_n. We conclude that the set of all matrices in T_n with one semi-column having finitely many 1s is meager, thus its complement in T_n, the set of all matrices with all semi-columns having infinitely many 1s, is non-meager. Therefore the same holds for the superset $\{A \in T_n : \Xi_2(A) = \textit{Yes}\}$. Denoting this set by R we obviously have $R = \bigcup_{r\in\mathbb{N}} R_r$ with $R_r := \{A \in R : \Gamma_k(A) = \textit{Yes}\ \forall k \geq r\}$, and there is an r_n such that $S_n := R_{r_n}$ is non-meager in T_n. Note that $\Gamma_{r_n,s}$ are General algorithms and $\Gamma_{r_n}(A) = \lim_{s\to\infty} \Gamma_{r_n,s}(A) = \textit{Yes}$ for all $A \in S_n$. Thus, Lemma 8.12 applies and yields an initial segment σ_n, such that

$$E^{T_n}(\sigma_n) \neq \emptyset \quad \text{and } \Gamma_{r_n}(A) = \textit{Yes} \text{ for all } A \in E^{T_n}(\sigma_n). \tag{8.28}$$

Now, let $T_{n+1} \subset T_n$ be the (closed) set of all matrices $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ in $E^{T_n}(\sigma_n)$ with the property that (8.27) holds with $\sigma = \sigma_n$. Letting $T_0 = \Omega_2$ concludes the construction. The nested sequence $\{\sigma_n\}$again defines a matrix $A \in \cap_{n=0}^{\infty} T_n$ with the property that A has finitely many but at least k non-zero entries in the each of its kth semi-column which gives $\Xi_2(A) = \textit{No}$, but, by (8.28), $\Gamma_k(A) = \textit{Yes}$ for infinitely many k, a contradiction.

Step III: $\mathrm{SCI}(\Xi_1, \Omega_1)_{\mathrm{A}} \leq 3$ and $\mathrm{SCI}(\Xi_2, \Omega_2)_{\mathrm{A}} \leq 3$. This can again be proved by defining an appropriate tower of height 3 directly. For Ξ_1 we define

$$\Gamma_{k,m,n}(\{a_{i,j}\}_{i,j\in\mathbb{N}}) = \textit{Yes} \quad \Leftrightarrow \quad |\{j = 1, \ldots, m : \sum_{i=1}^{n} a_{i,j} < m\}| < k.$$

For Ξ_2 we define

$$\Gamma_{k,m,n}(\{a_{i,j}\}_{i,j\in\mathbb{Z}}) = \textit{Yes} \quad \Leftrightarrow \quad |\{j = -m, \ldots, m : k < \sum_{i=1}^{n} a_{i,j} < m \text{ or } k < \sum_{i=-n}^{-1} a_{i,j} < m\}| = 0.$$

It is straightforward to show that these provide height three arithmetical towers. □

The lower bounds of the SCI of the decision problems Ξ_1 and Ξ_2 allow us to obtain the lower bounds of the SCI of spectra and essential spectra of operators.

9. Proofs of Theorem 5.3 and Theorem 5.5

Remark 9.1 (Fourier Transform)**.** In this section we require the Fourier transform on $\mathrm{L}^2(\mathbb{R}^d)$, which will be denoted by $\mathcal{F} : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d)$. Our definition of $\mathcal{F}$ is as follows:

$$[\mathcal{F}\psi](\xi) = \int_{\mathbb{R}^d} \psi(x) e^{-2\pi i x\cdot\xi}\, dx.$$

We write $\hat{\psi}$ instead of $\mathcal{F}\psi$ for brevity. With this definition $\mathcal{F}$ is unitary on $\mathrm{L}^2(\mathbb{R}^d)$.

Remark 9.2 (The Attouch–Wets Topology)**.** In (7.4), we introduced the Attouch–Wets metric d_{AW} on the space $\mathcal{M}$ of non-empty closed subsets of $\mathbb{C}$. Since it is not convenient to work with d_{AW} directly, we make a note of the following simple characterization of convergence w.r.t. d_{AW}. Let $A \subset \mathbb{C}$ and $A_n \subset \mathbb{C}$ be a sequence of closed and non-empty sets. Then:

$$d_{\mathrm{AW}}(A_n, A) \to 0 \quad \text{if and only if} \quad d_{\mathcal{K}}(A_n, A) \to 0 \text{ for any compact } \mathcal{K} \subset \mathbb{C}, \tag{9.1}$$

where

$$d_{\mathcal{K}}(S,T) = \max\left\{ \sup_{s\in S\cap\mathcal{K}} d(s,T), \sup_{t\in T\cap\mathcal{K}} d(t,S)\right\}, \tag{9.2}$$

where we use the convention that $\sup_{s\in S\cap\mathcal{K}} d(s,T) = 0$ if $S\cap\mathcal{K} = \emptyset$. We refer to [12, Chapter 3] for details and further discussion. Equivalently, we observe that

$$\begin{aligned} &d_{\mathrm{AW}}(A_n, A) \to 0 \\ &\qquad \text{if and only if} \\ &\forall \delta > 0,\ \mathcal{K} \subset \mathbb{C} \text{ compact}, \exists N \text{ s.t. } \forall n > N,\ A_n \cap \mathcal{K} \subset \mathcal{N}_\delta(A) \text{ and } A\cap\mathcal{K} \subset \mathcal{N}_\delta(A_n) \end{aligned} \tag{9.3}$$

where $\mathcal{N}_\delta(X)$ is the usual open δ-neighbourhood of the set X. In this section, we will use the notation $A_n \to A$ to denote this convergence since there is no room for confusion.

9.1. **The case of bounded potential V: The proof of Theorem 5.3.** We will split the proof of Theorem 5.3 into two sections:

a. $\mathrm{SCI}(\Xi_{\mathrm{sp}}, \Omega_{\phi,g})_{\mathrm{A}} = 1$*:* While the proof of this is somewhat long and technical (extra care has been taken to deal with Δ_1-information and arithmetic algorithms over $\mathbb{Q}$), it is done via similar steps to the proof of Theorem 4.4 in §8.3, namely through approximations of the resolvent norm. However, some work is needed to convert point samples of V into approximations of the relevant matrices with respect to a Gabor basis. Lemmas 9.6 and 9.7 are technical lemmas needed to achieve this, whereas Lemma 9.8 concerns the approximations obtained via discretizations of the relevant inner products (and is needed to gain the Σ_1^A classification).
b. *Error control and rest of proof:* Lemma 9.8 is used to prove $\{\Xi_{\mathrm{sp}}, \Omega_{\phi,g}\} \in \Sigma_1^A$ and we extend the argument in §8.3 to prove $\{\Xi_{\mathrm{sp}}, \Omega_{\phi,\mathrm{SA}}\} \in \Sigma_1^{A,\mathrm{eigv}}$. To prove the rest of the theorem, we argue that it is enough to prove $\{\Xi_{\mathrm{sp},\epsilon}, \Omega_\phi\} \in \Sigma_1^A$. This is done via Lemma 9.10, which uses the approximations of $\gamma(z)$ constructed in part (a).

Before we embark on the proof, the reader unfamiliar with Halton sequences may want to review this material. An excellent reference is [97] (see p. 29 for definition). We will also need the following definition and theorem to prove Theorem 5.3.

Definition 9.3. Let $\{t_1,\dots t_N\}$ be a sequence in $[0,1]^d$. Then we define the *star discrepancy* of $\{t_1,\dots t_N\}$ to be

$$D_N^*(\{t_1,\dots t_N\}) = \sup_{K\in\mathcal{K}} \left| \frac{1}{N}\sum_{k=1}^N \chi_K(t_k) - \nu(K)\right|,$$

where $\mathcal{K}$ denotes the family of all subsets of $[0,1]^d$ of the form $\prod_{k=1}^d [0,b_k)$, χ_K denotes the characteristic function on K, $b_k \in (0,1]$ and ν denotes the Lebesgue measure.

Theorem 9.4 ([97])**.** *If $\{t_k\}_{k\in\mathbb{N}}$ is the Halton sequence in $[0,1]^d$ in the pairwise relatively prime bases $b_1,\dots,b_d$, then*

$$D_N^*(\{t_1,\dots t_N\}) \le \frac{d}{N} + \frac{1}{N}\prod_{k=1}^d \left(\frac{b_k-1}{2\log(b_k)}\log(N) + \frac{b_k+1}{2}\right) \qquad N\in\mathbb{N}. \tag{9.4}$$

For a proof of this theorem, see [97], p. 29. Note that as the right-hand side of (9.4) is somewhat cumbersome, it is convenient to define the following constant.

Definition 9.5. Define $C^*(b_1,\dots,b_d)$ to be the smallest integer such that for all $N\in\mathbb{N}$

$$\frac{d}{N}+\frac{1}{N}\prod_{k=1}^{d}\left(\frac{b_k-1}{2\log(b_k)}\log(N)+\frac{b_k+1}{2}\right)\leq C^*(b_1,\dots,b_d)\frac{\log(N)^d}{N}$$

where $b_1,\dots,b_d$ are as in Theorem 9.4.

Further to these definitions, we shall require a Gabor basis, which is the core of the discretization to produce the tower of algorithms. In particular, let

$$\psi_{k,l}(x)=e^{2\pi ikx}\chi_{[0,1]}(x-l),\qquad k,l\in\mathbb{Z}. \tag{9.5}$$

It is well-known that $\psi_{k,l}$ form an orthonormal basis for $\mathrm{L}^2(\mathbb{R})$. Thus, by applying the Fourier transform,

$$\{\hat{\psi}_{k_1,l_1}\otimes\hat{\psi}_{k_2,l_2}\otimes\cdots\otimes\hat{\psi}_{k_d,l_d}:k_1,l_1,\dots,k_d,l_d\in\mathbb{Z}\} \tag{9.6}$$

forms an orthonormal basis for $\mathrm{L}^2(\mathbb{R}^d)$ since the Fourier transform $\mathcal{F}$ is unitary. Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an enumeration of the collection of functions above, define

$$\mathcal{S}=\mathrm{span}\{\varphi_j\}_{j\in\mathbb{N}} \tag{9.7}$$

and let

$$\theta:\mathbb{N}\ni j\mapsto(k_1,l_1)\times\ldots\times(k_d,l_d)\in\mathbb{Z}^{2d} \tag{9.8}$$

be the bijection used in this enumeration. Define

$$\begin{aligned}\tilde{k}(m,d)&:=\max\{|k_p|:(k_p,l_p)=\theta(j)_p,p\in\{1,\dots,d\},j\in\{1,\dots,m\}\},\\ \tilde{l}(m,d)&:=\max\{|l_p|:(k_p,l_p)=\theta(j)_p,p\in\{1,\dots,d\},j\in\{1,\dots,m\}\},\end{aligned} \tag{9.9}$$

and let

$$C_1(m,d,a):=d^2\left(4\frac{(\max\{\tilde{l}(m,d)^2+\tilde{l}(m,d)+1/3,1\})^2}{|a-\tilde{k}(m,d)|+1}\right)^d,\qquad m,d,a\in\mathbb{N}, \tag{9.10}$$

$$C_2(m,d):=2^d\left(2((\tilde{l}(m,d)+1)^4+\tilde{l}(m,d)^4)^2(2(\tilde{k}(m,d)+1)+2)\right)^d,\qquad m,d\in\mathbb{N}. \tag{9.11}$$

The quantities $C_1(m,d,a)$ and $C_2(m,d)$ may seem to come out of the blue. They stem from Lemma 9.6 and Lemma 9.7 that are technical lemmas needed to construct the tower of algorithms. However, $C_1(m,d,a)$ and $C_2(m,d)$ occur in the main proof, and thus it is advantageous to introduce them here to prepare the reader.

9.1.1. **Proof that** $\mathrm{SCI}(\Xi_{\mathrm{sp}},\Omega_{\phi,g})_{\mathrm{A}}=1$**.** The proof will make clear that we do not need to worry about the algorithm outputting the empty set - given m, simply compute $\Gamma_{j(m)}(V)$ with $j(m)\geq m$ minimal such that $\Gamma_{j(m)}(V)\neq\emptyset$.

Proof of $\mathrm{SCI}(\Xi_{\mathrm{sp}},\Omega_{\phi,g})_{\mathrm{A}}=1$. **Step I: Defining** $\Gamma_m(\{V_\rho\}_{\rho\in\Lambda_{\Gamma_m}(V)})$ **and** $\Lambda_{\Gamma_m}(V)$. To do so, recall $\mathcal{S}$ from (9.7). Note that since $\mathcal{D}(H)=\mathrm{W}^{2,2}(\mathbb{R}^d)$, it is easy to show that $\mathcal{S}$ is a core for H. Let P_m, $m\in\mathbb{N}$, be the projection onto $\mathrm{span}\{\varphi_j\}_{j=1}^m$, and let $z\in\mathbb{C}$. Define

$$S_m(V,z):=(-\Delta+V-zI)P_m\quad\text{and}\quad\tilde{S}_m(V,z):=(-\Delta+\overline{V}-\overline{z}I)P_m.$$

Let

$$\sigma_{\inf}(S_m(V,z)):=\min\{(\langle S_m(V,z)f,S_m(V,z)f\rangle)^{\frac{1}{2}}:f\in\mathrm{Ran}(P_m),\|f\|=1\}$$

and $\sigma_{\inf}(\tilde{S}_m(V,z)):=\min\{(\langle\tilde{S}_m(V,z)f,\tilde{S}_m(V,z)f\rangle)^{\frac{1}{2}}:f\in\mathrm{Ran}(P_m),\|f\|=1\}$, and define

$$\gamma_m(z):=\min\{\sigma_{\inf}(S_m(V,z)),\sigma_{\inf}(\tilde{S}_m(V,z))\}. \tag{9.12}$$

Note that if we could evaluate γ_m at any point z using only finitely many arithmetic operations of elements of the form $V(x)$, $x\in\mathbb{R}^d$, we could have defined a general algorithm as desired by using $\Upsilon^{1/m}_{B_m(0)}(\gamma_m)$ where

$\Upsilon^{1/m}_{B_m(0)}$ is defined in (8.8). Unfortunately, such evaluation is not possible (γ_m may depend on infinitely many samples of V), and we will now focus on finding an approximation to γ_m.

Let $S = \{t_k\}_{k\in\mathbb{N}}$, where $t_k \in [0,1]^d$ is a Halton sequence (see [97] p. 29 for definition) in the pairwise relatively prime bases $b_1,\dots,b_d$ (note that the particular choice of the b_js is not important). Define, for $a > 0$ and $N \in \mathbb{N}$, the discrete inner product

$$\langle f,u\rangle_{a,N} = \frac{(2a)^d}{N}\sum_{k=1}^{N} f^a(t_k)\overline{u^a(t_k)}, \qquad f,u \in \mathrm{L}^2(\mathbb{R}^d)\cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)), \tag{9.13}$$

where we have defined the rescaling function on $[0,1]^d$ by

$$f^a = f(a(2\cdot -1),\dots,a(2\cdot -1))|_{[0,1]^d}, \tag{9.14}$$

(we will throughout the proof use the superscript a on a function to indicate (9.14)), where $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)) = \{f : \mathrm{TV}(f|_{[-b,b]^d}) < \infty, \forall b > 0\}$ and $\mathrm{TV}(f|_{[-b,b]^d})$ denotes the total variation, in the sense of Hardy and Krause (see [97]), of f restricted to $[-b,b]^d$. Note that since $V \in \mathrm{L}^\infty(\mathbb{R}^d)\cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and any $f \in \mathrm{Ran}(P_m)$ is smooth we have that $S_m(V,z)f \in \mathrm{L}^2(\mathbb{R}^d)\cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$. Hence, we can define for $n,m \in \mathbb{N}$

$$\begin{aligned}\sigma_{\mathrm{inf},n}(S_m(V,z)) &:= \min\{(\langle S_m(V,z)f, S_m(V,z)f\rangle_{n,N(n)})^{\frac12} : f \in \mathrm{Ran}(P_m), \|f\| = 1\}\\ \sigma_{\mathrm{inf},n}(\tilde S_m(V,z)) &:= \min\{(\langle \tilde S_m(V,z)f, \tilde S_m(V,z)f\rangle_{n,N(n)})^{\frac12} : f \in \mathrm{Ran}(P_m), \|f\| = 1\},\end{aligned} \tag{9.15}$$

where $N(n) := \lceil n\phi(n)^4\rceil$ and where ϕ comes from the definition of Ω_ϕ. We also set

$$\begin{aligned}Z_m(z)_{ij} &= \langle S_m(V,z)\varphi_j, S_m(V,z)\varphi_i\rangle_{n,N(n)}, \quad i,j \le m,\\ \tilde Z_m(z)_{ij} &= \langle \tilde S_m(V,z)\varphi_j, \tilde S_m(V,z)\varphi_i\rangle_{n,N(n)}, \quad i,j \le m.\end{aligned} \tag{9.16}$$

We have the following expansion

$$\begin{aligned}\langle S_m(V,z)\varphi_j, S_m(V,z)\varphi_i\rangle_{n,N} =& \langle\Delta\varphi_j,\Delta\varphi_i\rangle_{n,N} - \langle V\varphi_j,\Delta\varphi_i\rangle_{n,N} - \langle\Delta\varphi_j, V\varphi_i\rangle_{n,N}\\ &+ \langle V\varphi_j, V\varphi_i\rangle_{n,N} - 2\Re(z)\langle\Delta\varphi_j,\varphi_i\rangle_{n,N}\\ &+ \langle 2\Re(z\overline{V})\varphi_j,\varphi_i\rangle_{n,N} + |z|^2\langle\varphi_j,\varphi_i\rangle_{n,N},\end{aligned} \tag{9.17}$$

with a similar expansion holding for the matrix entries of $\tilde Z_m(z)$. Recall that the φ_js are an enumeration of the Fourier transforms of the basis $\psi_{k,l}(x) = e^{2\pi ikx}\chi_{[0,1]}(x-l)$, $k,l \in \mathbb{Z}$. It is easy to derive closed-form expressions for $\hat\psi_{k,l}$ and $\frac{\partial^2\hat\psi_{k,l}}{\partial\xi^2}$, and these expressions are variations of products of exponential functions and functions of the form $x \mapsto 1/x^p$ for $p = 1,2,3$. It follows that the matrix entries of $Z_m(z)$ and $\tilde Z_m(z)$ also have closed-form expressions in terms of point evaluations of the potential V (at the Halton nodes - see (9.13)). Note that the Halton nodes are rational. Using (9.17), it follows that given Δ_1-information for Λ, we can compute in finitely many arithmetic operations and comparisons, approximations to $Z_m(z)$ to any required accuracy. The same holds true for $\tilde Z_m(z)$. From Proposition 8.2, it follows that $\sigma_{\mathrm{inf},n}(S_m(V,z))$ and $\sigma_{\mathrm{inf},n}(\tilde S_m(V,z))$ can be computed to any given accuracy using finitely many arithmetic operations and comparisons.

Consider the quantity

$$\begin{aligned}\tilde\beta(m,n) := &(m+1)mC_1(m,d,n)\\ &+ (2n)^d d^2\left(m^2 + \sigma^2\phi^2(n) + 2(\sigma m+1)(\phi(n)+1)\right)\\ &\times\left(1+\sigma^2+2\sigma\right)C_2(m,d)C^*(b_1,\dots,b_d)\frac{\log(N(n))^d}{N(n)},\end{aligned} \tag{9.18}$$

where $\sigma = 3^d - 2^{d+1} + 2$, $C_1(m,d,n)$ is defined in (9.10), $C_2(m,d)$ is defined in (9.11) and $C^*(b_1,\dots,b_d)$ is defined in Definition 9.5. The function $\tilde\beta$ may seem to come somewhat out of the blue. However, it stems from certain bounds in (9.36) (see also (9.37)) on errors of discrete integrals related to (9.15). For any m, n, we can compute an upper bound in $\mathbb{Q}$ for $\tilde{\beta}(m,n)$ accurate to $1/m^4$ using finitely many arithmetic operations over $\mathbb{Q}$. Denote such an approximation by $\tilde{\tau}(m,n)$ and set

$$n(m) := \min\left\{n : \tilde{\tau}(m,n) \leq \frac{1}{m^3}\right\}. \tag{9.19}$$

First, note that the choice of $N(n)$ in (9.18) implies that $\tilde{\beta}(m,n) \to 0$ as $n \to \infty$. Thus, $n(m)$ is well defined since $\tilde{\tau}(m,n) < m^{-3}$ for large n. Second, note that it is clear that $\tilde{\tau}$, and hence also $n(m)$, can be evaluated by using finitely many arithmetic operations and comparisons.

We now let $\zeta_m(z)$ be a non-negative real valued function with

$$0 \leq \zeta_m(z) - \min\{\sigma_{\mathrm{inf},n(m)}(S_m(V,z)), \sigma_{\mathrm{inf},n(m)}(\tilde{S}_m(V,z))\} \leq \frac{1}{m}. \tag{9.20}$$

Combining the above remarks shows that, given Δ_1-information for Λ, we can compute such an approximation $\zeta_m(z)$ for any $z \in \mathbb{C}$ in finitely many arithmetic operations and comparisons over $\mathbb{Q}$. We can now define

$$\Gamma_m(V) := \Upsilon^{1/m}_{B_m(0)}(\zeta_m),$$

where $\Upsilon^{1/m}_{B_m(0)}(\zeta_m)$ is defined in (8.8). We conclude this step by noting that Γ_m are arithmetic towers of algorithms using Δ_1-information for Λ.

Step II: We show that $\Gamma_m(V) \to \Xi_{\mathrm{sp}}(V)$, as $m \to \infty$. Note that, by the properties of the Attouch–Wets topology, and as discussed in Remark 9.2, it suffices to show that for any compact set $\mathcal{K} \subset \mathbb{C}$

$$d_{\mathcal{K}}(\Gamma_m(V), \Xi_{\mathrm{sp}}(V)) \longrightarrow 0, \quad m \to \infty, \tag{9.21}$$

where $d_{\mathcal{K}}$ is defined in (9.2). To show (9.21) we start by defining

$$\begin{aligned}\gamma(z) := \min\big\{ &\inf\{\|(-\Delta + V - zI)\psi\| : \psi \in \mathrm{W}^{2,2}(\mathbb{R}^d), \|\psi\| = 1\},\\ &\inf\{\|(-\Delta + \overline{V} - \overline{z}I)\psi\| : \psi \in \mathrm{W}^{2,2}(\mathbb{R}^d), \|\psi\| = 1\}\big\} = \|(-\Delta + V - zI)^{-1}\|^{-1},\end{aligned} \tag{9.22}$$

where we use the convention that $\|(-\Delta+V-zI)^{-1}\|^{-1} = 0$ when $z \in \mathrm{sp}(-\Delta+V)$ and proceed similarly to the proof of Theorem 4.4 with the following claim. Before we state the claim recall h from the definition of $\Upsilon^{\delta}_{K}(\zeta)$ in Step II of the proof of Theorem 4.4 in §8.3.

Claim: Let $\mathcal{K} \subset \mathbb{C}$ be any compact set, and let K be a compact set containing $\mathcal{K}$ such that $\mathrm{sp}(-\Delta+V) \cap K \neq \emptyset$ and $0 < \delta < \epsilon < 1/2$. Suppose that ζ is a function with $\|\zeta - \gamma\|_{\infty,\hat{K}} := \|(\zeta-\gamma)\chi_{\hat{K}}\|_{\infty} < \epsilon$ on $\hat{K} := (K + B_{h(\mathrm{diam}(K)+2\epsilon)+\epsilon}(0))$, where $\chi_{\hat{K}}$ denotes the characteristic function of $\hat{K}$ and h is the inverse of g. Finally, let u be defined as in (8.9). Then $\lim_{\xi\to 0} u(\xi) = 0$ and

$$d_{\mathcal{K}}(\Upsilon^{\delta}_{K}(\zeta), \mathrm{sp}(-\Delta+V)) \leq u(\epsilon). \tag{9.23}$$

To prove the claim, we first show that

$$\sup_{s \in \Upsilon^{\delta}_{K}(\zeta)\cap\mathcal{K}} \mathrm{dist}(s, \mathrm{sp}(-\Delta+V)) \leq u(\epsilon). \tag{9.24}$$

If $\Upsilon^{\delta}_{K}(\zeta)\cap\mathcal{K} = \emptyset$, then there is nothing to prove. Thus we assume that $\Upsilon^{\delta}_{K}(\zeta)\cap\mathcal{K} \neq \emptyset$. Let $z \in G^{\delta}(K)$ and recall $G^{\delta}(K)$, h_δ and $I_z = B_{h_\delta(\zeta(z))}(z) \cap (\delta(\mathbb{Z}+\mathrm{i}\mathbb{Z}))$ from the definition of $\Upsilon^{\delta}_{K}(\zeta)$ in Step II of the proof of Theorem 4.4 in §8.3. Notice that we may argue exactly as in (8.10) and deduce that $I_z \subset \hat{K}$. Suppose that $M_z \neq \emptyset$. Note that from

$$\|(-\Delta+V-zI)^{-1}\|^{-1} \geq g(\mathrm{dist}(z, \mathrm{sp}(H))),$$

the monotonicity of h, and the compactness of $\mathrm{sp}(-\Delta+V)\cap K \neq \emptyset$ there is a $y \in \mathrm{sp}(-\Delta+V)$ of minimal distance to z with $|z-y| \leq h(\gamma(z))$. Since $\|\zeta-\gamma\|_{\infty,\hat{K}} < \epsilon$, and by using the monotonicity of h, we get $|z-y| \leq h(\zeta(z)+\epsilon)$. Hence, at least one of the $v \in I_z$, say v_0, satisfies $|v_0-y| < h(\zeta(z)+\epsilon) - h(\zeta(z)) + 2\delta$. Thus, by noting that $\gamma(v_0) \leq \mathrm{dist}(v_0, \mathrm{sp}(-\Delta+V))$, and by the assumption that $\delta < \epsilon$, we get $\zeta(v_0) <$

$\gamma(v_0) + \epsilon < h(\zeta(z) + \epsilon) - h(\zeta(z)) + 3\epsilon$. By the definition of M_z, this estimate now holds for all points $w \in M_z$. Thus, we may argue precisely as in (8.11) and deduce that

$$\mathrm{dist}(w, \mathrm{sp}(-\Delta + V)) \leq h(h(\zeta(z) + \epsilon) - h(\zeta(z)) + 3\epsilon),$$

which yields (9.24). To see that

$$\sup_{t \in \mathrm{sp}(-\Delta+V)\cap\mathcal{K}} \mathrm{dist}(\Upsilon_K^\delta(\zeta), t) \leq u(\epsilon), \tag{9.25}$$

(where we assume that $\mathrm{sp}(-\Delta + V) \cap \mathcal{K} \neq \emptyset$) take any $y \in \mathrm{sp}(-\Delta + V) \cap \mathcal{K} \subset K$. Then there is a point $z \in G^\delta(K)$ with $|z - y| < \delta < \epsilon$, hence

$$\zeta(z) < \gamma(z) + \epsilon \leq \mathrm{dist}(z, \mathrm{sp}(-\Delta + V)) + \epsilon < 2\epsilon < 1.$$

Thus, M_z is not empty and contains a point which is closer to y than $h(\zeta(z)) + \epsilon \leq h(2\epsilon) + \epsilon \leq u(\epsilon)$, and this yields (9.25). The fact that $\lim_{\xi\to 0} u(\xi) = 0$ is shown in Step II of the proof of Theorem 4.4 in §8.3, and we have proved the claim.

Armed with this claim, we continue on the path to prove (9.21). We define

$$\gamma_{m,n}(z) := \min\{\sigma_{\mathrm{inf},n}(S_m(V, z)), \sigma_{\mathrm{inf},n}(\tilde{S}_m(V, z))\}. \tag{9.26}$$

Then $\left\|\zeta_m - \gamma_{m,n(m)}\right\|_\infty \leq 1/m$ where $n(m)$ is defined as in (9.19). By Lemma 9.8 (below), $\zeta_m \to \gamma$ locally uniformly, when $m \to \infty$. Let m_0 be large enough so that for all $m \geq m_0$, $\Gamma_m(V) \cap \mathcal{K} = \Upsilon^{1/m}_{B_{m_0}(0)}(\zeta_m) \cap \mathcal{K}$. Choose $K = B_{m_0}(0)$ and $\epsilon \in (0, 1/2)$ as in the claim. Then, by the claim, there is an $m_1 > m_0$ such that for every $m > m_1$, by (9.23), $d_{\mathcal{K}}(\Gamma_m(V), \Xi_{\mathrm{sp}}(V)) \leq u(\epsilon)$. Since $\lim_{\xi\to 0} u(\xi) = 0$ then (9.21) follows. ☐

To finish this step of the proof, we need to establish the convergence of the functions γ_m, ζ_m and $\gamma_{m,n}$.

Lemma 9.6. *Consider the functions $\gamma_{m,n}$ and γ_m defined in (9.26) and (9.12) respectively. Then $\gamma_{m,n} \to \gamma_m$, locally uniformly as $n \to \infty$.*

Proof. Note that we will be using the notation $\mathrm{TV}_{[-a,a]^d}(f) = \mathrm{TV}(f|_{[-a,a]^d})$. Let, for $s, t \in \{0, 1\}$, $i, j \leq m$ and $u \in \{V, \overline{V}, |V|^2\}$

$$I(u, \Delta^s\varphi_j, \Delta^t\varphi_i) = \int_{\mathbb{R}^d} u(x) \sum_{p\in\Phi(s),q\in\Phi(t)} h_{i,j,p,q}(x)\, dx,$$

where

$$\begin{aligned} h_{i,j,p,q}(x) := &\left(\hat{\psi}_{\theta(j)_1}(x_1) \cdots \frac{\partial^{\tilde{s}}\hat{\psi}_{\theta(j)_p}}{\partial x_p^{\tilde{s}}}(x_p) \cdots \hat{\psi}_{\theta(j)_d}(x_d)\right) \\ &\times \overline{\left(\hat{\psi}_{\theta(i)_1}(x_1) \cdots \frac{\partial^{\tilde{t}}\hat{\psi}_{\theta(i)_q}}{\partial x_q^{\tilde{t}}}(x_q) \cdots \hat{\psi}_{\theta(i)_d}(x_d)\right)}, \end{aligned} \tag{9.27}$$

and

$$\Phi(t) = \begin{cases} \{1, \ldots, d\}, & t = 1 \\ \{1\}, & t = 0. \end{cases} \tag{9.28}$$

Observe that by the definition of $\gamma_{m,n}$ and γ_m in (9.26) and (9.12) the lemma follows if we can show that

$$I(u, \Delta^s\varphi_j, \Delta^t\varphi_i) - \frac{(2n)^d}{N} \sum_{k=1}^N u^n(t_k) \sum_{p\in\Phi(s),q\in\Phi(t)} h^n_{i,j,p,q}(t_k)) \longrightarrow 0, \qquad n \to \infty, \tag{9.29}$$

where $N = N(n)$ is from (9.16), $i, j \le m$, $s, t \in \{0, 1\}$ and u is either $V, \overline{V}, |V|^2$ (recall the notation V^a from (9.14)). Note that, by the multi-dimensional Koksma–Hlawka inequality (Theorem 2.11 in [97]) it follows that

(9.30)

$$\left| I(u, \Delta^s\varphi_j, \Delta^t\varphi_i) - \frac{(2n)^d}{N}\sum_{k=1}^{N} u^n(t_k) \sum_{p\in\Phi(s), q\in\Phi(t)} h^n_{i,j,p,q}(t_k)\right| \le \left\| u \sum_{p\in\Phi(s), q\in\Phi(t)} h_{i,j,p,q}\chi_{R(n)}\right\|_{\mathrm{L}^1} + (2n)^d \cdot \mathrm{TV}_{[-n,n]^d}\left(u \sum_{p\in\Phi(s), q\in\Phi(t)} h_{i,j,p,q}\right) D^*_N(t_1, \dots, t_N),$$

where $R(n) = ([-n, n]^d)^c$. To bound the first part of the right-hand side of (9.30), note that

$$\left\| u \sum_{p\in\Phi(s), q\in\Phi(t)} h_{i,j,p,q}\chi_{R(n)}\right\|_{\mathrm{L}^1} \le \|u\|_\infty K_{i,j}(n), \tag{9.31}$$

where

$$K_{i,j}(n) := \sum_{p\in\Phi(s), q\in\Phi(t)} \left\langle \left| \chi_{([-n,n]^d)^c} \hat{\psi}_{\theta(j)_1} \cdots \frac{\partial^{\tilde{s}}\hat{\psi}_{\theta(j)_p}}{\partial x_p^{\tilde{s}}} \cdots \hat{\psi}_{\theta(j)_d}\right|, \left| \hat{\psi}_{\theta(i)_1} \cdots \frac{\partial^{\tilde{t}}\hat{\psi}_{\theta(i)_q}}{\partial x_q^{\tilde{t}}} \cdots \hat{\psi}_{\theta(i)_d}\right| \right\rangle,$$

(recall θ from (9.8)) where $\chi_{([-n,n]^d)^c}$ denotes the characteristic function on $([-n, n]^d)^c$. To bound $K_{i,j}(n)$, note that it follows by the definition of $\psi_{k,l}$ with $k, l \in \mathbb{Z}$ in (9.5) and some straightforward integration that for $1 \le p \le d$ and $(k_p, l_p) = \theta(j)_p$ we have

$$\left|\hat{\psi}_{k_p,l_p}(x_p)\right| \le \begin{cases} 1 & \text{when } k_p - 1 \le x_p \le k_p + 1,\\ \frac{1}{|x_p - k_p|+1} & \text{otherwise,}\end{cases} \tag{9.32}$$

$$\left|\frac{\partial^2\hat{\psi}_{k_p,l_p}}{\partial x_p^2}(x_p)\right| \le \begin{cases} l_p^2 + l_p + \frac{1}{3} & \text{when } k_p - 1 \le x_p \le k_p + 1,\\ \frac{l_p^2 + l_p + \frac{1}{3}}{|x_p - k_p|+1} & \text{otherwise.}\end{cases} \tag{9.33}$$

Hence, if

$$\tilde{k} = \tilde{k}(m, d) := \max\{|k_p| : (k_p, l_p) = \theta(j)_p, p \in \{1, \dots, d\}, j \in \{1, \dots, m\}\},$$
$$\tilde{l} = \tilde{l}(m, d) := \max\{|l_p| : (k_p, l_p) = \theta(j)_p, p \in \{1, \dots, d\}, j \in \{1, \dots, m\}\},$$

and $n > \tilde{k}$, then it follows that

(9.34)

$$K_{i,j}(n) \le d^2 \max_{\substack{p\in\Phi(s)\\ q\in\Phi(t)\\ s,t\in\{0,1\}}} \left\{ \left\langle \left| \chi_{([-n,n]^d)^c} \hat{\psi}_{\theta(j)_1} \cdots \frac{\partial^{2s}\hat{\psi}_{\theta(j)_p}}{\partial x_p^{2s}} \cdots \hat{\psi}_{\theta(j)_d}\right|, \left| \hat{\psi}_{\theta(i)_1} \cdots \frac{\partial^{2t}\hat{\psi}_{\theta(i)_q}}{\partial x_q^{2t}} \cdots \hat{\psi}_{\theta(i)_d}\right| \right\rangle \right\}$$
$$\le d^2 \left(4\frac{(\max\{\tilde{l}^2 + \tilde{l} + 1/3, 1\})^2}{|n - \tilde{k}| + 1}\right)^d =: C_1(m, d, n).$$

To bound the second part of the right-hand side of (9.30), observe that, by Lemma 9.7, we have

$$\begin{aligned}
&(2n)^d \cdot \mathrm{TV}_{[-n,n]^d}\left(u \sum_{p\in\Phi(s), q\in\Phi(t)} h_{i,j,p,q}\right)\\
&\le (2n)^d d^2 \max_{p\in\Phi(s), q\in\Phi(t)} \big(\|u\|_\infty \|h_{i,j,p,q}\|_\infty + \sigma^2 \mathrm{TV}_{[-n,n]^d}(u)\mathrm{TV}_{[-n,n]^d}(h_{i,j,p,q})\\
&\quad + \sigma\left(\mathrm{TV}_{[-n,n]^d}(u)\|h_{i,j,p,q}\|_\infty + \mathrm{TV}_{[-n,n]^d}(h_{i,j,p,q})\|u\|_\infty\right)\big)\\
&\le (2n)^d d^2 \max\left\{\|V\|_\infty, \|V^2\|_\infty, \mathrm{TV}_{[-n,n]^d}(V), \mathrm{TV}_{[-n,n]^d}(|V|^2)\right\}\left(1 + \sigma^2 + 2\sigma\right) C_2(m, d),
\end{aligned} \tag{9.35}$$

where $\sigma = 3^d - 2^{d+1} + 2$ and $C_2(m,d)$ is defined in (9.11). Thus, by (9.30), (9.31), (9.34), (9.35), Lemma 9.7 and Theorem 9.4 (recall that $\{t_k\}_{k\in\mathbb{N}}$ is a Halton sequence) we get

(9.36)
$$\begin{aligned}
&\left| I(u,\Delta^s\varphi_j,\Delta^t\varphi_i) - \frac{(2n)^d}{N}\sum_{k=1}^{N} V^n(t_k) \sum_{p\in\Phi(s),q\in\Phi(t)} h^n_{i,j,p,q}(t_k)\right| \\
&\leq \max\{\|V\|_\infty, \|V\|_\infty^2\}C_1(m,d,n) + (2n)^d d^2 \max\left\{\|V\|_\infty, \|V^2\|_\infty, \mathrm{TV}_{[-n,n]^d}(V), \mathrm{TV}_{[-n,n]^d}(|V|^2)\right\} \\
&\qquad \times \left(1+\sigma^2+2\sigma\right) C_2(m,d)\left(\frac{d}{N} + \frac{1}{N}\prod_{k=1}^{d}\left(\frac{b_k-1}{2\log(b_k)}\log(N) + \frac{b_k+1}{2}\right)\right) \\
&\leq \beta(\|V\|_\infty, m, n),
\end{aligned}$$

where the last inequality uses the bound on the total variation of V from (5.2) and

$$\begin{aligned}
\beta(\|V\|_\infty, m, n) &:= (\|V\|_\infty+1)\|V\|_\infty C_1(m,d,n) \\
&\quad + (2n)^d d^2\left(\|V\|_\infty^2 + \sigma^2\phi^2(n) + 2(\sigma\|V\|_\infty+1)(\phi(n)+1)\right) \\
&\quad \times \left(1+\sigma^2+2\sigma\right) C_2(m,d) C^*(b_1,\dots,b_d)\frac{\log(N)^d}{N}, \quad N(n) = \lceil n\phi(n)^4\rceil
\end{aligned} \tag{9.37}$$

(recall (9.15)) where $C^*(b_1,\dots,b_d)$ is defined in Definition 9.5. Finally, note that, by the definition of $C_1(m,d,n)$ and the fact that we have chosen $N(n)$ according to (9.37), it follows that $\beta(\|V\|_\infty, m, n) \to 0$ as $n\to\infty$. Hence, (9.29) follows via (9.37), and the proof is finished. ☐

Lemma 9.7. *For all $a>0$, $i,j \leq n_2$ and $m,n\leq d$:*

(i) $\mathrm{TV}(h^a_{i,j,m,n}) = \mathrm{TV}_{[-a,a]^d}(h_{i,j,m,n}) \leq C_2(m,d)$,
(ii) $\|h^a_{i,j,m,n}\|_\infty \leq C_2(m,d)$,
(iii) for $u\in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\sigma = 3^d - 2^{d+1}+2$ we have that

$$\begin{aligned}
\mathrm{TV}(u^a h^a_{i,j,p,q}) = \mathrm{TV}_{[-a,a]^d}(uh_{i,j,p,q}) &\leq \|u\|_\infty\|h_{i,j,p,q}\|_\infty + \sigma^2\mathrm{TV}_{[-a,a]^d}(u)\mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q}) \\
&\quad + \sigma\left(\mathrm{TV}_{[-a,a]^d}(u)\|h_{i,j,p,q}\|_\infty + \mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q})\|u\|_\infty\right),
\end{aligned}$$

(iv) $\mathrm{TV}_{[-a,a]^d}(|g|^2) \leq \|g\|_\infty^2 + \sigma^2\mathrm{TV}^2_{[-a,a]^d}(g) + 2\sigma\|g\|_\infty\mathrm{TV}_{[-a,a]^d}(g)$

where

$$C_2(m,d) := 2^d\left(2((\tilde{l}+1)^4+\tilde{l}^4)^2(2(\tilde{k}+1)+2)\right)^d,$$

and $\tilde{k},\tilde{l}$ are defined in (9.9).

Proof. To prove both (i) and (ii), we will use the easy facts that $\mathrm{TV}(h^a_{i,j,p,q}) = \mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q})$ and $\mathrm{TV}(g^a h^a_{i,j,p,q}) = \mathrm{TV}_{[-a,a]^d}(gh_{i,j,p,q})$. To prove (i) of the claim let us first recall (see, for example [97], p. 19) that when $\psi\in C^1([-a,a]^d)$ then

$$\mathrm{TV}_{[-a,a]^d}(\psi) = \sum_{k=1}^{d}\sum_{1\leq i_1<\ldots<i_k\leq d} V^{(k)}(\psi; i_1,\dots,i_k), \tag{9.38}$$

where $V^{(k)}(\psi;i_1,\dots,i_k) = V^{(k)}(\psi_{i_1,\dots,i_k})$ and

$$\psi_{i_1,\dots,i_k} : (y_1,\dots,y_k)\mapsto \psi(\tilde{y}_1,\dots,\tilde{y}_d), \qquad \tilde{y}_j = a,\ j\neq i_1,\dots,i_k, \quad \tilde{y}_{i_j} = y_j,$$

$$V^{(k)}(\varphi) = \int_{-a}^{a}\cdots\int_{-a}^{a}\left|\frac{\partial^k\varphi}{\partial x_1\cdots\partial x_k}\right| dx_1\dots dx_k, \qquad \varphi\in C^1([-a,a]^k).$$

Note that from (9.27) and (9.5) it follows that $h^a_{i,j,p,q} \in C^\infty([0,1]^d)$, so by the definition of h in (9.27) we have that, for $k \in \{1,\dots,d\}$ and $1 \le i_1 < \dots < i_k \le d$,

$$\begin{aligned} &V^{(k)}(h^a_{i,j,p,q}; i_1,\dots,i_k) \\ &\le \prod_{\mu=1}^{d} \max\Bigg[\max_{s,t=0,2}\int_{-a}^{a}\left|\frac{\partial}{\partial x_\mu}\left(\frac{\partial^s\hat\psi_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu)\overline{\frac{\partial^t\hat\psi_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)}\right)\right| dx_\mu, \\ &\qquad \max_{\substack{s,t=0,2\\ x_\mu\in[-a,a]}}\left|\frac{\partial^s\hat\psi_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu)\overline{\frac{\partial^t\hat\psi_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)}\right|\Bigg], \quad \forall k,p,q \le d. \end{aligned} \tag{9.39}$$

We will now focus on bounding the right-hand side of (9.39). Note that by using the definition of $\psi_{k,l}$ with $k,l\in\mathbb{Z}$ in (9.5) and some straightforward integration, it follows that for $1\le p\le d$ and $(k_p,l_p)=\theta(j)_p$ we have

$$\left|\frac{\partial\hat\psi_{k_p,l_p}}{\partial x_p}(x_p)\right| \le \begin{cases} l_p+\frac{1}{2} & \text{when } k_p-1\le x_p\le k_p+1,\\ \frac{l+\frac12}{|x_p-k_p|+1} & \text{otherwise,}\end{cases} \tag{9.40}$$

$$\left|\frac{\partial^3\hat\psi_{k_p,l_p}}{\partial x_p^3}(x_p)\right| \le \begin{cases} \frac{(l_p+1)^4-l_p^4}{4} & \text{when } k_p-1\le x_p\le k_p+1,\\ \frac{(l_p+1)^4-l_p^4}{4(|x_p-k_p|+1)} & \text{otherwise.}\end{cases} \tag{9.41}$$

Thus, by using (9.32), (9.33), (9.40) and (9.41) it follows that

$$\begin{aligned} &\max_{s,t=0,2}\int_{-a}^{a}\left|\frac{\partial}{\partial x_\mu}\left(\frac{\partial^s\hat\psi_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu)\overline{\frac{\partial^t\hat\psi_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)}\right)\right| dx_\mu \\ &\le 2\max_{s,t=0,1,2,3}\int_{-\infty}^{\infty}\left|\frac{\partial^s\hat\psi_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu)\overline{\frac{\partial^t\hat\psi_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)}\right| dx_\mu \\ &\le 2((\tilde l+1)^4+\tilde l^4)^2\left(2(\tilde k+1)+\int_{[-\infty,-1]\cup[1,\infty]}\frac{1}{y^2}\,dy\right) \\ &= 2((\tilde l+1)^4+\tilde l^4)^2\left(2(\tilde k+1)+2\right), \end{aligned} \tag{9.42}$$

where $\tilde k := \max\{|k_p| : (k_p,l_p)=\theta(j)_p, p\in\{1,\dots,d\}, j\in\{1,\dots,n\}\}$, $\tilde l := \max\{|l_p| : (k_p,l_p)=\theta(j)_p, p\in\{1,\dots,d\}, j\in\{1,\dots,n\}\}$. Moreover, by (9.32) and (9.33)

$$\max_{\substack{s,t=0,2\\ x_\mu\in[-a,a]}}\left|\frac{\partial^s\hat\psi_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu)\overline{\frac{\partial^t\hat\psi_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)}\right| \le \max\{\tilde l^2+\tilde l+1/3,1\}, \qquad i,j\le m, \quad 1\le\mu\le d. \tag{9.43}$$

Hence, from (9.39), (9.42) and (9.43) it follows that for $k\in\{1,\dots,d\}$ and $1\le i_1<\dots<i_k\le d$,

$$V^{(k)}(h^a_{i,j,p,q}; i_1,\dots,i_k) \le \left(2((\tilde l+1)^4+\tilde l^4)^2(2(\tilde k+1)+2)\right)^d$$

and thus, by (9.38) we get that

$$\begin{aligned} \mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q}) &\le \left(2((\tilde l+1)^4+\tilde l^4)^2(2(\tilde k+1)+2)\right)^d\sum_{k=1}^{d}\binom{d}{k} \\ &\le 2^d\left(2((\tilde l+1)^4+\tilde l^4)^2(2(\tilde k+1)+2)\right)^d, \end{aligned}$$

and thus, we have proved (i) in the claim.

To prove (ii) in the claim, we observe that by (9.5), (9.27) and (9.43), it follows that

$$\|h^a_{i,j,p,q}\|_\infty \le \prod_{\mu=1}^{d}\max_{\substack{s,t=0,2\\ x_\mu\in[-\infty,\infty]}}\left|\frac{\partial^s\hat\psi_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu)\overline{\frac{\partial^t\hat\psi_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)}\right| \le \left(\max\{\tilde l^2+\tilde l+1/3,1\}\right)^d,$$

for $i, j \leq m$ and $p, q \leq d$. The last part of the above inequality is bounded by $C_2(m, d)$, which yields the assertion.

To prove (iii) and (iv), we will use the fact (see [20]) that

$$\mathcal{A} = \{f \in \mathcal{M}([-a, a]^d) : \|f\|_\infty + \mathrm{TV}_{[-a,a]^d}(f) < \infty\},$$

where $\mathcal{M}([-a, a]^d)$ denotes the set of measurable functions on $[-a, a]^d$, is a Banach algebra when $\mathcal{A}$ is equipped with the norm $\|f\|_{\mathcal{A}} = \|f\|_\infty + \sigma\mathrm{TV}_{[-a,a]^d}(f)$, where $\sigma > 3^d - 2^{d+1} + 1$. We will let $\sigma = 3^d - 2^{d+1} + 2$. Hence, by the Banach algebra property of the norm and (i) and (ii),

$$\begin{aligned}\mathrm{TV}_{[-a,a]^d}(uh_{i,j,p,q}) &\leq \|u\|_\infty \|h_{i,j,p,q}\|_\infty + \sigma^2 \mathrm{TV}_{[-a,a]^d}(u)\mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q}) \\ &\quad + \sigma\left(\mathrm{TV}_{[-a,a]^d}(u)\|h_{i,j,p,q}\|_\infty + \mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q})\|u\|_\infty\right), \quad u \in \mathcal{A},\end{aligned}$$

finally proving (iii). The proof of (iv) is almost identical. □

Lemma 9.8. *Recall ζ_m defined in (9.20). Then, $\zeta_m \to \gamma$ locally uniformly, where γ is defined in (9.22). Furthermore, if $m \geq \|V\|_\infty$ then we have*

$$\zeta_m(z) \geq \gamma_m(z) - \frac{2 + |z|}{m},$$

where γ_m is defined in (9.12).

Proof. Observe that $\gamma_m \to \gamma$ locally uniformly as $m \to \infty$. Indeed, let $\mathcal{T} = \{\|(-\Delta + V + zI)\psi\| : \psi \in \mathrm{W}^{2,2}(\mathbb{R}^d), \|\psi\| = 1\}$. Then, since $\mathcal{S}$ is a core for H (recall $\mathcal{S}$ from Step I of the proof of $\mathrm{SCI}(\Xi_{\mathrm{sp}}, \Omega_{\phi,g})_{\mathrm{A}} = 1$) then every element in $\mathcal{T}$ can be approximated arbitrarily well by $\|(-\Delta + V + zI)\tilde{\varphi}\|$ for some $\tilde{\varphi} \in \mathcal{S}$, thus it follows from (9.22) that we have convergence. Note that the convergence must be monotonically from above by the definition of P_m, and thus Dini's Theorem assures the locally uniform convergence. Thus, it suffices to show that $|\zeta_m - \gamma_m| \to 0$ locally uniformly as $m \to \infty$.

Note that if we define, for $z \in \mathbb{C}$, the operator matrices

$$\begin{aligned} Z_m(z)_{ij} &= \langle S_m(V, z)\varphi_j, S_m(V, z)\varphi_i\rangle_{n,N}, \quad i, j \leq m, \\ \tilde{Z}_m(z)_{ij} &= \langle \tilde{S}_m(V, z)\varphi_j, \tilde{S}_m(V, z)\varphi_i\rangle_{n,N}, \quad i, j \leq m, \quad N = \lceil n\phi(n)^4 \rceil, \end{aligned} \tag{9.44}$$

where $n = n(m)$ is defined in (9.19) and

$$W_m(z)_{ij} = \langle S_m(V, z)\varphi_j, S_m(V, z)\varphi_i\rangle, \quad i, j \leq m,$$

$$\tilde{W}_m(z)_{ij} = \langle \tilde{S}_m(V, z)\varphi_j, \tilde{S}_m(V, z)\varphi_i\rangle, \quad i, j \leq m,$$

the desired convergence follows if we can show that $\|Z_m(z) - W_m(z)\|$ and $\|\tilde{Z}_m(z) - \tilde{W}_m(z)\|$ tend to zero as m tends to infinity for all z in some compact set. However, this follows from the choice of $n(m) = \min\{n : \tilde{\tau}(m, n) \leq \frac{1}{m^3}\}$ in (9.19). In particular, $\beta(m, m, n) = \tilde{\beta}(m, n) \leq \tilde{\tau}(m, n)$ and clearly $\beta(\|V\|_\infty, m, n) \leq \beta(m, m, n)$ for $\|V\|_\infty \leq m$ (recall β from (9.37)). We also have

$$\begin{aligned} \langle S_m(V, z)\varphi_j, S_m(V, z)\varphi_i\rangle_{n,N} = &\langle \Delta\varphi_j, \Delta\varphi_i\rangle_{n,N} - \langle V\varphi_j, \Delta\varphi_i\rangle_{n,N} - \langle \Delta\varphi_j, V\varphi_i\rangle_{n,N} \\ &+ \langle V\varphi_j, V\varphi_i\rangle_{n,N} - 2\Re(z)\langle \Delta\varphi_j, \varphi_i\rangle_{n,N} \\ &+ \langle 2\Re(z\overline{V})\varphi_j, \varphi_i\rangle_{n,N} + |z|^2\langle \varphi_j, \varphi_i\rangle_{n,N}. \end{aligned} \tag{9.45}$$

Thus it follows immediately by (9.36) that

$$\begin{aligned} \max\left\{ |Z_m(z)_{ij} - W_m(z)_{ij}|, |\tilde{Z}_m(z)_{ij} - \tilde{W}_m(z)_{ij}|\right\} &\leq (4(|z| + 1) + |z|^2)\beta(\|V\|_\infty, m, n) \\ &\leq \frac{4(|z| + 1) + |z|^2}{m^3}. \end{aligned}$$

Using the fact that the operator norm of a matrix is bounded by its Frobenius norm $\|\cdot\|_F$, it follows that for $z \in K \subset \mathbb{C}$, where K is compact, $\|Z_m(z) - W_m(z)\|_F = \mathcal{O}(\frac{1}{m^2})$ and $\|\tilde{Z}_m(z) - \tilde{W}_m(z)\|_F = \mathcal{O}(\frac{1}{m^2})$ for sufficiently large m. To see the explicit bound, note that the above shows for $\|V\|_\infty \leq m$ that

$$\gamma_m(z)^2 \leq \frac{4(|z|+1)+|z|^2}{m^2} + \zeta_m(z)^2 \leq \left(\zeta_m(z) + \frac{\sqrt{4(|z|+1)+|z|^2}}{m}\right)^2$$

Taking square roots and re-arranging gives the result. □

9.1.2. **Proof of the $\in \Sigma_1^A$ and $\in \Sigma_1^{A,\mathrm{eigv}}$ classifications in Theorem 5.3.** To show the Σ_1^A classification for $\{\Xi_{\mathrm{sp}}, \Omega_{\phi,g}\}$, consider $\hat{\Gamma}_m(A) = \Gamma_{m+\lceil \|V\|_\infty \rceil}(V)$ where we now use the fact that an upper bound on $\|V\|_\infty$ is included in the evaluation functions. From Lemma 9.8, if $z \in \hat{\Gamma}_m(A)$ then

$$\mathrm{dist}(z, \mathrm{sp}(-\Delta + V)) \leq g^{-1}\left(\zeta_{m+\lceil \|V\|_\infty \rceil}(z) + \frac{2+|z|}{m}\right).$$

This can be approximated from above to within an error that converges to zero as $m \to \infty$ using finitely many evaluations of the function g at rational points. Taking the maximum over all $z \in \hat{\Gamma}_m(A)$ gives us an error bound which converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $m \to \infty$. The following shows that this is enough for the Σ_1^A error control.

Lemma 9.9. *Let $\Xi : \Omega \to (\mathcal{C}(\mathbb{C}), d_{\mathrm{AW}})$ be a problem function and suppose that there is an arithmetic tower of algorithms $\{\Gamma_m\}$ for Ξ. Suppose also that there exists a function $E_m : \Gamma_m(A) \mapsto \mathbb{R}_{\geq 0}$ (which may depend on A) computed along with each Γ_m (using finitely many arithmetic operations and comparisons) and converging uniformly to zero on compact subsets, such that*

$$\mathrm{dist}(z, \Xi(A)) \leq E_m(z), \quad \forall z \in \Gamma_m(A).$$

Suppose also that $\Gamma_m(A)$ is finite for each m and A. Then we can compute in finitely many arithmetic operations and comparisons a sequence of non-negative numbers $b_m \to 0$ such that $\Gamma_m(A) \subset A_m$ for some $A_m \in \mathcal{C}(\mathbb{C})$ with $d_{\mathrm{AW}}(A_m, \Xi(A)) \leq b_m$. Hence, by taking subsequences if necessary, we can build an arithmetic Σ_1^A tower for $\{\Xi, \Omega\}$.

Proof. Let $a_m^n = \sup\{E_m(z) : z \in \Gamma_m(A) \cap B_n(0)\}$. Define

$$A_m^n = \big((\Xi(A) + B_{a_m^n}(0)) \cap B_n(0)\big) \cup (\Gamma_m(A) \cap \{z : |z| \geq n\}).$$

It is clear that $\Gamma_m(A) \subset A_m^n$ and given $\{\Gamma_m(A), E_m(A)\}$ (we assume $\Gamma_m(A) \neq \emptyset$), we can easily compute a lower bound n_1 such that $\Xi(A) \cap B_{n_1}(0) \neq \emptyset$. Compute this from $\Gamma_1(A)$ and then fix it. Suppose that $n \geq 4n_1$, and suppose that $|z| < \lfloor n/4 \rfloor$. Then the points in A_m^n and $\Xi(A)$ nearest to z must lie in $B_n(0)$ and hence $\mathrm{dist}(z, A_m^n) \leq \mathrm{dist}(z, \Xi(A))$ and $\mathrm{dist}(z, \Xi(A)) \leq \mathrm{dist}(z, A_m^n) + a_m^n$. It follows that

$$d_{\mathrm{AW}}(A_m^n, \Xi(A)) \leq a_m^n + 2^{-\lfloor n/4 \rfloor}.$$

We now choose a sequence $n(m)$ such that setting $A_m = A_m^{n(m)}$ and $b_m = a_n^{n(m)} + 2^{-\lfloor n(m)/4 \rfloor}$ proves the result. Clearly, it is enough to ensure that b_m is null. If $m < 4n_1$ then set $n(m) = 4n_1$, otherwise consider $4n_1 \leq k \leq m$. If such a k exists with $a_m^k \leq 2^{-k}$, then let $n(m)$ be the maximal such k, and finally, if no such k exists, then set $n(m) = 4n_1$. For a fixed n, $a_m^n \to 0$ as $m \to \infty$. It follows that for large m, we must have $a_m^{n(m)} \leq 2^{-n(m)}$ and that $n(m) \to \infty$. □

Finally, we extend the argument of §8.3 for the approximate eigenvectors.

Proof that $\{\Xi_{\mathrm{sp}}, \Omega_{\phi,\mathrm{SA}}\} \in \Sigma_1^{A,\mathrm{eigv}}$. We need only argue for the approximate eigenvectors, and we sketch the proof since it is a simple adaptation of the discrete case considered in §8.3. Consider a Schrödinger operator in $\Omega_{\phi,\mathrm{SA}}$ with potential V and $z \in \hat{\Gamma}_m(V)$, where $\hat{\Gamma}_m$ is the constructed Σ_1^A tower for $\Omega_{\phi,g}$. By taking subsequences if necessary, it suffices to show that we can compute a vector $\psi_m \in \mathbb{C}^m$ such that, for a given $\delta \in \mathbb{Q}_{>0}$ with $\delta < 1$,

$$\langle Z_m(z)\psi_m, \psi_m\rangle \leq \sqrt{\sigma_{\inf}(Z_m(z))} + \delta, \quad 1 - \delta < \|\psi_m\| < 1, \tag{9.46}$$

where $Z_m(z)$ is the Hermitian positive (semi-)definite matrix defined via (9.44). The vector ψ_m will then correspond to the first m coefficients with respect to the Gabor basis. To see why this is sufficient, note that if T denotes the infinite matrix corresponding to $-\Delta + V - zI$ (with respect to the Gabor basis) and P_m denotes the projection onto the span of the first m basis functions, then (9.46) implies that

$$\|TP_m\psi_m\|^2 = \langle T^*T\psi_m, \psi_m\rangle = \langle Z_m(z)\psi_m, \psi_m\rangle$$

and that $\sqrt{\sigma_{\inf}(Z_m(z))}$ is bounded above by a computable null sequence since $z \in \hat{\Gamma}_m(V)$. We can then adapt the proof of $\{\Xi_{\mathrm{sp}}, \Omega_f \cap \Omega_{\mathrm{N}}\} \in \Sigma_1^{A,\mathrm{eigv}}$, in §8.3 with suitable approximations of $Z_m(z)$ (which can be computed with error control using Δ_1 information by the above arguments) replacing the matrix $(P_{f(n)}\tilde{A}P_n)^*(P_{f(n)}\tilde{A}P_n)$. □

9.1.3. **Proof of the $\in \Pi_2^A$ classification in Theorem 5.3.** It is clear that none of the problems lie in Δ_1^G. Hence to finish the proof of Theorem 5.3, we must show that $\{\Xi_{\mathrm{sp},\epsilon}, \Omega_\phi\} \in \Sigma_1^A$ since by taking $\epsilon \downarrow 0$ this will show $\{\Xi_{\mathrm{sp}}, \Omega_\phi\} \in \Pi_2^A$ since we have $\Omega_{\phi,g} \subset \Omega_\phi$. Note that through the use of ζ_m and Lemma 9.8 we can compute, using finitely many arithmetic operations and comparisons for any z, a function $\hat{\gamma}_m(z)$ that converges uniformly to $\gamma(z)$ from (9.22) on any compact subset of $\mathbb{C}$ with $\hat{\gamma}_m(z) \geq \gamma(z)$. The next Lemma then says that this is enough.

Lemma 9.10. *Suppose that $\hat{\gamma}_m(z) \geq \gamma(z)$ converge uniformly to $\left\|(-\Delta + V - zI)^{-1}\right\|^{-1}$ as $m \to \infty$ on compact subsets of $\mathbb{C}$. Set*

$$\Gamma_m(V) = (B_m(0) \cap \frac{1}{m}(\mathbb{Z} + i\mathbb{Z})) \cap \{z : \hat{\gamma}_m(z) < \epsilon\}.$$

For large m, $\Gamma_m(V) \neq \emptyset$ so we can assume this without loss of generality. Also, $d_{\mathrm{AW}}(\Gamma_m(V), \mathrm{sp}_\epsilon(-\Delta + V)) \to 0$ as $m \to \infty$ and clearly $\Gamma_m(V) \subset \mathrm{sp}_\epsilon(-\Delta + V)$.

Proof. Since the pseudospectrum is non-empty, for large m, $\Gamma_m(V) \neq \emptyset$, so we may assume that this holds for all m without loss of generality. We use the characterization of the Attouch–Wets topology where it is enough to consider closed balls. Suppose that n is large such that $B_n(0) \cap \mathrm{sp}_\epsilon(-\Delta + V) \neq \emptyset$. Since $\Gamma_m(V) \subset \mathrm{sp}_\epsilon(-\Delta + V)$, we must show that given $\delta > 0$, there exists N_1 such that if $m > N_1$ then $\mathrm{sp}_\epsilon(-\Delta+V) \cap B_n(0) \subset \Gamma_m(V) + B_\delta(0)$. Suppose for a contradiction that this was false. Then there exists $z_j \in \mathrm{sp}_\epsilon(-\Delta+V) \cap B_n(0)$, $\delta > 0$ and $m_j \to \infty$ such that $\mathrm{dist}(z_j, \Gamma_{m_j}(V)) \geq \delta$. Without loss of generality, we can assume that $z_j \to z \in \mathrm{sp}_\epsilon(-\Delta + V)$. There exists some w with $\left\|(-\Delta + V - wI)^{-1}\right\|^{-1} < \epsilon$ and $|z - w| \leq \delta/2$. Assuming $m_j > n+\delta$, there exists $y_{m_j} \in (B_{m_j}(0) \cap \frac{1}{m_j}(\mathbb{Z}+i\mathbb{Z}))$ with $\left|y_{m_j} - w\right| \leq 1/m_j$. It follows that

$$\hat{\gamma}_{m_j}(y_{m_j}) \leq \left|\hat{\gamma}_{m_j}(y_{m_j}) - \gamma(y_{m_j})\right| + \left|\gamma(w) - \gamma(y_{m_j})\right| + \left\|(-\Delta + V - wI)^{-1}\right\|^{-1}.$$

But γ is continuous and $\hat{\gamma}_{m_j}$ converges uniformly to γ on compact subsets. Hence for large m_j, $\hat{\gamma}_{m_j}(y_{m_j}) < \epsilon$ so that $y_{m_j} \in \Gamma_{m_j}(V)$. But $\left|y_{m_j} - z\right| \leq |z - w| + \left|y_{m_j} - w\right| \leq \delta/2 + 1/m_j$ which is smaller than δ for large m_j. This gives the required contradiction. □

9.2. **The case of unbounded potential V: The proof of Theorem 5.5.** In this section, we prove Theorem 5.5 on the SCI of spectra and pseudospectra of Schrödinger operators with unbounded potentials. First of all, we will build the Δ_2^A algorithms. Let us outline the steps of the proof first:

a. *Compactness of the resolvent:* The assumptions on the potential imply that the operator H has a compact resolvent $R(H, z)$ (see Proposition 9.21). Therefore, the spectrum is countable, consisting of eigenvalues with finite-dimensional invariant subspaces.

b. *Finite-dimensional approximations:* The main part of the proof centers around showing that it is possible to construct, with finitely many evaluations of V, square matrices $\widetilde{H}_n$ whose resolvents (when suitably embedded into the large space) converge to $R(H, z_0)$ in norm at a suitable point z_0 (see Theorem 9.23). Note that this technique is very different from the techniques used so far in the paper and is only possible due to compactness.

c. *Convergence of the spectrum and pseudospectrum:* We use the convergence at z_0 to show convergence at other points z in the resolvent set.

Once this is done, we prove that neither problem lies in $\Sigma_1^G \cup \Pi_1^G$.

We start with a general discussion as the argument is otherwise independent of the particular setup. In the end, we demonstrate the construction of the matrices $\widetilde{H}_n$ and the convergence of the resolvents. We assume the following:

(i) Assumptions on the operator A**:** Given a closed densely defined operator A in a Hilbert space $\mathcal{H}$ such that at $z_0 \in \mathbb{C}$ the resolvent operator $R(z_0) = (A - z_0)^{-1}$ is compact $R(z_0) \in \mathcal{K}(\mathcal{H})$. Thus $\mathrm{sp}(A) = \{\lambda_j\}$, the spectrum of A, is at most countable with no finite accumulation points.

(ii) Assumptions on the approximations A_n**:** Suppose A_n is a finite rank approximation to A such that if E_n is the orthogonal projection onto the range of A_n, then $A_n = A_n E_n$. We put further $\mathcal{H}_n = E_n\mathcal{H}$ and denote by $\widetilde{A}_n$ the matrix representing A_n when restricted to the invariant subspace $\mathcal{H}_n$ w.r.t. some orthonormal basis. Now, take the resolvent $(A_n E_n - zE_n)^{-1}$ of this restriction, extend it to $\mathcal{H}_n^\perp$ by zero, and denote this extension by $R_n(z)$. Then $R_n(z) = R_n(z)E_n$, and $R_n(z) = (A_n - z)^{-1} + (I - E_n)z^{-1}$ for all $z \neq 0$ for which the inverse exists. Finally we assume that $R_n(z_0)$ exist and

$$\lim_{n\to\infty} \|R_n(z_0) - R(z_0)\| = 0. \tag{9.47}$$

9.2.1. **Convergence of the spectrum and pseudospectrum.** The first step is to conclude that if the finite rank approximations to the resolvent converge in operator norm at one point z_0, then they also converge locally uniformly away from the spectrum of A. To that end, denote by $U_r(\mu)$ the open disc at center μ and radius r.

Proposition 9.11. *Suppose $R(z)$ and $R_n(z)$ are as above and satisfy* (9.47)*. Let $\mathcal{K} \subset \mathbb{C}$ be compact, $r > 0$ and define $\mathcal{K}_r = \mathcal{K}\backslash\bigcup_j U_r(\lambda_j)$. Then for large enough n, $R_n(z)$ exists for all $z \in \mathcal{K}_r$ and $\sup_{z\in\mathcal{K}_r} \|R_n(z) - R(z)\| \to 0$ as $n \to \infty$.*

Proof. Clearly $R(z) = R(z_0)(I - (z - z_0)R(z_0))^{-1}$ and $R_n(z) = R_n(z_0)(I - (z - z_0)R_n(z_0))^{-1}$ for all z in which $R(z)$, resp. $R_n(z)$, exist. By (9.47) it suffices to prove the existence of $R_n(z)$ and

$$\sup_{z\in\mathcal{K}_r} \|(I - (z - z_0)R_n(z_0))^{-1} - (I - (z - z_0)R(z_0))^{-1}\| \to 0.$$

However, we know that $(I - (z - z_0)R(z_0))^{-1}$ is meromorphic in the whole plane and hence analytic in the compact set $\mathcal{K}_r$ and in particular uniformly bounded. But this means that it is sufficient to show that the inverses converge, which in turn is immediate from (9.47) since

$$\sup_{z\in\mathcal{K}_r} \|(I-(z-z_0)R_n(z_0))-(I-(z-z_0)R(z_0))\| \leq \|R_n(z_0)-R(z_0)\| + \sup_{z\in\mathcal{K}_r} |z-z_0|\ \|R_n(z_0)-R(z_0)\|.$$

To see that this suffices, write $T_n(z) = (I - (z - z_0)R_n(z_0))$, $T(z) = (I - (z - z_0)R(z_0))$ and $T_n(z) = T(z)[I + T(z)^{-1}(T_n(z) - T(z))]$. Then for large enough n and $z \in \mathcal{K}_r$ by a Neumann series argument

$$\|T_n(z)^{-1} - T(z)^{-1}\| \leq \|T(z)^{-1}\|\ [(1 - \|T(z)^{-1}\|\|T_n(z) - T(z)\|)^{-1} - 1].$$

□

Proposition 9.12. *Let $\mathcal{K} \subset \mathbb{C}$ be compact and $\delta > 0$. Then, for all large enough n, $\mathrm{sp}(A)\cap\mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}(A_n))$ and $\mathrm{sp}(A_n) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}(A))$.*

Proof. Since the eigenvalues are precisely the poles of the resolvents, the claim follows immediately from the previous proposition. □

The last proposition gives the convergence of the spectra. The discussion on pseudospectra is more involved. We need to know that the resolvent norm is not constant in any open set. The following is a theorem due to J. Globevnik, E.B. Davies, and E. Shargorodsky, which we formulate here as a lemma:

Lemma 9.13 ([62] and [40]). *Suppose A is a closed and densely defined operator in $\mathcal{H}$ such that the resolvent $R(z) = (A-z)^{-1}$ is compact. Let $\Omega \subset \mathbb{C}$ be open and connected, and assume that, for all $z \in \Omega$, $\|R(z)\| \leq M$. Then, for all $z \in \Omega$, $\|R(z)\| < M$. This is particularly true if $\mathcal{H}$ is finite-dimensional.*

The theorem in [40] is formulated for Banach spaces X with the extra assumption that X or its dual are complex strictly convex, a condition which holds for Hilbert spaces. The case $\mathcal{H}$ being of finite dimension is already settled by [62]. We put $\gamma(z) = 1/\|R(z)\|$ and $\gamma_n(z) = 1/\|R_n(z)\|$ and summarise the properties of γ and γ_n as follows:

Lemma 9.14. *If (i) and (ii) hold, then $\gamma_n(z) \to \gamma(z)$ uniformly on compact sets. Neither γ, nor γ_n is constant in any open set, and they have local minima only where they vanish. Additionally, $\gamma(z) \leq \mathrm{dist}(z, \mathrm{sp}(A))$. Consequently,*

$$\mathrm{sp}_\epsilon(A) = \{z \,:\, \gamma(z) \leq \epsilon\} = \mathrm{cl}\{z \,:\, \gamma(z) < \epsilon\}, \quad \mathrm{sp}_\epsilon(A_n) = \{z \,:\, \gamma_n(z) \leq \epsilon\} = \mathrm{cl}\{z \,:\, \gamma_n(z) < \epsilon\}.$$

Proof. Note that $\gamma(z) \leq \mathrm{dist}(z, \mathrm{sp}(A))$ follows from a reformulation of a general property of resolvents. Next, notice that $\|R_n(z)\| = \|R(A_n, z)\|$ and that the norms of resolvents are subharmonic away from spectra and therefore γ and γ_n cannot have proper local minima, except when they vanish. Furthermore, they cannot be constant in an open set by Lemma 9.13.

To conclude the local uniform convergence, let M be such that along the curve $\{|z| = M\}$ there are no eigenvalues of A and choose $\mathcal{K}$ as the set $\{|z| \leq M\}$. Choose any ϵ, small enough so that the discs $\{|z - \lambda_j|\} \leq \epsilon/3$ separate the eigenvalues inside $\mathcal{K}$. By Proposition 9.11, we may assume that n is large enough so that for $z \in \mathcal{K}_{\epsilon/3}$ (recall $\mathcal{K}_r$ from Proposition 9.11) we have $|\gamma_n(z) - \gamma(z)| \leq \epsilon/3$. On the other hand, if $|z - \lambda_j| \leq \epsilon/3$ then $\gamma(z) \leq \epsilon/3$ and, since γ_n has to vanish also somewhere in that disc (again for large enough n) and $\gamma_n(z) \leq \mathrm{dist}(z, \mathrm{sp}(A_n|_{\mathcal{H}_n}))$, we have $\gamma_n(z) \leq 2\epsilon/3$ in that disc, hence $|\gamma_n(z) - \gamma(z)| \leq \gamma_n(z) + \gamma(z) \leq \epsilon$. Thus we have $|\gamma_n(z) - \gamma(z)| \leq \epsilon$ for all $z \in \mathcal{K}$.

Finally, to justify the equivalence of the characterizations of pseudospectra, notice that the level sets $\{z \,:\, \gamma(z) = \epsilon\}$ and $\{z \,:\, \gamma_n(z) = \epsilon\}$ cannot contain open subsets or isolated points. □

Lemma 9.15. *Assume φ_n and φ are continuous non-negative functions in $\mathbb{C}$ which have local minima only when they vanish, are not constant in any open set, and φ_n converges to φ uniformly in compact sets. Set $\mathcal{S} := \{z \,:\, \varphi(z) \leq 1\}$ and $\mathcal{S}_n := \{z \,:\, \varphi_n(z) \leq 1\}$. Let $\mathcal{K}$ be compact and $\delta > 0$. Then the following hold for all large enough n: $\mathcal{S} \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathcal{S}_n)$, $\mathcal{S}_n \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathcal{S})$, where $\mathcal{N}_\delta(\cdot)$ denotes the open δ neighbourhood.*

Proof. Consider $\mathcal{S} \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathcal{S}_n)$, and assume that the left-hand side is not empty. Due to compactness of $\mathcal{S} \cap \mathcal{K}$ there are points $z_i \in \mathcal{S} \cap \mathcal{K}$ for $i = 1, \dots, m$ such that $\mathcal{S} \cap \mathcal{K} \subset \bigcup_{i=1}^m U_{\delta/2}(z_i)$. Notice that $\varphi(z_i) \leq 1$. If $\varphi(z_i) < 1$, set $y_i = z_i$. Otherwise, $\varphi(z_i) = 1$, in which case z_i cannot be a local minimum, but since φ is not constant in any open set, there exists a point $y_i \in U_{\delta/2}(z_i)$ such that $\varphi(y_i) < 1$. But since φ_n converges uniformly in compact sets to φ we conclude that for all large enough n and all i we have $\varphi_n(y_i) < 1$. Hence $z_i \in \mathcal{N}_{\delta/2}(\mathcal{S}_n)$ and so $\mathcal{S} \cap \mathcal{K} \subset \bigcup_{i=1}^m U_{\delta/2}(z_i) \subset \mathcal{N}_\delta(\mathcal{S}_n)$.

Consider now $\mathcal{S}_n \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathcal{S})$,. If it did not hold, there would exist a sequence $\{n_j\}$ and points $z_{n_j} \in \mathcal{S}_{n_j} \cap \mathcal{K}$ such that $z_{n_j} \notin \mathcal{N}_\delta(\mathcal{S})$. Suppose $z_{n_{j_k}} \to \hat{z}$. Then $\mathrm{dist}(\hat{z}, \mathcal{S}) \geq \delta$ as well. However, writing $\varphi(\hat{z}) \leq |\varphi(\hat{z}) - \varphi(z_{n_{j_k}})| + |\varphi(z_{n_{j_k}}) - \varphi_{n_{j_k}}(z_{n_{j_k}})| + \varphi_{n_{j_k}}(z_{n_{j_k}})$ we obtain $\varphi(\hat{z}) \leq 1$ as the first term on the right tends to zero because φ is continuous, the second term converges to zero as φ_n approximate φ uniformly in compact sets, and $\varphi_{n_{j_k}}(z_{n_{j_k}}) \leq 1$. Hence, $\hat{z} \in \mathcal{S} \cap \mathcal{K}$, which is a contradiction. □

Note that the same argument for Lemma 9.15 holds when replacing ≤ 1 by $\leq \epsilon$ in the definitions of $\mathcal{S}$ and $\mathcal{S}_n$. Combining the results of this section, we can state the following result.

Proposition 9.16. *Let $\mathcal{K} \subset \mathbb{C}$ be compact and $\delta > 0$. Then, for all large enough n,*

$$\mathrm{sp}_\epsilon(A) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}_\epsilon(A_n)), \qquad \mathrm{sp}_\epsilon(A_n) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}_\epsilon(A)).$$

9.2.2. **The general algorithms.** Here A, A_n are operators in $\mathcal{H}$ as in (i), (ii) above, while $\widetilde{A}_n$ is the matrix representing A_n when restricted to the finite-dimensional invariant subspace $\mathcal{H}_n = E_n\mathcal{H}$. In particular $\|R_n(z)\| = \|(\widetilde{A}_n - z)^{-1}\|$. Denoting by $\sigma_{\inf}$ the smallest singular value of a square matrix we have $\gamma_n(z) = 1/\|R_n(z)\| = \sigma_{\inf}(\widetilde{A}_n - zI)$. Let $r > 0$ and define $G_r := B_r(0) \cap (\frac{1}{2r}(\mathbb{Z} + i\mathbb{Z}))$. Suppose the matrices $\widetilde{A}_n$ are available with Δ_1-information. From Proposition 8.2, it follows that we can compute, in finitely many arithmetic operations and comparisons over $\mathbb{Q}$, an approximation to $\gamma_n(z)$ from above, accurate to $1/n^2$, and taking values in $\mathbb{Q}_{\geq 0}$. Call this approximation $\hat{\gamma}_n$ and let $\epsilon_n \in \mathbb{Q}$ be an approximation of ϵ from below accurate to $1/n^2$ and define Γ_n^1 and Γ_n^2 by

$$\Gamma_n^1(A) = \left\{z \in G_n : \hat{\gamma}_n(z) \leq \frac{1}{n}\right\}, \qquad \Gamma_n^2(A) = \{z \in G_n : \hat{\gamma}_n(z) \leq \epsilon_n\}, \tag{9.48}$$

which we shall prove to be the towers of algorithms for Ξ_{sp} and $\Xi_{\mathrm{sp},\epsilon}$ (as defined in Theorem 5.5), respectively. Observe that $\Gamma_n^1(A)$ and $\Gamma_n^2(A)$ can be executed in a finite number of arithmetic operations over $\mathbb{Q}$ using Δ_1-information. Also, our proof will show that $\Gamma_n^i(A) \neq \emptyset$ for large n. Hence, by our usual trick of searching for minimal $n(m) \geq m$ such that this is so, we can assume that this holds for all n without loss of generality.

Proposition 9.17. *The algorithms satisfy the following:*

$$\Gamma_n^1(A) \longrightarrow \mathrm{sp}(A), \qquad \Gamma_n^2(A) \longrightarrow \mathrm{sp}_\epsilon(A), \quad n \to \infty. \tag{9.49}$$

Proof. We begin with the second part of (9.49). It suffices to show that given δ and a compact ball $\mathcal{K}$, for large n:

$$\text{(i) } \Gamma_n^2(A) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}_\epsilon(A)), \qquad \text{(ii) } \mathrm{sp}_\epsilon(A) \cap \mathcal{K} \subset \mathcal{N}_\delta(\Gamma_n^2(A)).$$

Note that $\Gamma_n^2(A) \subset \mathrm{sp}_\epsilon(\widetilde{A}_n) \cap G_n$ and hence, the first inclusion follows immediately from Proposition 9.16. To see (ii), we argue by contradiction and suppose not. Then by possibly passing to an increasing subsequence $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ there is a sequence $z_n \in (\mathrm{sp}_\epsilon(A) \cap \mathcal{K}) \setminus \mathcal{N}_\delta(\Gamma_n^2(A))$ for all n. Since $\mathrm{sp}_\epsilon(A) \cap \mathcal{K}$ is a compact set, by possibly extracting a subsequence, we have that $z_n \to z_0 \in \mathrm{sp}_\epsilon(A) \cap \mathcal{K}$. Consider the open ball $U_{\delta/3}(z_0)$ which must contain all z_n for n sufficiently large. Since $\gamma(z)$ is continuous, positive, not constant in any open set and without nontrivial local minima, it follows that $\mathrm{sp}_\epsilon(A)$ equals the closure of its interior points. In particular $\mathrm{int}(\mathrm{sp}_\epsilon(A)) \cap U_{\delta/3}(z_0) \neq \emptyset$. Suppose then $r > 0$ and y_0 are such that the closure of the open ball $U_r(y_0)$ is inside this open set: $B_r(y_0) \subset \mathrm{int}(\mathrm{sp}_\epsilon(A)) \cap U_{\delta/3}(z_0)$. We claim that $\{z : \hat{\gamma}_n(z) \leq \epsilon\} \cap U_r(y_0) = U_r(y_0)$ for all large enough n. Indeed, since $U_r(y_0)$ bounded away from the boundary of the pseudospectrum of A, we have $\gamma(z) \leq \epsilon - s$ for some $s > 0$ and for all $z \in U_r(y_0)$. The claim follows from the locally uniform convergence of γ_n and hence of $\hat{\gamma}_n$. By the definition of G_n we have that $U_r(y_0) \subset \mathcal{N}_{\delta/3}(U_r(y_0) \cap G_n)$ for large n, so, by the claim, $U_r(y_0) \subset \mathcal{N}_{\delta/3}(\{z : \hat{\gamma}_n(z) \leq \epsilon\} \cap G_n)$. Hence, since $U_r(y_0) \subset U_{\delta/3}(z_0)$, it follows that

$$z_n \in U_{\delta/3}(z_0) \subset \mathcal{N}_{2\delta/3}(U_r(y_0)) \subset \mathcal{N}_\delta(\{z : \hat{\gamma}_n(z) \leq \epsilon\} \cap G_n),$$

for large n, contradicting $z_n \notin \mathcal{N}_\delta(\Gamma_n^2(A))$. To prove the first part of (9.49) we argue as follows. Given $\delta > 0$ and compact $\mathcal{K}$, we need to show that for large n:

$$\text{(iii) } \mathrm{sp}(A) \cap \mathcal{K} \subset \mathcal{N}_\delta(\{z : \hat{\gamma}_n(z) \leq 1/n\} \cap G_n) \qquad \text{(iv) } \{z : \hat{\gamma}_n(z) \leq 1/n\} \cap G_n \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}(A)).$$

For notational convenience, we let $a_n = 1/n - 1/n^2$.

To show (iii), we start by defining $\widetilde{G}_n := \frac{1}{2n}(\mathbb{Z} + i\mathbb{Z})$ and note that for $\lambda_j \in \mathrm{sp}(\widetilde{A}_n)$ we have that $\mathcal{N}_{a_n}(\{\lambda_j\}) \cap \widetilde{G}_n \neq \emptyset$ for large n. Hence, $\mathrm{sp}(\widetilde{A}_n) \subset \mathcal{N}_{1/n}\left(\mathcal{N}_{a_n}\left(\mathrm{sp}(\widetilde{A}_n)\right) \cap \widetilde{G}_n\right)$. Since $\mathcal{N}_{a_n}(\mathrm{sp}(\widetilde{A}_n)) \subset \mathrm{sp}_{a_n}(\widetilde{A}_n)$, it follows that $\mathrm{sp}(\widetilde{A}_n) \subset \mathcal{N}_{1/n}\left(\mathrm{sp}_{a_n}(\widetilde{A}_n) \cap \widetilde{G}_n\right)$. Now by the first part of Proposition 9.12 we have that $\mathrm{sp}(A) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2}(\mathrm{sp}(\widetilde{A}_n))$ for large n. Thus, combining the previous observations, we have for large n that

$$\mathrm{sp}(A) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2+1/n}\left(\mathrm{sp}_{a_n}(\widetilde{A}_n) \cap \widetilde{G}_n\right) \subset \mathcal{N}_{\delta/2+1/n}\left(\{z : \hat{\gamma}_n(z) \leq 1/n\} \cap \widetilde{G}_n\right).$$

However, since $\mathcal{K}$ is bounded we have that there exists an $r > 0$ such that if $\lambda \in \widetilde{G}_n \cap U_r(0)^c$ then $\mathcal{N}_\delta(\{\lambda\}) \cap \mathrm{sp}(A) \cap \mathcal{K} = \emptyset$ for all n. Hence, $\mathrm{sp}(A) \cap \mathcal{K} \subset \mathcal{N}_\delta\left(\{z : \hat{\gamma}_n(z) \leq 1/n\} \cap G_n\right)$ as desired.

To see (iv), let $r > 0$ be so large that $\mathcal{N}_\delta(U_r(0)^c) \cap \mathcal{K} = \emptyset$. Note that $\mathrm{sp}_\epsilon(A) \to \mathrm{sp}(A)$ as $\epsilon \to 0$. Thus, $\mathrm{sp}_{\epsilon_1}(A) \cap B_r(0) \subset \mathcal{N}_{\delta/2}(\mathrm{sp}(A))$ for a sufficiently small ϵ_1. Also, by the second part of Proposition 9.16 it follows that $\mathrm{sp}_{\epsilon_1}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2}(\mathrm{sp}_{\epsilon_1}(A))$ for large n. However, by the choice of r we have that $\mathrm{sp}_{\epsilon_1}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2}(\mathrm{sp}_{\epsilon_1}(A) \cap B_r(0))$. Clearly, $\mathrm{sp}_{1/n}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathrm{sp}_{\epsilon_1}(\widetilde{A}_n) \cap \mathcal{K}$ for large n. Thus, by patching the above inclusions together, we get that

$$\{z : \hat{\gamma}_n(z) \leq 1/n\} \cap G_n \cap \mathcal{K} \subset \mathrm{sp}_{1/n}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathrm{sp}_{\epsilon_1}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2}(\mathrm{sp}_{\epsilon_1}(A) \cap B_r(0)) \subset \mathcal{N}_\delta(\mathrm{sp}(A)),$$

for large n, as desired. This finishes the proof of Proposition 9.17. □

Next, we pass from these general considerations to the Schrödinger case.

9.2.3. **Compactness of the resolvent.** We first show that the resolvent of the Schrödinger operator $H \in \Omega_\infty$ is compact. To prove this, we recall some well-known lemmas and definitions from [83].

Definition 9.18. An operator A on the Hilbert space $\mathcal{H}$ is *accretive* if the $\mathrm{Re}\langle Ax, x\rangle \geq 0$ for $x \in \mathcal{D}(A)$. It is called m-accretive if no proper accretive extension exists. If A (possibly after shifting with a scalar) is m-accretive and additionally there exists $\beta < \pi/2$ such that $|\arg\langle Ax, x\rangle| \leq \beta$ for all $x \in \mathcal{D}(A)$, then A is m-sectorial.

Lemma 9.19 ([83, VI-Theorem 3.3]). *Let A be m-sectorial with $B = \mathrm{Re}\, A$. A has compact resolvent if and only if B has.*

Lemma 9.20 ([83, V-Theorem 3.2]). *If T is closed and the complement of $\mathrm{Num}(T)$ is connected, then for every ζ in the complement of the closure of $\mathrm{Num}(T)$ the following hold: the kernel of $T - \zeta$ is trivial and the range of $T - \zeta$ is closed with constant codimension.*

Proposition 9.21. *Suppose V is continuous $\mathbb{R}^d \to \mathbb{C}$ satisfying the following: $V(x) = |V(x)|e^{i\varphi(x)}$ such that $|V(x)| \to \infty$ as $x \to \infty$, and there exist non-negative θ_1, θ_2 such that $\theta_1 + \theta_2 < \pi$ and $-\theta_2 \leq \varphi(x) \leq \theta_1$. Denote by h the operator $h = -\Delta + V$ with domain $\mathcal{D}(h) = C_c^\infty(\mathbb{R}^d)$ and put in $\mathrm{L}^2(\mathbb{R}^d)$ $H = h^{**}$. Then $H = -\Delta + V$ is a densely defined operator with compact resolvent, whose spectrum lies in the sector $\{z : \arg(z) \in [-\theta_2, \theta_1]\}$.*

Proof. The proof goes as follows: First, the numerical range of H lies in a sector with opening $2\beta < \pi$. Then, we turn the sector into the symmetric position around the positive real axis to get the operator $a(\alpha)$. It is clearly enough to show that $A(\alpha) = a(\alpha)^{**}$ is an m-sectorial operator with half-angle $\beta = (\theta_1 + \theta_2)/2$ which has a compact resolvent. Next, since the numerical range of $a(\alpha)$ is not the whole plane, the operator is closable. Then, we conclude that every point away from the numerical range belongs to the resolvent set. This is done based on the fact that the adjoint shares the same key properties as $A(\alpha)$. Then, the compactness of the resolvent follows by considering the resolvent of the real part of $A(\alpha)$.

Here is the notation. Put $\alpha = (\theta_1 - \theta_2)/2$ so that $|\alpha| < \pi/2$. Then with

$$\vartheta(x) = \varphi(x) - \alpha \tag{9.50}$$

we have $a(\alpha) := e^{-i\alpha}h = -e^{-i\alpha}\Delta + |V(x)|e^{i\vartheta(x)}$ and after extending $A(\alpha) = a(\alpha)^{**}$, in particular $H(\alpha) := \mathrm{Re}A(\alpha) = -\cos\alpha\,\Delta + \cos\vartheta(x)|V(x)|$.

We claim that the operator $A(\alpha) := e^{-i\alpha}H$ is m-sectorial with half-angle $\beta = (\theta_1+\theta_2)/2$. Indeed, it is immediate that the numerical range satisfies the following $\mathrm{Num}(a(\alpha)) \subset \{z = re^{i\theta} : |\theta| \le \beta,\ r \ge 0\}$, which is not the whole complex plane, and we can therefore (by [83, V-Theorem 3.4 on p. 268]) consider the extended closed operator $A(\alpha)$ instead. The next thing is to conclude that points away from this closed sector are in the resolvent set of $A(\alpha)$. Take any point $\zeta = re^{i\varphi}$ with $\beta < |\varphi| \le \pi$, $r > 0$. We need to conclude that $\zeta \notin \mathrm{sp}(A(\alpha))$. Since the complement of $\mathrm{Num}(A(\alpha))$ is connected, the following holds (by Lemma 9.20): the operator $A(\alpha) - \zeta$ has closed range with constant codimension. Thus, we need that the range is the whole space. Put for that purpose $T = A(\alpha) - \zeta$. Suppose there is $g \neq 0$ such that $g \in \mathrm{Ran}(T)^{\perp}$. Then for all $f \in \mathcal{D}(T)$ we have $\langle Tf, g\rangle = 0$ which means, as $\mathcal{D}(T)$ is dense, that $T^*g = 0$. However, that is not the case as $A(\alpha)^* - \overline{\zeta}$ is also closed whose complement of the numerical range is connected and hence does not have a non-trivial kernel.

The proof of Proposition 9.21 can now be completed by invoking Lemma 9.19 since it is well known ([101], Theorem XIII.67) that (since $\alpha < \pi/2$) the self-adjoint operator $H(\alpha)$ has compact resolvent when the potential $|V(x)|$ tends to infinity with x. □

We shall next consider the discretization of H and $A(\alpha)$. It shall be clear that the discrete versions have their numerical ranges inside the same sectors, where the numerical range of an operator T is denoted by $\mathrm{Num}(T)$. Thus, all resolvents can be estimated using the fact that if $(T-\zeta)^{-1}$ is regular outside the closure of $\mathrm{Num}(T)$, then $\|(T-\zeta)^{-1}\| \le 1/\mathrm{dist}(\zeta, \mathrm{Num}(T))$.

9.2.4. **Discretizing the Schrödinger operator.** We shall show how to assemble the matrices $\widetilde{H}_n$ mentioned above. The underlying Hilbert space is again $\mathrm{L}^2(\mathbb{R}^d)$ and we start with approximating the Laplacian. Let $1 \le j \le d$, $t \in \mathbb{R}$ and define $U_{j,t}$ to be the one-parameter unitary group of translations

$$U_{j,t}\psi(x_1,\dots,x_d) = \psi(x_1,\dots,x_j - t,\dots,x_d)$$

and let P_j be the infinitesimal generator of $U_{j,t}$ so that $U_{j,t} = e^{\mathrm{i}tP_j}$ and $P_j = \lim_{t\to 0}\frac{1}{\mathrm{i}t}(U_{j,t} - I)$. Thus, defining $\Phi_n(x) = \frac{n}{\mathrm{i}}(e^{\mathrm{i}\frac{1}{n}x} - 1)$ with $n \in \mathbb{N}$ and $x \in \mathbb{R}$, it follows that

$$|\Phi_n|^2(P_j)\psi(x) = n^2(-\psi(x_1,\dots,x_j + 1/n,\dots x_d) - \psi(x_1,\dots,x_j - 1/n,\dots x_d) + 2\psi(x)) \tag{9.51}$$

is the discretized Laplacian in the j direction. The full discretized Laplacian is therefore $\sum_{j=1}^{d}|\Phi_n|^2(P_j)$. Now, we replace V with an appropriate approximation. Consider the lattice $(\frac{1}{n}\mathbb{Z})^d$ as a subset of $\mathbb{R}^d$ and for $y \in (\frac{1}{n}\mathbb{Z})^d$ define the box

$$Q_n(y) = \left\{x = (x_1,\dots,x_d) : x_j \in \left[y_j - \frac{1}{2n}, y_j + \frac{1}{2n}\right),\ 1 \le j \le d\right\}. \tag{9.52}$$

Let $S_n = [-\lfloor\sqrt{n}\rfloor, \lfloor\sqrt{n}\rfloor]^d \subset \mathbb{R}^d$ and define E_n to be the orthogonal projection onto the subspace

$$\left\{\psi \in \mathrm{L}^2(\mathbb{R}^d) : \psi = \sum_{y\in(\frac{1}{n}\mathbb{Z})^d\cap S_n} \alpha_y\chi_{Q_n(y)},\ \alpha_y \in \mathbb{C}\right\}, \tag{9.53}$$

where $\chi_{Q_n(y)}$ denotes the characteristic function on $Q_n(y)$. Define the approximate potential as

$$V_n(x) = \begin{cases} V(y) & x \in Q_n(y)\cap S_n \text{ for some } y \in (\frac{1}{n}\mathbb{Z})^d, \\ 0 & \text{otherwise.}\end{cases}$$

Note that $V_n = E_nV_nE_n$, but that, generally, $V_n \neq E_nVE_n$. Finally, we define the approximate Schrödinger operator $H_n : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d)$ defined as

$$H_n = E_n\sum_{j=1}^{d}|\Phi_n|^2(P_j)E_n + V_n. \tag{9.54}$$

Remark 9.22. Note that the restriction $H_n|_{\mathrm{Ran}(E_n)}$ of H_n to the image of E_n has a matrix representation $\widetilde{H}_n \in \mathbb{C}^{m\times m}$ (where $m = \dim(\mathrm{Ran}(E_n))$) defined as follows. First, for $y_1, y_2 \in (\frac{1}{n}\mathbb{Z})^d \cap S_n$,

$$\langle |\Phi_n|^2(P_j) E_n n^{d/2}\chi_{Q_n(y_1)}, n^{d/2}\chi_{Q_n(y_2)}\rangle = \begin{cases} 2n^2 & y_1 = y_2 \\ -n^2 & y_1 - y_2 = \pm 1/n e_j \\ 0 & \text{otherwise} \end{cases}$$

and $\langle V_n n^{d/2}\chi_{Q_n(y_1)}, n^{d/2}\chi_{Q_n(y_2)}\rangle = V(y_1)$ when $y_1 = y_2$ and zero otherwise. Thus, we can form the matrix representation of $H_n|_{\mathrm{Ran}(E_n)}$ with respect to the orthonormal basis $\{n^{d/2}\chi_{Q_n(y)}\}_{y\in(\frac{1}{n}\mathbb{Z})^d\cap S_n}$. It is important to note that calculating the matrix elements of $\widetilde{H}_n$ requires knowledge only of $\{V_f\}_{f\in\Lambda_n}$ where we have $\Lambda_n := \{f_y : y \in (n^{-1}\mathbb{Z})^d \cap S_n\}$ and $V_{f_y} = f_y(V) = V(y)$.

9.2.5. **Proof that** $\{\Xi_{\mathrm{sp}}, \Omega_\infty\} \in \Delta_2^A, \{\Xi_{\mathrm{sp},\epsilon}, \Omega_\infty\} \in \Delta_2^A$**.** We have so far shown that Assumption (i) holds, and we are left to show that the discretization we have chosen satisfies Assumption (ii). In particular, we must demonstrate that our discretization satisfies (9.47). That is the topic of the following theorem.

Theorem 9.23. *Let $V \in C(\mathbb{R}^d)$ be sectorial as defined in* (5.3) *satisfying $|V(x)| \to \infty$ as $|x| \to \infty$, and let $h = -\Delta + V$ with $\mathcal{D}(h) = C_c^\infty(\mathbb{R}^d)$ and let $H = h^{**}$. Let H_n be as in* (9.54)*. Then there exists z_0 such that $\|(H - z_0)^{-1} - (H_n - z_0)^{-1}E_n\| \to 0$, as $n \to \infty$.*

Note that we immediately have

$$\text{Theorem 9.23 } + \text{ Proposition 9.17 } \Rightarrow \{\Xi_{\mathrm{sp}}, \Omega_\infty\} \in \Delta_2^A, \{\Xi_{\mathrm{sp},\epsilon}, \Omega_\infty\} \in \Delta_2^A.$$

Thus, the rest of the section is devoted to proving Theorem 9.23.

We shall treat the discretizations similarly to the continuous case, namely by "rotating" the operator into a symmetric position with respect to the real axis, and then, by taking the real part, we are dealing with a sequence of self-adjoint invertible operators. Before we prove this theorem, we will need a couple of lemmas. We recall the following definition.

Definition 9.24 (Collectively compact)**.** A set $\mathcal{T} \subset B(\mathcal{H})$ is called *collectively compact* if the set $\{Tx : T \in \mathcal{T}, \|x\| \leq 1\}$ has compact closure.

Lemma 9.25. *Let $\{K_n\}$ be a collectively compact operator sequence and $K_n^* \to 0$ strongly. Then $\|K_n\| \to 0$.*

Proof. It is well known that on any compact set $\mathcal{B}$ the strong convergence $K_n^* \to 0$ turns into norm convergence: $\sup\{\|K_n^*x\| : x \in \mathcal{B}\} \to_n 0$. Since $\mathcal{B} := \mathrm{cl}\{K_n x : \|x\| \leq 1, n \in \mathbb{N}\}$ is compact, we get

$$\|K_n\|^2 = \|K_n^* K_n\| = \sup\{\|K_n^* K_n x\| : \|x\| \leq 1\} \leq \sup\{\|K_n^* y\| : y \in \mathcal{B}\} \to 0 \quad \text{as} \quad n \to \infty.$$

□

We also need a modification of Lemma 9.19.

Lemma 9.26. *Let $\{A_n\}$ be m-sectorial with common semi-angle $\beta < \pi/2$ and denote $B_n = \mathrm{Re}\, A_n$. Assume that $\{E_n\}$ is a sequence of orthogonal projections, converging strongly to identity and such that $A_n E_n = E_n A_n E_n$ and $B_n E_n = E_n B_n E_n$. Assume further that $\{B_n^{-1}\}$ is uniformly bounded. If $\{B_n^{-1}E_n\}$ is collectively compact, then so is $\{A_n^{-1}E_n\}$.*

Proof. Denote by $B_n^{1/2}$ the unique self-adjoint non-negative square root of B_n. By [83, VI-Theorem 3.2 on p.337] for each A_n there exists a bounded symmetric operator C_n satisfying $\|C_n\| \leq \tan(\beta)$ and such that $A_n = B_n^{1/2}(1 + \mathrm{i}\, C_n)B_n^{1/2}$. Writing

$$A_n^{-1} = \int_0^\infty e^{-tA_n}\, dt$$

we conclude that $E_nA_n^{-1}E_n = A_n^{-1}E_n$ and likewise for B_n^{-1}. Assume now that $\{B_n^{-1}E_n\}$ is collectively compact. But then so is $\{(B_n + t)^{-1}E_n\} = \{B_n^{-1}E_n(I + tB_n^{-1})^{-1}E_n\}$ and writing, compare [83, V (3.43) on p.282],

$$B_n^{-1/2}E_n = \frac{1}{\pi}\int_0^\infty t^{-1/2}(B_n + t)^{-1}E_n dt$$

we see that $\{B_n^{-1/2}E_n\}$ is also collectively compact and $B_n^{-1/2}E_n = E_nB_n^{-1/2}E_n$. Finally $\{A_n^{-1}E_n\}$ is then collectively compact as well since $A_n^{-1}E_n$ is of the form $B_n^{-1/2}E_nT_n$ with T_n uniformly bounded. □

Proof of Theorem 9.23. Note that it is clear from the definition of H_n and the assumption on V that $\mathrm{Num}(H_n) \subset \{re^{i\rho} : -\theta_2 \leq \rho \leq \theta_1, r \geq 0\}$ for all n. Thus, since H_n is bounded and by Proposition 9.21 we can choose any point $z_0 \in \mathbb{C}$ such that z_0 has a positive distance d to the closed sector $\{re^{i\rho} : -\theta_2 \leq \rho \leq \theta_1, r \geq 0\}$, and both $R(H, z_0) = (H-z_0)^{-1}$ and $R(H_n, z_0) = (H_n-z_0)^{-1}$ for every n will exist. Moreover, $R(H_n, z_0)$ are uniformly bounded for all n, since for every x, $\|x\| = 1$,

$$\|(H_n - z_0)x\| \geq |\langle (H_n - z_0)x, x\rangle| \geq |\langle H_nx, x\rangle - z_0| \geq d.$$

Note that by Lemma 9.25 it suffices to show that (i) $R(H_n, z_0)^*E_n \to R(H, z_0)^*$ strongly, and (ii) $\{R(H_n, z_0)E_n - R(H, z_0)\}$ is collectively compact, which follows if we can show that $\{R(H_n, z_0)E_n\}$ is collectively compact.

To see (i) observe that $C_c^\infty(\mathbb{R}^d)$ is a common core for H and for H_n. Hence by [83, VIII-Theorem 1.5 on p.429], the strong resolvent convergence $R(H_n, z_0)^* \to R(H, z_0)^*$ will follow if we show that $H_n^*\psi \to H^*\psi$ as $n \to \infty$ for any $\psi \in C_c^\infty(\mathbb{R}^d)$. Then the strong convergence $R(H_n, z_0)^*E_n \to R(H, z_0)^*$ follows as well. Note that

$$\|H_n^*\psi - H^*\psi\| \leq \left\| \sum_{j=1}^d |\Phi_n|^2(P_j)E_n\psi - \sum_{j=1}^d P_j^2\psi \right\| + \|(\overline{V}_n - \overline{V})\psi\|. \tag{9.55}$$

Also, $|\Phi_n|^2(P_j) = n(\tau_{-1/ne_j} - I)n(\tau_{1/ne_j} - I)$, where $\tau_z\psi(x) = \psi(x - z)$ and $\{e_j\}$ is the canonical basis for $\mathbb{R}^d$. Moreover, for $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$E_n\psi = \sum_{y\in(\frac{1}{n}\mathbb{Z})^d\cap S_n} (\Psi_n * \psi)(y)\chi_{Q_n(y)}, \quad \Psi_n = \rho_n \otimes \ldots \otimes \rho_n, \quad \rho_n = n\chi_{[-\frac{1}{2n},\frac{1}{2n})},$$

where S_n was defined in (9.53). Thus, it follows from easy calculus manipulations and basic properties of convolution that $|\Phi_n|^2(P_j)E_n\psi = \sum_{y\in(\frac{1}{n}\mathbb{Z})^d}(\Psi_n * \tilde{\rho}_1 *_j \tilde{\rho}_2 *_j \psi'')(y)\chi_{Q_n(y)}$, where $\tilde{\rho}_1 = n\chi_{[-1/n,0]}$, $\tilde{\rho}_2 = n\chi_{[0,1/n]}$ and $*_j$ denotes the convolution operation in the jth variable. By standard properties of the convolution we have that $\Psi_n * \tilde{\rho}_1 *_j \tilde{\rho}_2 *_j \psi'' \to \psi''$ uniformly as $n \to \infty$. Thus, since $\psi \in C_c^\infty(\mathbb{R}^d)$, the first part of the right-hand side of (9.55) tends to zero as $n \to \infty$. Due to the continuity of V and the bounded support of ψ it also follows easily that $\|(\overline{V}_n - \overline{V})\psi\| \to 0$ as $n \to \infty$.

To see (ii), we use the same trick as in the proof of Proposition 9.21. In particular, first set $z_0 = -e^{i\alpha}$ (which is clearly in the resolvent set of H_n for $\alpha = (\theta_1 - \theta_2)/2$) then let $A_n(\alpha) = e^{-i\alpha}(H_n - z_0)$ and further $H_n(\alpha) = \mathrm{Re}\ A_n(\alpha)$. Note that, by Lemma 9.26, we would be done if we could show that $\{H_n(\alpha)^{-1}\}$ is uniformly bounded and $\{H_n(\alpha)^{-1}E_n\}$ is collectively compact as that would yield collective compactness of $\{A_n(\alpha)^{-1}E_n\}$ and hence of $\{R(H_n, z_0)E_n\}$. To establish the uniform bound, note that

$$H_n(\alpha) = \cos\alpha\ E_n \sum_{j=1}^d |\Phi_n|^2(P_j)E_n + \cos\vartheta(x)|V_n(x)| + 1, \tag{9.56}$$

where ϑ is defined in (9.50). Thus $\|H_n(\alpha)^{-1}\| \leq 1$ and by applying Lemma 9.27 we are now done. □

Lemma 9.27. *Let $H_n(\alpha)$ be given by (9.56). Then the set $\{H_n(\alpha)^{-1}E_n\}$ is collectively compact.*

Proof. We shall show that if we choose an arbitrary sequence $\{\psi_n\} \subset \mathrm{L}^2(\mathbb{R}^d)$ satisfying $\|\psi_n\| \le 1$, then the sequence $\{\varphi_n\}$ where $\varphi_n = H_n(\alpha)^{-1}E_n\psi_n$, is relatively compact in $\mathrm{L}^2(\mathbb{R}^d)$. The compactness argument is based on Rellich's criterion.

Lemma 9.28 (Rellich's criterion ([101] Theorem XIII.65))**.** *Let $F(x)$ and $G(\omega)$ be two measurable non-negative functions becoming larger than any constant for all large enough $|x|$ and $|\omega|$. Then*

$$S = \{\varphi \ : \ \int |\varphi(x)|^2 dx \le 1, \ \int F(x)|\varphi(x)|^2 dx \le 1, \ \int G(\omega)|\mathcal{F}\varphi(\omega)|^2 d\omega \le 1\}$$

is a compact subset of $\mathrm{L}^2(\mathbb{R}^d)$.

To prove Lemma 9.27, we proceed as follows. First, we conclude that $\{\varphi_n\}$ is a bounded sequence itself. Then, to be able to define suitable functions F, G, we need to approximate the sequence by another one of the form $\Psi_n * \varphi_n$. This approximation shall satisfy $\lim_{n\to\infty} \|\Psi_n * \varphi_n - \varphi_n\| = 0$, and this is very similar to the standard result on local uniform convergence of mollifications of continuous functions. Then the Rellich's criterion holds for $\Psi_n * \varphi_n$ with $F(x)$ essentially given by $|V(x)|$ and $G(\omega)$ by $|\omega|^2$. We conclude that the sequence $\{\Psi_n * \varphi_n\}$ is relatively compact. But since $\lim_{n\to\infty} \|\Psi_n * \varphi_n - \varphi_n\| = 0$, the sequence $\{\varphi_n\}$ is relatively compact as well, completing the argument.

More precisely, since $|\vartheta(x)| \le \alpha < \pi/2$ we have from (9.56)

$$|\langle H_n(\alpha)\varphi_n, \varphi_n\rangle| \ge \cos\alpha \left(\left\langle \sum_{j=1}^d |\Phi_n|^2(P_j)\varphi_n, \varphi_n \right\rangle + \langle |V_n|\varphi_n, \varphi_n\rangle \right) + \|\varphi_n\|^2. \tag{9.57}$$

However, $|\langle H_n(\alpha)\varphi_n, \varphi_n\rangle|$ is bounded not only from below but also from above. Indeed, $|\langle H_n(\alpha)\varphi_n, \varphi_n\rangle| = |\langle E_n\psi_n, \varphi_n\rangle| \le \|H_n(\alpha)^{-1}E_n\|\|\psi_n\|^2$. Thus, we conclude first from (9.57) that the sequence $\{\varphi_n\}$ is bounded. Next, given (9.57), there exist constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$

$$\left\langle \sum_{j=1}^d |\Phi_n|^2(P_j)\varphi_n, \varphi_n \right\rangle \le C_1, \qquad \langle |V_n|\varphi_n, \varphi_n\rangle \le C_2. \tag{9.58}$$

First, we use the bound in the first part of (9.58). Letting $\mathcal{F}$ denote the Fourier transform, we have that $(\mathcal{F}\Phi_n(P_j)\varphi_n)(\omega) = \Phi_n(\omega_j)(\mathcal{F}\varphi_n)(\omega)$, for a.e. ω and for $1 \le j \le d$. Letting $\Theta_n(\omega) = \frac{\sin(\omega/2n)}{\omega/2n}$, an application of the Fourier transform to (9.58) along with Plancherel's theorem yield

$$\int_{\mathbb{R}^d} |(\mathcal{F}\varphi_n)(\omega)|^2 \sum_{1\le j\le d} |\omega_j \Theta_n(\omega_j)|^2 \, d\omega \le C_1.$$

Moreover, since $|\Theta_n(\omega)| \le 1$ for all ω, we get

$$\int_{\mathbb{R}^d} |\omega|^2 |\Theta_n(\omega_1)\cdots\Theta_n(\omega_d)|^2 |(\mathcal{F}\varphi_n)(\omega)|^2 \, d\omega \le C_1. \tag{9.59}$$

We now define the approximation $\Psi_n * \varphi_n$. Let $\Psi_1(z) = \chi_{[-1/2,1/2]^d}(z)$ and further $\Psi_n(z) = n^d\Psi_1(nz)$, where $\chi_A(z)$ is the usual characteristic function for the set A. We shall prove below that $\lim_{n\to\infty} \|\Psi_n * \varphi_n - \varphi_n\| = 0$, which in particular shows that the sequence $\{\Psi_n * \varphi_n\}$ is bounded. Observe then that $(\mathcal{F}\Psi_n)(\omega) = \Theta_n(\omega_1)\cdots\Theta_n(\omega_d)$. Therefore we obtain from (9.59) $\int_{\mathbb{R}^d} |\omega|^2 |\mathcal{F}(\Psi_n * \varphi_n)(\omega)|^2 d\omega \le C_1$, which shows that we can choose $G(\omega)$ to be (a constant times) $|\omega|^2$.

We must still establish the growth function $F(x)$ for $\Psi_n * \varphi_n$. Consider φ_n. It is of the form $\varphi_n = (E_n + E_nB_nE_n)^{-1}E_n\psi_n$ and hence $E_n\varphi_n = \varphi_n$. Therefore φ_n vanishes outside S_n, and we can essentially replace V_n by V in the inequality in the last part of (9.58). To that end, put $F(x) = \min_{|y|\ge|x|} |V(y)|$. Then, with some constant C_3

$$\int_{\mathbb{R}^d} F(x)|(\Psi_n * \varphi_n)(x)|^2 dx \le C_3. \tag{9.60}$$

In view of the bounds (9.59), (9.60) and since the sequence $\{\Psi_n * \varphi_n\}_{n\in\mathbb{N}}$ is bounded in L^2, Rellich's criterion implies that $\{\Psi_n * \varphi_n\}_{n\in\mathbb{N}}$ is a relatively compact sequence and it therefore follows that

$\{\varphi_n\}_{n\in\mathbb{N}}$ is relatively compact, thus finishing the proof. Hence, our only remaining obligation is to show that $\lim_{n\to\infty}\|\Psi_n * \varphi_n - \varphi_n\| = 0$. This result is very similar to the standard result on local uniform convergence of mollifications of continuous functions.

Let $z \in \mathbb{R}^d$ and define the shift operator τ_z on $\mathrm{L}^2(\mathbb{R}^d)$ by $\tau_z f(x) = f(x-z)$. Now observe that by Minkowski's inequality for integrals, it follows that

$$\|\Psi_n * \varphi_n - \varphi_n\| \leq \int_{\mathbb{R}^d} \|\tau_{\frac{1}{n}z}\varphi_n - \varphi_n\| |\Psi_1(z)|\, dz = \int_{[-1/2,1/2]^d} \|e^{i\frac{z_d}{n}P_d}\dots e^{i\frac{z_1}{n}P_1}\varphi_n - \varphi_n\|\, dz. \tag{9.61}$$

The claim follows from an ϵ/d argument and (9.61) combined with the dominated convergence theorem (recall that $\{\varphi_n\}$ is bounded): we need to show that for fixed $z \in [-1/2,1/2]^d$ and for any $1 < j \leq d$,

$$\lim_{n\to\infty}\left\|e^{i\frac{z_j}{n}P_j}\dots e^{i\frac{z_1}{n}P_1}\varphi_n - e^{i\frac{z_{j-1}}{n}P_{j-1}}\dots e^{i\frac{z_1}{n}P_1}\varphi_n\right\| = 0, \quad \lim_{n\to\infty}\left\|e^{i\frac{z_1}{n}P_1}\varphi_n - \varphi_n\right\| = 0. \tag{9.62}$$

Since $e^{i\frac{z_j}{n}P_j}e^{i\frac{z_k}{n}P_k} = e^{i\frac{z_k}{n}P_k}e^{i\frac{z_j}{n}P_j}$ and $\|e^{i\frac{z_j}{n}P_j}\cdots e^{i\frac{z_1}{n}P_1}\| \leq 1$ for $1 \leq j,k \leq d$, (9.62) will follow if we can show that $\|(e^{i\frac{z_j}{n}P_j} - I)\varphi_n\| \to 0$ as $n \to \infty$. Note that, by the choice of the projections E_n, it follows that for $1 \leq j \leq d$, $|((e^{i\frac{z_j}{n}P_j} - I)\varphi_n)(x)| \leq |((e^{i\frac{1}{n}P_j} - I)\varphi_n)(x)|$, for $0 \leq z_j \leq 1/2$ and $x \in \mathbb{R}^d$. Also,

$$|((e^{i\frac{z_j}{n}P_j} - I)\varphi_n)(x)| \leq |((e^{-i\frac{1}{n}P_j} - I)\varphi_n)(x)|, \qquad -1/2 \leq z_j < 0.$$

However the bound $\sum_{1\leq j\leq d}\|\Phi_n(P_j)\varphi_n\|^2 \leq C_1$ implies that $\lim_{n\to\infty}\|(e^{\pm i\frac{1}{n}P_j} - I)\varphi_n)\| = 0$, which proves the claim. ☐

9.2.6. **Proof that neither problem lies in $\Sigma_1^G \cup \Pi_1^G$.** Finally, we shall complete the proof of Theorem 5.5 by showing that $\{\Xi_{\mathrm{sp}}, \Omega_\infty\} \notin \Sigma_1^G \cup \Pi_1^G$ and $\{\Xi_{\mathrm{sp},\epsilon}, \Omega_\infty\} \notin \Sigma_1^G \cup \Pi_1^G$.

Proof. **Step I:** $\{\Xi_{\mathrm{sp}}, \Omega_\infty\} \notin \Sigma_1^G$. Suppose for a contradiction that there exists a Σ_1^G tower Γ_n which solves the computational problem $\{\Xi_{\mathrm{sp}}, \Omega_\infty\}$. Now let V be any (real-valued) positive potential in the class Ω_∞ such that the corresponding Schrödinger operator is self-adjoint and has a unique ground state (the operator must be bounded below). Call the associated operator H_0. For instance, in one dimension, this could be the quantum harmonic oscillator $V(x) = x^2$, and examples in arbitrary dimension (the harmonic oscillator in $d > 1$ dimensions does not have a unique ground state) are well known in the physics literature. In this case, let ϕ_0 be the normalized ground state and E be the orthogonal complement of the span of this function intersected with the domain of H_0. Assume that $H_0\phi_0 = c\phi_0$. Denoting the standard $\mathrm{L}^2(\mathbb{R}^d)$ inner product by $\langle\cdot,\cdot\rangle$, it follows that there exists some $\eta > 0$ such that

$$\langle H_0\phi, \phi\rangle \geq (c+\eta)\|\phi\|^2, \quad \forall \phi \in E.$$

There exists n such that there is a point $z_n \in \Gamma_n(V)$ with $|z_n - c| \leq \eta/20$ and such that $\Gamma_n(V)$ guarantees there is a point in the spectrum $\Xi_{\mathrm{sp}}(V)$ of distance at most $\eta/20$ to z_n. Hence $\Gamma_n(V)$ guarantees there is a point in the spectrum $\Xi_{\mathrm{sp}}(V)$ of of distance at most $\eta/10$ from c. There also exists a finite set $S = \{x^1, ..., x^{M(n)}\}$ such that the computation of $\Gamma_n(V)$ only depends on the potential V evaluated at points in S. Let V_m be a sequence of real-valued continuous potentials such that $0 \leq V_m(x) \leq 1$, $V_m(x^j) = 0$ $\forall x^j \in S$ and such that V_m converges pointwise almost everywhere to 1 as $m \to \infty$. By construction and the definition of a general algorithm (Definition 7.3) we must have for all $a \in \mathbb{R}_+$ that $\Gamma_n(V + aV_m) = \Gamma_n(V)$. In particular, this includes the guarantee of a point in the spectrum $\Xi_{\mathrm{sp}}(V + aV_m)$ of distance at most $\eta/10$ from c. We will show that this gives rise to a contradiction for a choice of $a \in \mathbb{R}_+$ and m.

Indeed, choose m large such that $\langle V_m\phi_0, \phi_0\rangle \geq \frac{10}{11}$, and set $a = \eta/2$. It is well known that the minimum of the spectrum $\Xi_{\mathrm{sp}}(V + aV_m)$ is given by

$$\inf_{\phi\in\mathcal{D}(H_0):\|\phi\|=1}\langle(H_0 + aV_m)\phi, \phi\rangle.$$

In particular, $H_0 + aV_m$ and H_0 have the same domain as V_m is bounded. Now let $\phi \in \mathcal{D}(H_0)$ of norm 1. Without loss of generality by a change of phase, we can write $\phi = \delta\phi_0 + \sqrt{1-\delta^2}\phi_1$, with $\phi_1 \in E$ of unit norm and $\delta \in [0,1]$. Using the fact that $H_0\phi_0 = c\phi_0$ and H_0 is self-adjoint and $\langle \phi_0, \phi_1\rangle = 0$, we have that

$$\begin{aligned}\langle (H_0 + aV_m)\phi, \phi\rangle &= \delta^2 c + (1-\delta^2)\langle H_0\phi_1, \phi_1\rangle + \delta^2 a\langle V_m\phi_0, \phi_0\rangle \\ &\quad + a(1-\delta^2)\langle V_m\phi_1, \phi_1\rangle + 2\mathrm{Re}(a\delta\sqrt{1-\delta^2}\langle V_m\phi_0, \phi_1\rangle) \\ &\geq c + (1-\delta^2)\eta + \frac{10}{11}\delta^2 a - 2a\delta\sqrt{1-\delta^2},\end{aligned}$$

where we have used that V_m is positive to throw away the $\langle V_m\phi_1, \phi_1\rangle$ term. It follows that the minimum of the spectrum of $H_0 + aV_m$ is at least

$$c + \inf_{\delta\in[0,1]} \eta(1 - (1-5/11)\delta^2 - \delta\sqrt{1-\delta^2}) > c + \frac{\eta}{10},$$

yielding the required contradiction.

Step II: $\{\Xi_{\mathrm{sp}}, \Omega_\infty\} \notin \Pi_1^G$. We argue as in Step I, but now the proof is less involved. Suppose for a contradiction that there exists a Π_1^G tower Γ_n which solves the computational problem $\{\Xi_{\mathrm{sp}}, \Omega_\infty\}$. We let H_0, V, ϕ_0, and E be as in Step I, where we also assume as before that $H_0\phi_0 = c\phi_0$. We also assume that $c \geq 0$ and $V(x) \geq 1$.

Arguing as before, there exists some n such that $\Gamma_n(V)$ guarantees that the spectrum is disjoint from the interval $[c-3/2, c-1/2]$. Again, there exists a finite set $S = \{x^1, ..., x^{M(n)}\}$ such that the computation of $\Gamma_n(V)$ only depends on the potential V evaluated at points in S. Let V_m be a sequence of real-valued continuous potentials such that $-1 \leq V_m(x) \leq 0$, $V_m(x^j) = 0\ \forall x^j \in S$ but now such that V_m converges pointwise almost everywhere to -1 as $m \to \infty$. We must have $V + V_m \in \Omega_\infty$ since we assume the pointwise inequality $V(x) \geq 1$. By construction and the definition of a general algorithm (Definition 7.3), we must have that $\Gamma_n(V + V_m) = \Gamma_n(V)$. In particular, this includes the guarantee that the spectrum of $H_0 + V_m$ is disjoint from the interval $[c-3/2, c-1/2]$. But we have that

$$\langle (H_0 + V_m - (c-1))\phi_0, \phi_0\rangle = \langle V_m\phi_0, \phi_0\rangle + 1 \to 0,$$

as $m \to \infty$. It follows for some large m that $\|R(c-1, H_0 + V_m)\|^{-1} \leq 1/4$ and hence that the spectrum of $H_0 + V_m$ intersects the interval $[c-3/2, c-1/2]$, since the operator is self-adjoint. But this contradicts the Π_1^G guarantee.

Step III: $\{\Xi_{\mathrm{sp},\epsilon}, \Omega_\infty\} \notin \Pi_1^G \cup \Sigma_1^G$. The arguments are the same as in Steps I and II. The pseudospectrum is simply the ϵ ball neighborhood of the spectrum in these self-adjoint cases. The arguments work once we scale the operators by N/ϵ for some large N to gain the relevant separations. □

10. Smale's problem on roots of polynomials and Doyle-McMullen towers

In this section, we recall the definition of a tower of algorithms from [45]. We will name this type of tower a Doyle–McMullen tower and demonstrate how the results in [94] and [45] can be put into the framework of the SCI. In particular, we will demonstrate how the construction of the Doyle–McMullen tower in [45] can be viewed as a tower of algorithms defined in Definition 7.5. Note that one can compute zeros of a polynomial if one allows arithmetic operations and radicals and can pass to a limit. However, what if one cannot use radicals but rather iterations of a rational map? A natural choice for such a rational map would be Newton's method. The only problem is that the iteration may not converge, which motivated the question by Smale quoted in the introduction.

As we now know from [94], the answer is no. However, the results in [45] show that the quartic and the quintic can be solved with several rational maps and limits, while this is not the case for higher degree polynomials. Below, we first quote their results and then specify a particular tower of height three in the form that it can be viewed as a tower of algorithms in the sense of this paper.

10.1. **Doyle–McMullen towers.** A *purely iterative algorithm* [109] is a rational map [2]

$$T : \mathbb{P}_d \to \mathrm{Rat}_m,\ p \mapsto T_p$$

which sends any polynomial p of degree $\leq d$ to a rational function T_p of a certain degree m. An important example of a purely iterative algorithm is *Newton's method*. Furthermore, Doyle and McMullen call a purely iterative algorithm *generally convergent* if

$$\lim_{n\to\infty} T_p^n(z) \text{ exists for } (p, z) \text{ in an open dense subset of } \mathbb{P}_d \times \hat{\mathbb{C}}.$$

Here $T_p^n(z)$ denotes the nth iterate $T_p^n(z) = T_p(T_p^{n-1}(z))$ of T_p. For instance, Newton's method is generally convergent *only* when $d = 2$. However, given a cubic polynomial $p \in \mathbb{P}_3$ one can define an appropriate rational function $q \in \mathrm{Rat}_3$ whose roots coincide with the roots of p, and for which Newton's method *is* generally convergent (see [94], Proposition 1.2). In [45], the authors provide a definition of a tower of algorithms, which we quote verbatim:

Definition 10.1 (Doyle–McMullen tower)**.** A tower of algorithms is a finite sequence of generally convergent algorithms, linked together serially, so the output of one or more can be used to compute the input to the next. The final output of the tower is a single number, computed rationally from the original input and the outputs of the intermediate generally convergent algorithms.

Theorem 10.2 (McMullen [94]; Doyle and McMullen [45])**.** *For $\mathbb{P}_d$ there exists a generally convergent algorithm only for $d \leq 3$. Towers of algorithms exist additionally for $d = 4$ and $d = 5$ but not for $d \geq 6$.*

Note that, as shown in [107], there are generally convergent algorithms if, in addition, one allows the operation of complex conjugation. In the following, we present how the Doyle–McMullen towers can be recast as a general tower as defined in Definition 7.5.

10.2. **A height three tower for the quartic.** In the following $X, Y, \ldots$ denote variables in the polynomials while $x, y, \dots \in \mathbb{C}$. We build the tower following the standard reduction path, see, e.g., [43]. Given

$$p(X) := X^4 + a_1X^3 + a_2X^2 + a_3X + a_4$$

one first transforms the equation by a change of variable $Y = X + a_1/4$ to arrive at the polynomial

$$q(Y) := Y^4 + b_2Y^2 + b_3Y + b_4,$$

which one writes, with the help of a parameter z, as $q(Y) = (Y^2 + z)^2 - r(Y, z)$ where

$$r(Y, z) = (2z - b_2)Y^2 - b_3Y + z^2 - b_4.$$

Here one wants a value of z such that $r(Y, z)$ becomes a square which requires the discriminant to vanish: $4(2z - b_2)(z^2 - b_4) - b_3^2 = 0$. Viewing this as a polynomial in Z, making a change of variable $W = Z + (1/6)b_2$ and scaling the polynomial to monic we arrive at asking for a root of

$$s(W) := W^3 + c_2W + c_3. \tag{10.1}$$

As all these are rational computations on the coefficients of p, we shall not express them explicitly.

We denote by $N(f, \xi_0)$ the function in Newton's iteration with initial value ξ_0:

$$\xi_{j+1} := N(f(\xi_j)) \text{ where } N(f(\xi)) = \xi - \frac{f(\xi)}{f'(\xi)}$$

and further by N_j the mapping from initial data to the j^{th} iterate $N_j : (f, \xi_0) \mapsto \xi_j$. We shall apply Newton's iteration to the rational function [45]

$$t(W) := \frac{s(W)}{3c_2W^2 + 9c_3W - c_2^2}.$$

[2] I.e., it is a rational map of the coefficients of p.

Thus $w_j = N_j(t, w_0)$ denotes the j^{th} iterate w_j for a zero for $s(w) = 0$. This iteration converges in an open, dense set of initial data. Denote $w_\infty := \lim_{j\to\infty} w_j$. Now we change the variable $Z = W - (1/6)b_2$ and, denoting by z_j and z_∞ the corresponding values, we obtain $r(Y, z_\infty)$ as a square:

$$r(Y, z_\infty) = (2z_\infty - b_2)\left(Y - \frac{b_3}{2(2z_\infty - b_2)}\right)^2.$$

To find a zero of $q(Y)$, we shall need to have a generally convergent iteration for $\sqrt{2z - b_2}$. Thus, we set $u_j(V) := V^2 + b_2 - 2z_j$ and apply Newton's method for this, starting with initial guess v_0 and iterating k times and set $v_{k,j} := N_k(u_j, v_0)$. From $q(Y) = (Y^2 + z_\infty)^2 - r(Y, z_\infty) = 0$, we move to solve one of the factors

$$Q(Y) = Y^2 + z_\infty - \sqrt{2z_\infty - b_2}\left(Y - \frac{b_3}{2(2z_\infty - b_2)}\right) = 0.$$

However, we can do this only based on approximative values for the parameters, so we set

$$Q_{k,j}(Y) = Y^2 + z_j - v_{k,j}\left(Y - \frac{b_3}{2(2z_j - b_2)}\right) = 0.$$

Now apply Newton's iteration to this, say n times, using starting value y_0 and denote the output by $y_{n,k,j}$:

$$y_{n,k,j} = N_n(Q_{k,j}, y_0).$$

Finally, we set $x_{n,k,j} = y_{n,k,j} - a_1/4$ in order to get an approximation to a root of p. Suppose now $j = n_1, k = n_2, n = n_3$. If $n_1 \to \infty$ then $w_{n_1} \to w_\infty$ and hence $z_{n_1} \to z_\infty$, too. It is natural to denote $u(V) := V^2 + b_2 - 2z_\infty$ and correspondingly $v_{n_2} := N_{n_2}(u, v_0)$ and

$$Q_{n_2}(Y) = Y^2 + z_\infty - v_{n_2}\left(Y - \frac{b_3}{2(2z_\infty - b_2)}\right) = 0.$$

Then, in an obvious manner $x_{n_3,n_2} = N_{n_3}(Q_{n_2}, y_0) - a_1/4$. Then we have $\lim_{n_1\to\infty} x_{n_3,n_2,n_1} = x_{n_3,n_2}$. If we denote $x_{n_3} = N_{n_3}(Q, y_0) - a_1/4$, then clearly $\lim_{n_2\to\infty} x_{n_3,n_2} = x_{n_3}$. Finally $x_\infty = \lim_{n_3\to\infty} x_{n_3}$ is a root of p.

The link to the SCI. One unique feature of these towers, which are built on generally convergent algorithms, is the following: in addition to the polynomial p, the initial values for the iterations have to be read into the process via evaluation functions. Denoting the initial values for the three different Newton's iterations by $d_0 = (w_0, v_0, y_0) \in \mathbb{C}^3$ we can now put this Doyle–McMullen tower in the form of a general tower as defined in Definition 7.5, with the slight weakening that, for each $p \in \mathbb{P}_4$, the tower might converge only at a dense subset of initial values. In particular, set

$$\Gamma_{n_3} : \mathbb{P}_4 \times \mathbb{C}^3 \to \mathbb{C}, \text{ by } (p, d_0) \mapsto x_{n_3},$$
$$\Gamma_{n_3,n_2} : \mathbb{P}_4 \times \mathbb{C}^3 \to \mathbb{C} \text{ by } (p, d_0) \mapsto x_{n_3,n_2},$$
$$\Gamma_{n_3,n_2,n_1} : \mathbb{P}_4 \times \mathbb{C}^3 \to \mathbb{C} \text{ by } (p, d_0) \mapsto x_{n_3,n_2,n_1}.$$

Thus, if we let $\Omega = \mathbb{P}_4 \times \mathbb{C}^3$ and $\Xi, \mathcal{M}$ be as in Example 7.1 (III), and complement Λ by the mappings that read w_0, v_0, y_0 from the input, then by the construction above and Theorem 10.2 we have that

$$\mathrm{SCI}(\Xi, \Omega)_{\mathrm{DM}} \in \{2, 3\}.$$

10.3. **A height three tower for the quintic.** Let

$$p(X) = X^5 + a_1X^4 + a_2X^3 + a_3X^2 + a_4X + a_5$$

be the given quintic. Doyle and McMullen [45] give a generally convergent algorithm for the quintic in Brioschi form. Thus, one needs first to bring the general quintic to Brioschi form, then apply the iteration, and finally construct at least one root for $p(X)$. In the following, we outline a path for doing this, which follows L. Kiepert [85] except that the Brioschi quintic is solved by Doyle–McMullen iteration rather than by using Jacobi sextic. This path can be found in [86].

One begins applying a Tschirnhaus transformation $Y = X^2 - uX + v$ to arrive into *principal* form

$$q(Y) = Y^5 + b_3Y^2 + b_4Y + b_5.$$

Here v is obtained from a linear equation, but to solve u, one needs to solve a quadratic equation $Q(U) = U^2 + \alpha U + \beta$, where the coefficients α, β are rational expressions of the coefficients of $p(X)$ (see, for example, p. 100, eq. (6.2-9) in [86]).

Here is the first application of Newton's method. We are given an initial value u_0 and iterate j times $u_j = N_j(Q, u_0)$. We may assume that v is known exactly, but we only have an approximation u_j to make the transformation. So, suppose the Newton iteration converges to u_∞. Thus, we make the transformation using u_j and *force* the coefficients $b_{2,j} = b_{1,j} = 0$ while keeping the others as they appear. The transformation being continuous yields polynomials

$$q_j(Y) = Y^5 + b_{3,j}Y^2 + b_{4,j}Y + b_{5,j},$$

whose roots shall converge to those of $q(Y)$. The next step is transforming $q_j(Y)$ into Brioschi form. Let the Brioschi form corresponding to the exact polynomial $q(Y)$ be denoted by $B(Z)$

$$B(Z) = Z^5 - 10CZ^3 + 45C^2Z - C^2 = 0, \tag{10.2}$$

while with $B_j(Z)$ we denote the exact Brioschi form corresponding to $q_j(Y)$. The transformation from $q(Y)$ to $B(Z)$ is of the form

$$Y = \frac{\lambda + \mu Z}{(Z^2/C) - 3}. \tag{10.3}$$

Here λ satisfies a quadratic equation with coefficients being polynomials of the coefficients in the principal form (p. 107, eq. (6.3-28) in [86]). Let us denote that quadratic by $R(L)$ when it comes from $q(Y)$ and by $R_j(L)$ when it comes from $q_j(Y)$, respectively. Thus, here we meet our second application of Newton's method. So, we denote by

$$\lambda_{k,j} := N_k(R_j, \lambda_0)$$

the output of iterating k times for a solution of $R_j(L) = 0$. And, in a natural manner, we denote also

$$\lambda_k = N_k(R, \lambda_0) \quad \text{and} \quad \lambda = \lim_{k\to\infty} N_k(R, \lambda_0).$$

The corresponding values of $\mu_{k,j}, \mu_k$, and μ are then obtained by simple substitution (p. 107, eq. (6.3-30) in [86]). The Tschirnhaus transformation with exact values (λ, μ) transforms the equation not yet to the Brioschi form with just one parameter C but such that the constant term may be different. However, the last step is just a simple scaling, and then one is in the Brioschi form (10.2). However, when we apply the transformation with the approximated values $(\lambda_{k,j}, \mu_{k,j})$ or with (λ_k, μ_k) we do not arrive at the Brioschi form. So, we *force* the coefficients of the fourth and second powers to vanish and replace the coefficients of the first power to match the coefficients in the third power. Finally, after scaling the constant terms, we have the Brioschi quintics $B_{k,j}$ and B_k, e.g.

$$B_{k,j}(Z) = Z^5 - 10C_{k,j}Z^3 + 45C_{k,j}^2Z - C_{k,j}^2 = 0. \tag{10.4}$$

Provided that the Newton iterations converge, that is, the initial values (u_0, λ_0) are generic, these quintics converge to the exact one.

Here, we apply the generally convergent iteration by Doyle and McMullen [45]. They specify a rational function

$$T_C(Z) = z - 12\frac{g_C(Z)}{g_C'(Z)}$$

where g is a polynomial of degree 6 in the variable C and of degree 12 in Z. Starting from an initial guess w_o from an initial guess $w_{n+1} = T_C(T_C(w_n))$ to convergence and applying T_C still once, we obtain, after a finite rational computation with these two numbers, two roots of the Brioschi, say z_I and z_{II}. If applied to the approximative quintics and if the iteration is truncated after n steps, together with the corresponding post-processing, we have obtained e.g. a pair $(z_{I,n,k,j}, z_{II,n,k,j})$.

What remains is to invert the Tschirnhaus transformations. Suppose z is a root of the exact Brioschi form (10.2). Then the corresponding root of the principal quintic is obtained immediately from (10.3)

$$ty = \frac{\lambda + \mu z}{(z^2/C) - 3}.$$

Naturally, we can only apply this using approximated values for the parameters. Finally, one needs to transform the (approximative roots) of the principal quintic to (approximative) roots for the original general quintic $p(X)$. This is done by a rational function $X = r(Y)$ where $r(Y)$ is of second order in Y and the coefficients are polynomials of the coefficients if the original $p(X)$ and u and v (p. 127, eq. (6.8-3) in [86]). Again, we would be using only approximative values u_j instead of the exact u. In any case, we obtain a pair of approximations to the roots of the original quintic. If we put $n_1 = j, n_2 = k$ and $n_3 = n$, then this pair could be denoted by $(x_{I,n_3,n_2,n_1}, x_{II,n_3,n_2,n_1})$.

The link to the SCI. In the same way as with the quartic, we assume that the initial value $d_0 = (u_0, \lambda_0, w_0) \in \mathbb{C}^3$ is generic so that all iterations converge for large enough values and since the transformations are continuous functions of the parameters in it, all necessary limits exist and match with each other. The functions Γ_{n_3,n_2,n_1} can then be identified in a natural manner:

$$\begin{aligned}
\Gamma_{n_3} &: \mathbb{P}_5 \times \mathbb{C}^3 \to \mathbb{C}^2, \text{ by } (p, d_0) \mapsto (x_{I,n_3}, x_{II,n_3}),\\
\Gamma_{n_3,n_2} &: \mathbb{P}_5 \times \mathbb{C}^3 \to \mathbb{C}^2 \text{ by } (p, d_0) \mapsto (x_{I,n_3,n_2}, x_{II,n_3,n_2}),\\
\Gamma_{n_3,n_2,n_1} &: \mathbb{P}_5 \times \mathbb{C}^3 \to \mathbb{C}^2 \text{ by } (p, d_0) \mapsto (x_{I,n_3,n_2,n_1}, x_{II,n_3,n_2,n_1}),
\end{aligned}$$

where $(x_{I,n_3,n_2}, x_{II,n_3,n_2})$ and (x_{I,n_3}, x_{II,n_3}) are the limits as $n_1 \to \infty$ and $n_2 \to \infty$ respectively. These limits exist for initial values in an open dense subset of $\mathbb{C}^3$. Hence, we let $\Omega = \mathbb{P}_5 \times \mathbb{C}^3$, and $\Xi, \mathcal{M}, \Lambda$ be as in case of the quartic. Then, by the construction above and Theorem 10.2 we have, again in a slightly weakened sense, that

$$\mathrm{SCI}(\Xi, \Omega)_{\mathrm{DM}} \in \{2, 3\}.$$

10.4. **Particular initial guesses and height one towers.** The special feature of the above-mentioned Doyle–McMullen towers is that they address whether one can achieve convergence to the roots of a polynomial p for (almost) arbitrary initial guesses. With a slight change of perspective, one might also ask how large the SCI gets if one applies purely iterative algorithms *after a suitable clever choice* of initial values. Indeed, the answer to this question is very satisfactory: For polynomials of arbitrary degree, one can compute the whole set of roots (more precisely: approximate it in the sense of the Hausdorff distance) by a tower of height one which just consists of Newton's method.

The key tool for the choice of the initial values is the main theorem of [78]:

Theorem 10.3 (Hubbard, Schleicher and Sutherland [78]). *For every $d \geq 2$ there is a set S_d consisting of at most $1.11d \log^2 d$ points in $\mathbb{C}$ with the property that for every polynomial p of degree d and every root z of p there is a point $s \in S_d$ such that the sequence of Newton iterates $\{s_n\}_{n\in\mathbb{N}} := \{N_p^n(s)\}_{n\in\mathbb{N}}$ converges to z. In particular, the proof is constructive, and these sets S_d can easily be computed.*

A further important property of Newton's method is that, in the case of convergence, the speed is at least linear: If $z_n := N_p^n(s)$ tend to a root z of p then there exists a constant c such that $|z_n - z| \leq c/n$. Finally, we have the following.

Proposition 10.4. *Let p be a polynomial of degree d, $\epsilon > 0$ and $z_n := N_p^n(s)$. If $|z_n - z_{n+1}| < \frac{\epsilon}{d}$ then there is a root z of p with $|z_n - z| < \epsilon$.*

Proof. We have $\left|\frac{p(z_n)}{p'(z_n)}\right| = |z_n - z_{n+1}| < \frac{\epsilon}{d}$, hence $|p(z_n)| < \frac{\epsilon|p'(z_n)|}{d}$. Decompose $p(x) = a\Pi_{i=1}^d(x-x_i)$, notice that $p'(x) = a\sum_{j=1}^d \Pi_{i=1,i\neq j}^d(x-x_i)$, choose j such that $|\Pi_{i=1,i\neq j}^d(z_n - x_i)|$ is maximal, and conclude that

$$|a\Pi_{i=1}^d(z_n - x_i)| = |p(z_n)| < \frac{\epsilon|p'(z_n)|}{d} \leq \epsilon|a\Pi_{i=1,i\neq j}^d(z_n - x_i)|,$$

thus $|z_n - x_j| < \epsilon$. Now $z = x_j$ is a root as asserted. □

Let p be a polynomial of degree d. For each $s \in S_d$ let s_n denote the nth Newton iterates of s, and define

$$\Gamma_n(p) := \left\{ s_n : s \in S_d, |s_n - s_{n+1}| < \frac{1}{\sqrt{n}} \right\}. \tag{10.5}$$

Then $(\Gamma_n(p))$ converges to the set $\mathcal{Z}(p)$ of all zeros of p in the Hausdorff metric. Indeed, let z be a zero of p. By Theorem 10.3 there is an initial value $s \in S_d$ such that $s_n = N_p^n(s)$ tend to z with at least linear speed, i.e.

$$|s_n - s_{n+1}| \leq |s_n - z| + |s_{n+1} - z| \leq \frac{2c}{n} < \frac{1}{\sqrt{n}}$$

for all large n, hence $s_n \in \Gamma_n(p)$ for all large n. Conversely, each $s_n \in \Gamma_n(p)$ has the property that its distance to the set $\mathcal{Z}(p)$ is less than $\epsilon = \frac{d}{\sqrt{n}}$ by Proposition 10.4.

Therefore we define $\Omega_d = \mathbb{P}_d$ to be the set of polynomials of degree d, $\mathcal{M}$ the set of finite subsets of $\mathbb{C}$ equipped with the Hausdorff metric, and $\Xi : \Omega_d \to \mathcal{M}$ be the mapping that sends $p \in \Omega_d$ to the set of its zeros. Further, Λ_d shall consist of the evaluation functions that read the coefficients of the polynomial $p \in \Omega_d$, and the constant functions with the values $s \in S_d$. Note again that these values can be effectively constructed.

Theorem 10.5. *Consider $(\Xi, \Omega_d, \mathcal{M}, \Lambda_d)$ as above. Then, the algorithms* (10.5) *define an arithmetic tower of height one for the computation of the roots of each input polynomial p with error control. Thus this tower realises $\{\Xi, \Omega_d, \mathcal{M}, \Lambda_d\} \in \Sigma_1^A$. Moreover, this tower employs Newton's Method, i.e., a purely iterative algorithm.*

11. Computational Examples

This section aims to demonstrate that the new towers of algorithms developed to yield the sharp classifications in the SCI hierarchy yield efficient, implementable algorithms. In the case of Σ_1^A classifications, up to a user-specified error tolerance, they will never produce incorrect output. This fact makes the algorithms particularly suited for computer-assisted proofs. Moreover, they provide the first computations of spectra of several types of operators that before were out of reach, as many quantum problems are not exactly solvable [2]. Convergent algorithms that never make mistakes are highly desirable, and the reader may consult [37] to see the algorithms used in practice for large-scale problems in physics.

11.1. **Toeplitz operators.** Toeplitz and Laurent operators are familiar test objects because their spectra are very well understood [25, 26]. In this first example, we are concerned with operators that are banded with known growth on their resolvents. In particular, the problem of computing the spectrum lies in Σ_1^A and has $\mathrm{SCI} = 1$. Since the problem is not in Π_1^G, we monitor the changes of $\Gamma_n(A)$ as $n \to \infty$. This is common practice in computations when error control is not available. In particular, we choose an $\epsilon > 0$ and $K \in \mathbb{N}$ and stop the iteration when

$$\max\{E_n(A), d(\Gamma_n(A), \Gamma_{n+k}(A))\} \leq \epsilon \text{ for all } k \leq K. \tag{11.1}$$

Here $E_n(A)$ refers to the error guarantee $\Gamma_n(A) \subset \mathrm{sp}(A) + B_{E_n(A)}(0)$ provided by the algorithm. To visualize the convergence, we tested the tower of height one on the shift operator in Figure 3. Note that it is crucial to know the SCI of the problem so that one can apply the tower of algorithms with the correct height. In particular, solving this problem with a tower of height two would make the computation incredibly complex. Compare, for example, with the experiment in §11.5.

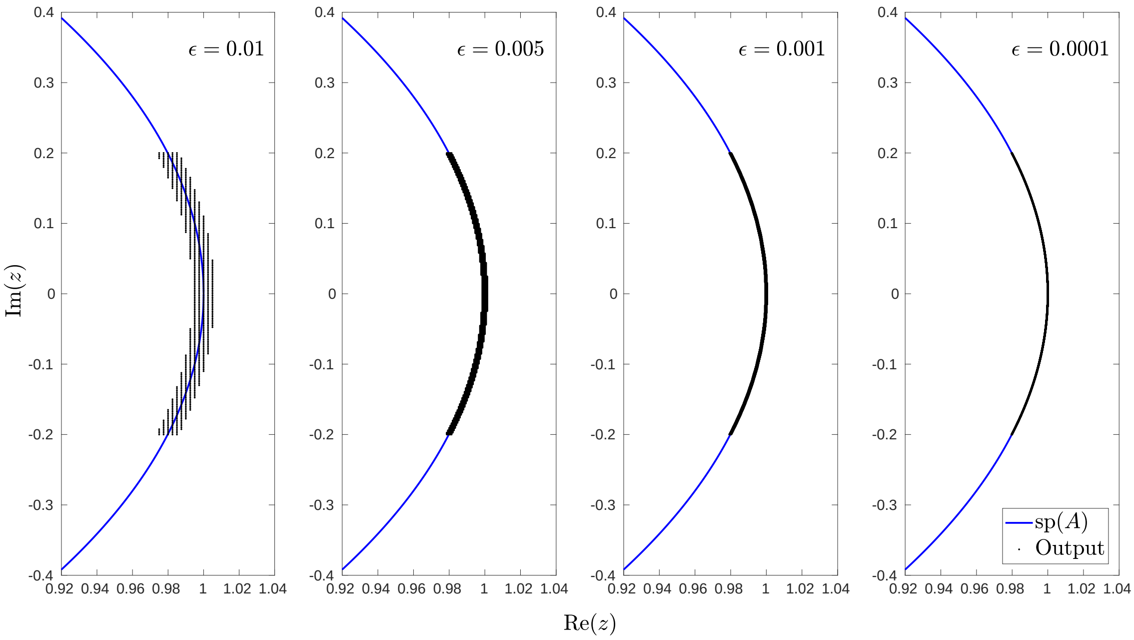

FIGURE 3. The figure shows a $\Gamma_n(A) \cap K$ (black) for a compact set $K \subset \mathbb{C}$ on top of a part of $\mathrm{sp}(A)$ (blue) for different increasing values of n corresponding to the chosen ϵ, where A is the shift operator on $l^2(\mathbb{Z})$.

11.2. **Spectra and approximate eigenvectors of operators on aperiodic tilings.** Quasicrystals,[3] and more generally aperiodic systems, have generated considerable interest due to their often exotic physical/spectral properties [106, 111]. However, the lack of reliable algorithms has limited the insight obtained from computations. We present the first rigorous spectral computational study with error bounds on an Ammann–Beenker tiling, a standard 2D model of a quasicrystal [3, 112]. Such models are difficult to deal with due to the lack of translational symmetry. The tiling has eight-fold rotational symmetry, shown in Figure 5 (left), which has been found to occur in real quasicrystals, e.g., in [118]. We consider a magnetic Hamiltonian

$$(H\psi)_a = -\sum_{a\sim b} e^{\mathrm{i}\alpha_{b,a}}\psi_b, \tag{11.2}$$

where the $a \sim b$ means vertices a and b are connected by an edge, and the summation is over connected sites. A constant perpendicular magnetic field with potential $\mathbf{A}(x, y, z) = (0, xB, 0)$ with $B \in \mathbb{R}$ is applied, leading to the Peierls phase factor between sites a and b:

$$\alpha_{b,a} = \int_b^a \mathbf{A} \cdot d\mathbf{l},$$

where $\mathbf{l}$ is the arclength. Figure 4 shows the output of the algorithm providing Σ_1^A classification computing spectra of the Hamiltonian in (11.2) for different values of $B \in [0, 2\pi]$ using the stopping criterion (11.1) and an error tolerance of 10^{-2} . The algorithm correctly leaves out the gaps in the spectrum, avoiding spectral pollution. We also show the output of the finite section method, which suffers from severe spectral pollution. One can study quasiperiodic tilings via periodic approximates [46]. However, it is not clear how these approximations affect the spectrum [93], and in the case of a magnetic field, this imposes severe restrictions on the values of B allowed [114]. In contrast, there is no such limitation for the new algorithm, which also

[3]Discovered in 1982 by D. Shechtman who was awarded the Nobel prize in 2011 for his discovery.

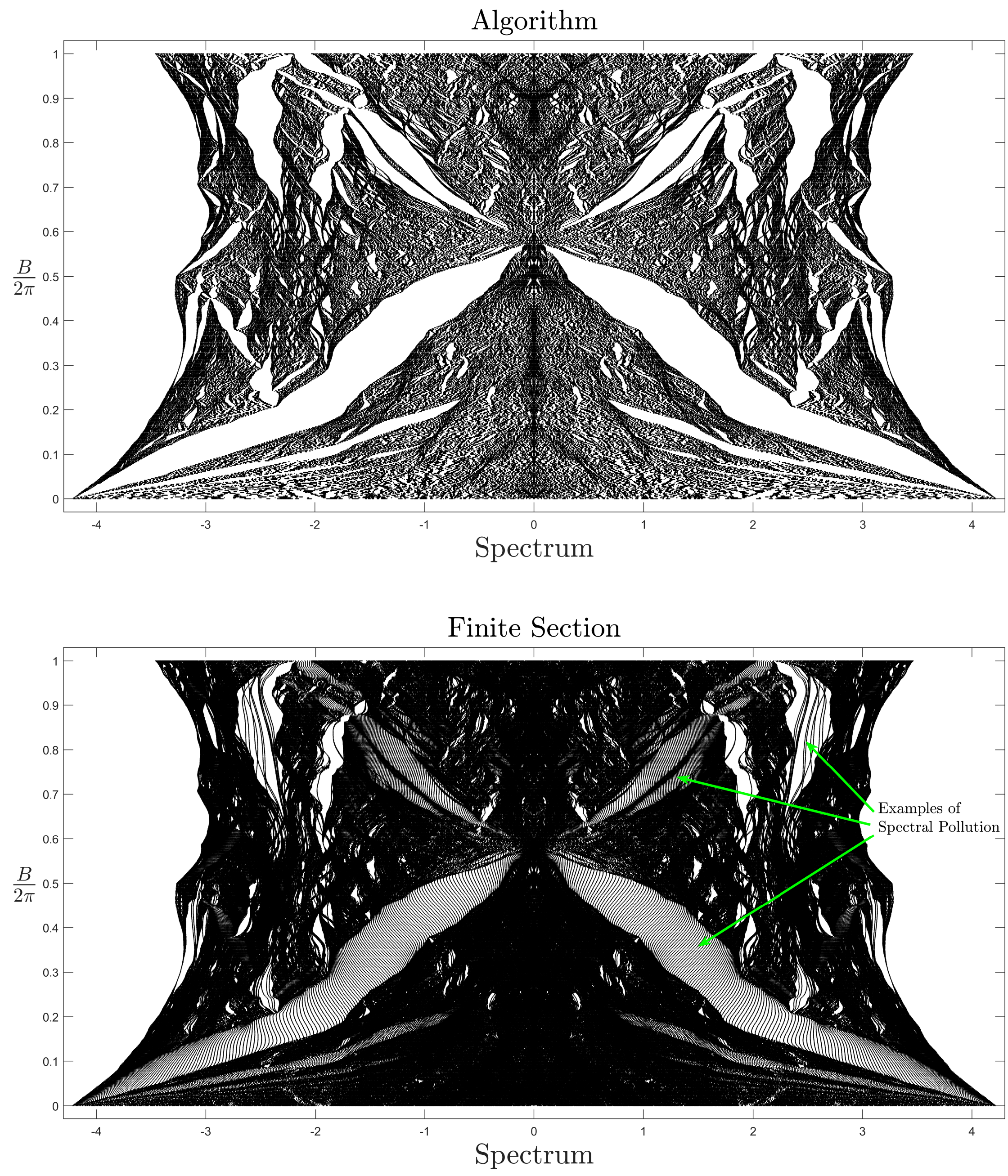

FIGURE 4. Top: Output of the algorithm providing Σ_1^A classification computing spectra of the Hamiltonian in (11.2) with error tolerance parameter 10^{-2} and different strengths of the magnetic field. The algorithm correctly leaves out the gaps and shows the fractal nature of the spectrum. Bottom: Output of the finite section method (4000 basis sites) showing severe spectral pollution.

provides rigorous error bounds and is guaranteed to converge. The new algorithm can also be used for non-constant magnetic fields. Finally, in Figure 5, we have also shown approximate eigenvectors for different values of B.

11.3. **Non-Hermitian Hamiltonians.** Non-Hermitian Hamiltonians have been standard in open systems. However, they have also found their way to quantum mechanics of closed systems due to the seminal work

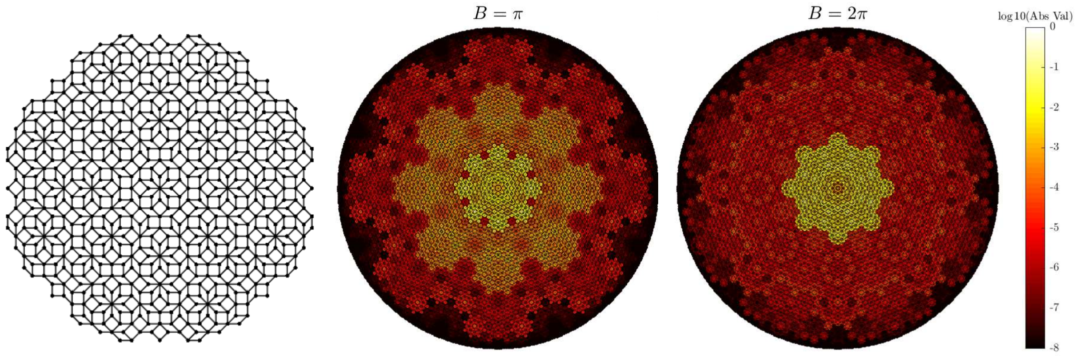

FIGURE 5. Left: Finite portion of the Ammann–Beenker tiling. The vertices correspond to the sites. Middle and Right: Approximate states (eigenvectors) ψ corresponding to the value $\lambda = 0$ for $B = \pi, 2\pi$ (logarithm of absolute value shown). These have bounds of $\|(H-\lambda)\psi\|$ by 3.3×10^{-7} and 1.5×10^{-6} respectively and were computed using 10^5 basis sites.

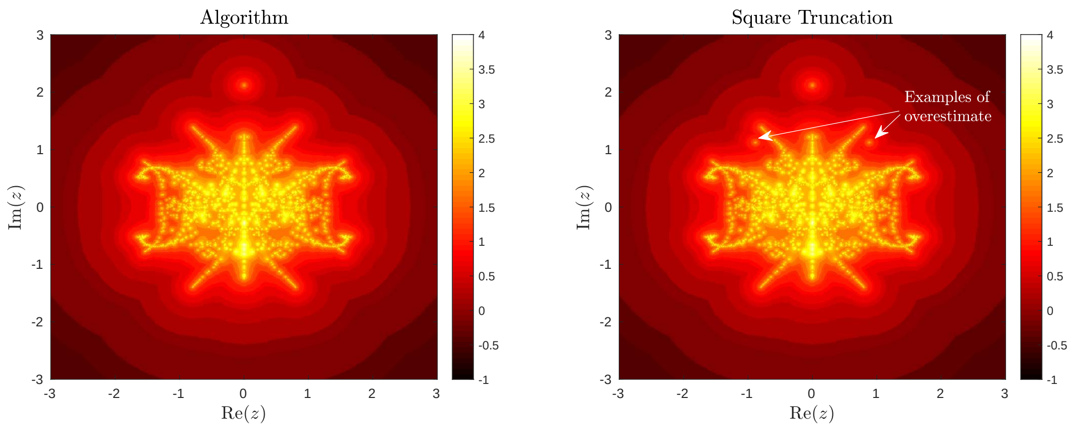

FIGURE 6. Left: Pseudospectra for the operator given by (11.3) computed by the algorithm providing Σ_1^A classification of the problem of computing pseudospectra. The colorbars correspond to the logarithm (base 10) of the resolvent norm (truncated at 4 for visibility). Right: failed attempt of computing pseudospectra with classical square truncation of the operator.

of C. Bender [16, 17]. There are also other variants of non-Hermitian quantum mechanics pioneered by N. Hatano and D. R. Nelson [74, 75]. The non-self-adjointness makes spectral computations incredibly difficult, and algorithms are typically unavailable for rigorous computations. As an example of computing pseudospectra and to demonstrate generality, we consider a non-normal operator A on $l^2(\mathbb{N})$ given by

$$(Ax)_n = \begin{cases} x_{n-1} + \mathrm{i}\sin(n)x_n - x_{n+1}, & \text{if } n+1 \text{ is prime} \\ x_{n-1} + \mathrm{i}\sin(n)x_n + x_{n+1}, & \text{otherwise,} \end{cases} \tag{11.3}$$

with the convention that $x_0 = 0$. Figure 6 shows pseudospectra computed using the new algorithm providing Σ_1^A classification and attempts of computing pseudospectra using square finite section truncations for 2000 basis vectors. We see that taking square truncations gives rise to overestimates of the resolvent norm, resulting in incorrect spectral information.

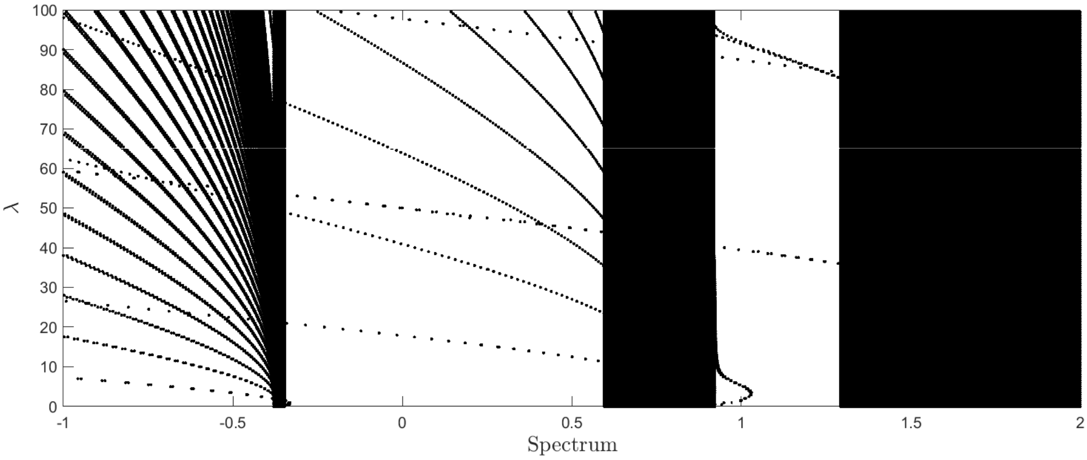

FIGURE 7. A portion of the computed spectrum of the one-dimensional Schrödinger operator with potential V_λ from (11.4) computed with an algorithm, providing the Σ_1^A classification, with error bound $\epsilon = 0.01$.

11.4. **Schrödinger operator on $\mathbb{R}$.** We now test the algorithm that computes spectra of Schrödinger operators

$$H = -\Delta + V, \qquad V : \mathbb{R}^d \to \mathbb{R},$$

acting on $\mathrm{W}^{2,2}(\mathbb{R})$ (i.e. a continuum model) with bounded potential. This example demonstrates the power of the Σ_1^A classification of Theorem 5.3. Recall that our algorithm uses only evaluations of the potential itself. As the class of problems considered are $\in \Sigma_1^A$ and have SCI $= 1$, we shall use the stopping criterion in (11.1) with $\epsilon = 0.01$. We chose the slowly decaying potentials

$$V_\lambda(x) = \cos(x) + \lambda \frac{\sin(x)}{x}. \tag{11.4}$$

When $\lambda = 0$, the operator is periodic, and computation of the spectrum reduces to computing spectra of two differential operators on a compact interval (with periodic and anti-periodic boundary conditions), which have compact resolvent. However, when $\lambda \neq 0$, eigenvalues appear in the gaps of the essential spectrum (and below), and methods based on finite section produce spectral pollution. Additionally, the slow decay of the potential makes this extremely difficult to detect via other means. In Figure 7, we display the computation of a portion of the spectrum in $[-1, 2]$ for various choices of λ and the error bound $\epsilon = 0.01$. The algorithm allows us to track eigenvalues in the gaps with error control guaranteeing the error bound.

11.5. **The operator $f(Q)$.** If we consider the multiplication operator $(Qg)(x) = xg(x)$ on $\mathrm{L}^2(\mathbb{R})$, then, for a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, the spectrum of $f(Q)$ is the range of the function f. In this example we use $f(x) = \frac{i(\exp(-2\pi i x)-1)}{2\pi x}$. To create an infinite matrix representation of $f(Q)$, we first consider the following Gabor basis for $\mathrm{L}^2(\mathbb{R})$:

$$e^{2\pi i m x}\chi_{[0,1]}(x-n), \qquad m, n \in \mathbb{Z},$$

(where χ is the characteristic function) and then chose some enumeration of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{N}$ to obtain a basis $\{\psi_j\}$ that is just indexed over $\mathbb{N}$. To get our basis, we let $\varphi_j = \mathcal{F}\psi_j$, where $\mathcal{F}$ is the Fourier Transform. Finally, we obtain the infinite matrix representation $A_{ij} = \langle f(Q)\varphi_j, \varphi_i\rangle$. Note that this becomes a full infinite matrix; however, we know the growth of the resolvent of the operator. Thus, this is a problem in the class Σ_2^A with SCI $= 2$. As there are now two limits, our algorithm depends on two parameters, namely m and n, and we compute $\Gamma_{n,m}(A)$. This means the stopping criterion from (11.1) becomes as follows. Choose

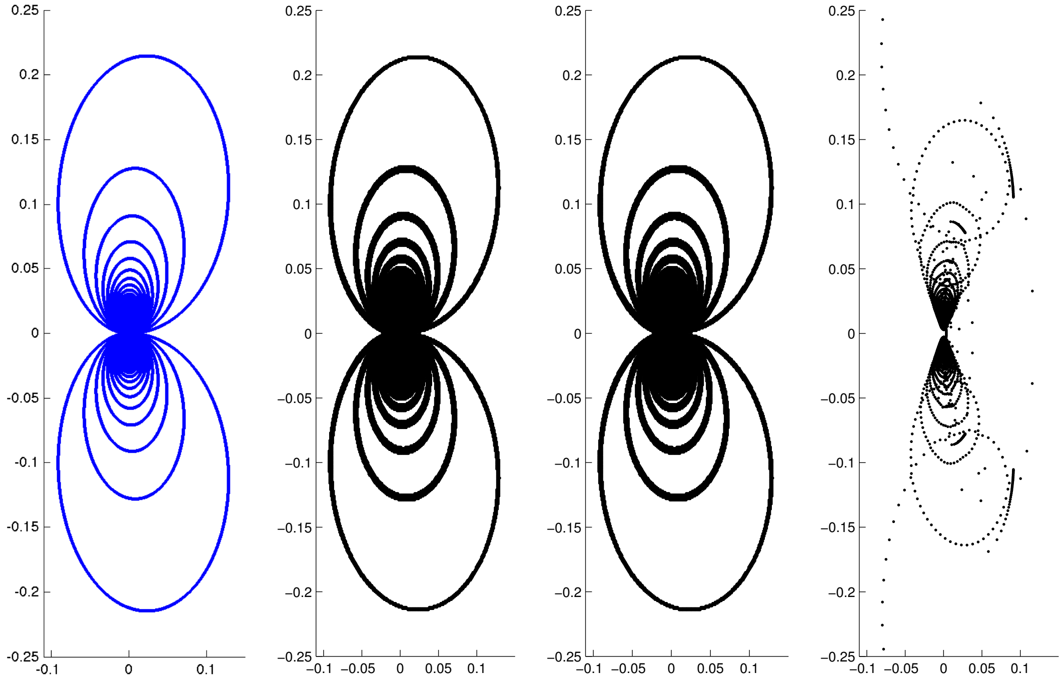

FIGURE 8. The left figure is a zoomed in part of $\mathrm{sp}(f(Q))$. The two following figures are $\Gamma_{n,m}(A)$ and $\Gamma_{n+p,m+s}(A)$ (restricted to the zoomed-in part) to visualize the stopping criterion in (11.5). A smaller ϵ can be chosen to get a better approximation. A is the matrix representation of $f(Q)$. The right figure is the result of the finite section method trying to compute $\mathrm{sp}(f(Q))$.

$\epsilon > 0$ and $K \in \mathbb{N}$. Define, for any $n, l \in \mathbb{N}$,

$$\begin{aligned} \tilde{\Gamma}_n(A) &:= \Gamma_{n,m}(A), \quad m = \min\{p : d(\Gamma_{n,p}(A), \Gamma_{n,p+k}(A) \leq \epsilon \text{ for all } k \leq K\} \\ \tilde{\Gamma}(A) &:= \tilde{\Gamma}_l(A), \quad l = \min\{p : d(\tilde{\Gamma}_p(A), \tilde{\Gamma}_{p+k}(A) \leq \epsilon \text{ for all } k \leq K\}, \end{aligned} \tag{11.5}$$

and let the output be $\tilde{\Gamma}(A)$. This stopping criterion is a generalization of (11.1) and extends in an obvious way to several limits. Note, however, how incredibly more complex it gets by adding one more limit. In Figure 8, we have plotted $\Gamma_{n,m}(A)$ and $\Gamma_{n+p,m+s}(A)$ visualizing an output based on the two limit stopping criterion in (11.5). We also plotted the result of the finite section method. As we are computing within the class of problems with SCI $= 2$, there is, of course, no way that the finite section method could work.

APPENDIX A. PROOF OF PROPOSITION 7.15 AND GENERALIZATIONS

A.1. **Proof of Proposition 7.15 parts (i) and (ii).** Let $(\mathcal{M}, d)$ be a metric space with the Attouch–Wets or Hausdorff topology induced by another metric space $(\mathcal{M}', d_{\mathcal{M}'})$. For the Attouch–Wets topology and any fixed $x_0 \in \mathcal{M}'$ we set

$$d_{\mathrm{AW}}(C_1, C_2) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, \sup_{d_{\mathcal{M}'}(x_0,x) \leq n} |\mathrm{dist}(x, C_1) - \mathrm{dist}(x, C_2)|\},$$

for $C_1, C_2 \in \mathrm{Cl}(\mathcal{M}')$, where $\mathrm{Cl}(\mathcal{M}')$ denotes the set of non-empty closed subsets of $\mathcal{M}'$. If $\mathcal{M}' = \mathbb{C}$ with the usual metric, we take $x_0 = 0$. Using the notation of §7, we have the following 'sandwich' lemma.

Lemma A.1. *Suppose that $(\mathcal{M}, d)$ is the Hausdorff or Attouch–Wets topology induced by a metric space $(\mathcal{M}', d_{\mathcal{M}'})$. Let $\epsilon > 0$. Suppose also that $A, A', B, B', C \in \mathcal{M}$ with $A \subset_{\mathcal{M}'} A'$, $C \subset_{\mathcal{M}'} B'$, $d(C, A') \leq \epsilon$ and $d(B, B') \leq \epsilon$. Then $d(A, C) \leq d(A, B) + 2\epsilon$.*

Proof. Suppose first that $(\mathcal{M}, d)$ is the Hausdorff topology. If $x \in C$ then $x \in B'$ and $\mathrm{dist}(x, A) \leq d(B', A) \leq d(A, B) + \epsilon$. On the other hand, if $x \in A$ then $x \in A'$ and $\mathrm{dist}(x, C) \leq d(A', C) \leq \epsilon$. The result now follows.

Suppose now that $(\mathcal{M}, d)$ is the Attouch–Wets topology and let $x \in \mathcal{M}'$. Since $C \subset_{\mathcal{M}'} B'$ we must have

$$\mathrm{dist}(x, A) - \mathrm{dist}(x, C) \leq \mathrm{dist}(x, A) - \mathrm{dist}(x, B') \leq |\mathrm{dist}(x, A) - \mathrm{dist}(x, B)| + |\mathrm{dist}(x, B) - \mathrm{dist}(x, B')| .$$

Similarly, since $A \subset_{\mathcal{M}'} A'$ we must have

$$\mathrm{dist}(x, C) - \mathrm{dist}(x, A) \leq \mathrm{dist}(x, C) - \mathrm{dist}(x, A') \leq |\mathrm{dist}(x, C) - \mathrm{dist}(x, A')| .$$

It follows that

$$|\mathrm{dist}(x, A) - \mathrm{dist}(x, C)| \leq |\mathrm{dist}(x, A) - \mathrm{dist}(x, B)| + |\mathrm{dist}(x, B) - \mathrm{dist}(x, B')| + |\mathrm{dist}(x, C) - \mathrm{dist}(x, A')| .$$

The result now follows. ☐

Proposition A.2. *Let $(\mathcal{M}, d)$ be either a metric space with the Attouch–Wets or Hausdorff topology induced by another metric space $(\mathcal{M}', d_{\mathcal{M}'})$ or a totally ordered metric space with order respecting metric. Suppose we have a computational problem*

$$\Xi : \Omega \to \mathcal{M},$$

with a corresponding convergent Σ_k^α tower $\Gamma^1_{n_k,...,n_1}$ and a corresponding convergent Π_k^α tower $\Gamma^2_{n_k,...,n_1}$ (either both arithmetic or both general). Suppose also that $1 \leq k \leq 3$ and that, in the case of arithmetic towers, we can compute for every $A \in \Omega$ the distance $d(\Gamma^1_{n_k,...,n_1}(A), \Gamma^2_{n_k,...,n_1}(A))$ to arbitrary precision using finitely many arithmetic operations and comparisons. Then $\{\Xi, \Omega\} \in \Delta_k^\alpha$.

Remark A.3. This proposition essentially says that we can combine the two notions of error control Π_k and Σ_k to reduce the number of limits needed by one.

Proof of Proposition A.2. **Step I:** For $k = 1$ and the case that $(\mathcal{M}, d)$ is either a metric space with the Attouch–Wets or Hausdorff topology, this is a trivial consequence of Lemma A.1. Let δ_{n_1} be an approximation of

$$d(\Gamma^1_{n_1}(A), \Gamma^2_{n_1}(A)) + 2 \cdot 2^{-n_1}$$

from above to accuracy $1/n_1$. Note that suitable approximations can easily be generated using approximations of $d(\Gamma^1_{n_1}(A), \Gamma^2_{n_1}(A))$. Let $\epsilon > 0$, then simply choose $n_1 \in \mathbb{N}$ minimal such that $\delta_{n_1} \leq \epsilon$. In the case that $(\mathcal{M}, d)$ is totally ordered with order respecting metric

$$d(\Gamma^1_{n_1}(A), \Xi(A)) \leq d(\Gamma^1_{n_1}(A), \Gamma^2_{n_1}(A)),$$

and we can take n_1 large such that the right-hand side is less than the given ϵ (recall, we can compute the right-hand side to arbitrary precision). Set $\Gamma(A) = \Gamma^1(A)$, then we have

$$d(\Gamma_(A), \Xi(A)) \leq \epsilon.$$

Step II: For larger k, we use the same idea, but we must be careful to ensure the first $k-1$ limits exist. For the rest of the proof, $\tilde{d}$ will denote an approximation of d to accuracy $1/n_1$ (which by assumption can always be computed).

We first deal with the case $k = 2$. Let $\epsilon > 0$ and consider the intervals $J^1_\epsilon = [0, \epsilon]$ and $J^2_\epsilon = [2\epsilon, \infty)$. Let $\delta_{n_2,n_1}(A)$ be an approximation of

$$d(\Gamma^1_{n_2,n_1}(A), \Gamma^2_{n_2,n_1}(A)) + 2 \cdot 2^{-n_2}$$

from above to accuracy $1/n_1$. Again, note that we can easily construct such approximations. It is clear that $\lim_{n_1\to\infty}\delta_{n_2,n_1}(A)=d(\Gamma^1_{n_2}(A),\Gamma^2_{n_2}(A))+2\cdot 2^{-n_2}=:\delta_{n_2}(A)$ and that $d(\Gamma^1_{n_2}(A),\Xi(A))\leq\delta_{n_2}(A)$ (again appealing to Lemma A.1 if we are in the case of the Attouch–Wets or Hausdorff topologies). Given n_1,n_2, let $l(n_2,n_1)\leq n_1$ be maximal such that $\delta_{n_2,l}(A)\in J^1_\epsilon\cup J^2_\epsilon$. If no such l exists or $\delta_{n_2,l}(A)\in J^1_\epsilon$ then define $\mathrm{Osc}(\epsilon;n_1,n_2,A)=1$ otherwise define $\mathrm{Osc}(\epsilon;n_1,n_2,A)=0$. Since $\delta_{n_2,n_1}(A)$ cannot oscillate infinitely often between the two intervals J^1_ϵ and J^2_ϵ, it follows that

$$\mathrm{Osc}(\epsilon;n_2,A):=\lim_{n_1\to\infty}\mathrm{Osc}(\epsilon;n_1,n_2,A)$$

exists. Define $\Gamma^\epsilon_{n_1}(A)$ as follows. Choose $j\leq n_1$ minimal such that $\mathrm{Osc}(\epsilon;n_1,j,A)=1$ if such a j exists, and define $\Gamma^\epsilon_{n_1}(A)=\Gamma_{j,n_1}(A)$. If no such j exists then define $\Gamma^\epsilon_{n_1}(A)=C_0$ where C_0 is any fixed member of $(\mathcal{M},d)$. In particular, $\Gamma^\epsilon_{n_1}$ is a type α algorithm. Now for large n_2, we must have $\delta_{n_2}(A)<\epsilon$ and hence $\mathrm{Osc}(\epsilon;n_2,A)=1$. It follows that $\Gamma^\epsilon(A)=\lim_{n_1\to\infty}\Gamma^\epsilon_{n_1}(A)$ exists and is equal to $\Gamma^1_N(A)$ where $N\in\mathbb{N}$ is minimal with $\mathrm{Osc}(\epsilon;N,A)=1$. It follows that $d(\Gamma^\epsilon(A),\Xi(A))\leq 2\epsilon$.

We will use the $\Gamma^\epsilon_{n_1}(A)$ to construct a height one tower. Observe first of all that by our assumptions we can compute $\tilde{d}(\Gamma^{\epsilon_1}_m(A),\Gamma^{\epsilon_2}_n(A))$ for $m,n\in\mathbb{N}$ and $\epsilon_1,\epsilon_2>0$. Given n_1, choose $j=j(n_1)\leq n_1$ maximal such that for all $1\leq l\leq j$ we have

$$\tilde{d}(\Gamma^{2^{-j}}_{n_1}(A),\Gamma^{2^{-l}}_{n_1}(A))\leq 4(2^{-j}+2^{-l}). \tag{A.1}$$

If no such j exists then set $\Gamma_{n_1}=C_0$, otherwise set $\Gamma_{n_1}(A)=\Gamma^{2^{-j(n_1)}}_{n_1}(A)$. Again, this is easily seen to be a type α algorithm. Pick any $N\in\mathbb{N}$, then by the convergence of the $\Gamma^\epsilon_{n_1}(A)$ and $d(\Gamma^\epsilon(A),\Xi(A))\leq 2\epsilon$, (A.1) must hold for $j=N$ and $1\leq l\leq N$ if n_1 is large enough. Hence by definition of $j(n_1)$,

$$\limsup_{n_1\to\infty} d(\Gamma_{n_1}(A),\Xi(A))\leq\limsup_{n_1\to\infty} d(\Gamma^{2^{-N}}_{n_1}(A),\Xi(A))+2^{3-N}\leq 2^{4-N}.$$

Since N was arbitrary, we must have convergence to $\Xi(A)$.

Step III: We now deal with $k=3$. The strategy will be similar to the $k=2$ case but now we construct $\Gamma^\epsilon_{n_2,n_1}(A)$ such that $\Gamma^\epsilon_{n_2}(A):=\lim_{n_1\to\infty}\Gamma^\epsilon_{n_2,n_1}(A)$ exists and is 3ϵ close to $\Xi(A)$ for large n_2, but may not converge in $(\mathcal{M},d)$. Using this, we will construct a height two type α tower.

As in Step II, let $\epsilon>0$ and consider the intervals $J^1_\epsilon=[0,\epsilon]$ and $J^2_\epsilon=[2\epsilon,\infty)$. Let $\delta_{n_3,n_2,n_1}(A)$ be an approximation of

$$d(\Gamma^1_{n_3,n_2,n_1}(A),\Gamma^2_{n_3,n_2,n_1}(A))+2\cdot 2^{-n_3},$$

from above to accuracy $1/n_1$. Again, we have

$$\lim_{n_2\to\infty}\lim_{n_1\to\infty}\delta_{n_3,n_2,n_1}(A)=d(\Gamma^1_{n_3}(A),\Gamma^2_{n_3}(A))+2\cdot 2^{-n_3}=:\delta_{n_3}(A)$$

exists with $d(\Gamma^1_{n_3}(A),\Xi(A))\leq\delta_{n_3}(A)$. Given n_1,n_2 and j, let $l(j,n_2,n_1)\leq n_1$ be maximal such that $\delta_{j,n_2,l}(A)\in J^1_\epsilon\cup J^2_\epsilon$. If no such l exists or $\delta_{j,n_2,l}(A)\in J^1_\epsilon$ then define $\mathrm{Osc}(\epsilon;n_1,n_2,j,A)=1$ otherwise define $\mathrm{Osc}(\epsilon;n_1,n_2,j,A)=0$. Arguing as in Step I, we have

$$\mathrm{Osc}(\epsilon;n_2,j,A):=\lim_{n_1\to\infty}\mathrm{Osc}(\epsilon;n_1,n_2,j,A)$$

exists. Now consider $\mathrm{Osc}(\epsilon;n_1,n_2,j,A)$ for $j\leq n_2$. If such a j exists with $\mathrm{Osc}(\epsilon;n_1,n_2,j,A)=1$ then let $j(n_1,n_2)$ be the minimal such j and set $\Gamma^\epsilon_{n_2,n_1}(A)=\Gamma^1_{j(n_1,n_2),n_2,n_1}(A)$. Otherwise set $\Gamma^\epsilon_{n_2,n_1}(A)=C_0$, where again C_0 is some fixed member of $(\mathcal{M},d)$. Since we only deal with finitely many $j\leq n_2$, it is clear that $\Gamma^\epsilon_{n_2,n_1}$ is a type α algorithm. Furthermore, we must have that $\Gamma^\epsilon_{n_2}(A):=\lim_{n_1\to\infty}\Gamma^\epsilon_{n_2,n_1}(A)$ exists and is defined as follows. Let $j(n_2)\leq n_2$ be minimal with $\mathrm{Osc}(\epsilon;n_2,j,A)=1$ (if such a j exists). If such a j exists then $\Gamma^\epsilon_{n_2}(A)=\Gamma^1_{j(n_2),n_2}(A)$, otherwise $\Gamma^\epsilon_{n_2}(A)=C_0$.

Now there exists $N\in\mathbb{N}$ such that $\delta_N(A)<\epsilon/2$ and hence $\delta_{N,n_2}(A)<\epsilon$ for large n_2. But this implies that $\mathrm{Osc}(\epsilon;n_2,N,A)=1$. Hence for n_2 large, we must have $j(n_2)\leq N$. If $\delta_l(A)>2\epsilon$ then for large n_2 we must have $\delta_{l,n_2}(A)>2\epsilon$ and hence $\mathrm{Osc}(\epsilon;n_2,l,A)=0$. As n_2 increases, $j(n_2)$ may not converge.

However, the above arguments show that for large n_2 it can take only finitely many values, say in the set $S = \{s_1, ..., s_m\}$, all of which must have $\delta_{s_i}(A) \leq 2\epsilon$. It follows that for large n_2 we must have

$$d(\Gamma^{\epsilon}_{n_2}(A), \Xi(A)) \leq 3\epsilon. \tag{A.2}$$

Now we get to work using these 'towers' (which do not necessarily converge in the last limit) and the trick to avoid oscillations. Define

$$F(n_1, n_2, j, l, A) := \tilde{d}(\Gamma^{2^{-j}}_{n_2,n_1}(A), \Gamma^{2^{-l}}_{n_2,n_1}(A)),$$
$$F(n_2, j, l, A) := \lim_{n_1\to\infty} F(n_1, n_2, j, l, A) = d(\Gamma^{2^{-j}}_{n_2}(A), \Gamma^{2^{-l}}_{n_2}(A))$$

and the intervals $J^1_{j,l} = [0, 4(2^{-j}+2^{-l})]$, $J^2_{j,l} = [8(2^{-j}+2^{-l}), \infty)$. Given j, l, n_1 and n_2, let $i(j, l, n_2, n_1) \leq n_1$ be maximal such that $F(i, n_2, j, l, A) \in J^1_{j,l} \cup J^2_{j,l}$. If no such i exists or if it does and $F(i, n_2, j, l, A) \in J^1_{j,l}$ then define $\widehat{\mathrm{Osc}}(n_1, n_2, j, l, A) = 1$ otherwise define $\widehat{\mathrm{Osc}}(n_1, n_2, j, l, A) = 0$. Choose $j = j(n_1, n_2) \leq n_2$ maximal such that for all $1 \leq l \leq j$ we have $\widehat{\mathrm{Osc}}(n_1, n_2, j, l, A) = 1$. If no such j exists then set $\Gamma_{n_2,n_1} = C_0$, otherwise set $\Gamma_{n_2,n_1}(A) = \Gamma^{2^{-j(n_1,n_2)}}_{n_2,n_1}(A)$. Again, this is easily seen to be a type α algorithm.

Arguing as before, we have the existence of

$$\widehat{\mathrm{Osc}}(n_2, j, l, A) := \lim_{n_1\to\infty} \widehat{\mathrm{Osc}}(n_1, n_2, j, l, A).$$

Now define $h = h(n_2) \leq n_2$ maximal such that for all $1 \leq l \leq h$ we have $\widehat{\mathrm{Osc}}(n_2, h, l, A) = 1$. If no such h exists then we must have

$$\Gamma_{n_2}(A) := \lim_{n_1\to\infty} \Gamma_{n_2,n_1}(A) = C_0,$$

otherwise we must have

$$\Gamma_{n_2}(A) := \lim_{n_1\to\infty} \Gamma_{n_2,n_1}(A) = \Gamma^{2^{-h(n_2)}}_{n_2}(A).$$

By (A.2), for any fixed j, l we have $\widehat{\mathrm{Osc}}(n_2, j, l, A) = 1$ for large n_2 and hence $h(n_2)$ exists for large n_2 and diverges to ∞. Now let $N \in \mathbb{N}$ then it follows that

$$\begin{aligned}\limsup_{n_2\to\infty} d(\Gamma^{2^{-h(n_2)}}_{n_2}(A), \Xi(A)) &\leq \limsup_{n_2\to\infty} d(\Gamma^{2^{-N}}_{n_2}(A), \Xi(A)) + d(\Gamma^{2^{-h(n_2)}}_{n_2}(A), \Gamma^{2^{-N}}_{n_2}(A))\\ &\leq 3 \cdot 2^{-N} + \limsup_{n_2\to\infty} 8(2^{-h(n_2)} + 2^{-N}) \leq 11 \cdot 2^{-N}.\end{aligned}$$

Since N was arbitrary, we must have convergence to $\Xi(A)$. □

Proof of Proposition 7.15 parts (i) and (ii). The statement regarding intersections follows directly from Proposition A.2 and the following remark - no assumptions on being able to compute distances between the output of algorithms is necessary when considering general towers. For the sharpness result in (i), we deal with $X = \Sigma$, and $X = \Pi$ follows from an identical argument. Suppose that $\Delta^G_k \not\ni \{\Xi, \Omega\} \in \Sigma^\alpha_k$. If $\{\Xi, \Omega\} \in \Pi^\alpha_k$, we would have $\{\Xi, \Omega\} \in \Sigma^\alpha_k \cap \Pi^\alpha_k \subset \Sigma^G_k \cap \Pi^G_k = \Delta^G_k$, a contradiction. □

A.2. **Proof of Proposition 7.15 part (iii).** To prove this part, we consider the following alternative definition in the case that $\mathcal{M} = \{0, 1\}$. Note that if we restricted to recursivity in the Turing sense with Ξ describing subsets of $\mathbb{N}$, this would correspond to the classical arithmetical hierarchy.

Definition A.4 (SCI hierarchy, $\mathcal{M} = \{0, 1\}$ (alternative definition))**.** Suppose that $\mathcal{M} = \{0, 1\}$. We define the following

(i) We say that $\Xi : \Omega \to \mathcal{M}$ permits a representation by an alternating quantifier form of length m if

$$\Xi = (Q_m n_m) \cdots (Q_1 n_1)\Gamma_{n_m,\ldots,n_1},$$

where (Q_i) is a list of alternating quantifiers $(\forall)$ and $(\exists)$, and all $\Gamma_{n_m,\ldots,n_1} : \Omega \to \mathcal{M}$ are general algorithms in the sense of Definition 7.3.

(ii) We say that $\{\Xi, \Omega\}$ is Σ_m if an alternating quantifier form of length m exists with Q_m being $(\exists)$, and that $\{\Xi, \Omega\}$ is Π_m if an alternating quantifier form of length m exists with Q_m being $(\forall)$.

(iii) We say that $\{\Xi, \Omega\}$ is Δ_m if $\{\Xi, \Omega\}$ is Σ_m and Π_m.

It is not clear from the wordings of Definition 7.11 and Definition A.4 that they are equivalent. However, the next proposition provides the link.

Proposition A.5 (The SCI hierarchy encompasses the arithmetical hierarchy)**.** *When $\mathcal{M} = \{0,1\}$, Definition 7.11 and Definition A.4 are equivalent, and hence the SCI encompasses generalizations of the arithmetical hierarchy. This also holds for arithmetic towers, which extend the arithmetical hierarchy to arbitrary domains.*

This immediately implies part (iii) or Proposition 7.15, and hence the rest of this subsection is devoted to proving Proposition A.5.

Remark A.6. In classical hierarchies the Δ_k class is defined by $\Delta_k = \Sigma_k \cap \Pi_k$. This is not the case in the SCI hierarchy. The Δ_k^α classes form the core of the hierarchy, and only when there is an extra structure on the metric space does it make sense to define the Σ_k^α and the Π_k^α. Moreover, in the general SCI hierarchy, we may have that

$$\Delta_k^\alpha \neq \Sigma_k^\alpha \cap \Pi_k^\alpha.$$

Of course, in the special cases of the SCI hierarchy, such as the arithmetical hierarchy, then $\Delta_k = \Sigma_k \cap \Pi_k$. Also, we show that $\Delta_k^\alpha = \Sigma_k^\alpha \cap \Pi_k^\alpha$ for $k = 1,2,3$ and $\alpha = G, A$ in the computational spectral problem case, however, there is no reason that this should hold for $k > 3$ in general. Moreover, classical hierarchies have that $\Sigma_k \setminus \Delta_{k-1} \neq \emptyset$ and $\Pi_k \setminus \Delta_{k-1} \neq \emptyset$. This does not have to be the case in general SCI hierarchies. Indeed, one may have that

$$\Sigma_k^\alpha \setminus \Delta_{k-1}^\alpha = \emptyset \text{ or } \Pi_k^\alpha \setminus \Delta_{k-1}^\alpha = \emptyset.$$

This happens, for example, in the SCI hierarchy for the computational spectral problem.

To prove Proposition A.5, we make the following definition corresponding to the SCI hierarchy in the main text.

Definition A.7 (Limit forms)**.** If $\mathcal{M} = \{0,1\}$, we define the following with respect to a given type of tower of algorithms (arithmetical, radical general, etc.):

(i) We say that $\{\Xi, \Omega\}$ is $\tilde{\Sigma}_m$ if there exists a height m tower solving the computational problem such that the final limit is monotonic from below. We say that $\{\Xi, \Omega\}$ is $\tilde{\Pi}_m$ if a height m tower solves the computational problem such that the final limit is monotonic from above.

(ii) We say that $\{\Xi, \Omega\}$ is $\tilde{\Delta}_{m+1}$ if there exists a height m tower solving the computational problem.

The following theorem demonstrates how the SCI framework can be viewed, in the special case of $\mathcal{M} = \{0,1\}$, as a generalization of the Arithmetical Hierarchy to arbitrary computational problems. In particular, one can define a hierarchy for any kind of tower. Here, we do this for a general tower, which can obviously be done for any tower. We will call the hierarchy described below a *General Hierarchy*.

Proposition A.8 (General Hierarchy)**.** *Suppose that $\mathcal{M} = \{0,1\}$. Following Definitions A.4 and A.7, for any $m \geq 1$ we have that*

$$\tilde{\Sigma}_m = \Sigma_m, \quad \tilde{\Pi}_m = \Pi_m \quad \textit{and} \quad \tilde{\Delta}_m = \Delta_m.$$

Proof of Proposition A.8. **Step I:** We show that if $\mathrm{SCI}(\Xi, \Omega)_{\mathrm{G}} \leq m$ then Ξ is Δ_{m+1}. Let $p = \lim_i p_i$. Then

$$p = \textit{true} \quad \Leftrightarrow \quad \forall n \exists k (k \geq n \wedge p_k) \quad \Leftrightarrow \quad \exists n \forall k (k \leq n \vee p_k).$$

Further, let $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $k \mapsto (\varphi_1(k), \varphi_2(k))$ be a bijection which enumerates all pairs of natural numbers, and note that

$$\exists n \exists m(p_{n,m}) \Leftrightarrow \exists k(p(\varphi_1(k), \varphi_2(k))), \qquad \forall n \forall m(p_{n,m}) \Leftrightarrow \forall k(p(\varphi_1(k), \varphi_2(k))),$$

for any family $(p_{n,m})_{n,m\in\mathbb{N}} \subset \mathcal{M}$. Thus, every limit in a tower of height m can be converted alternately into an expression with two quantifiers ($\forall\exists$ or $\exists\forall$), and then $m-1$ doubles $\exists\exists$ or $\forall\forall$ can be replaced by single quantifiers. This easily gives the claim.

Step II: We show that if Ξ is Σ_m or Π_m then $\mathrm{SCI}(\Xi, \Omega)_{\mathrm{G}} \leq m$. In fact, we show that $\Sigma_m \subset \tilde{\Sigma}_m$ and $\Pi_m \subset \tilde{\Pi}_m$. As a start let $(p_i) \subset \mathcal{M}$ be a sequence. Then

$$(\forall i(p_i)) = \textit{true} \quad \Leftrightarrow \quad \left(\lim_{n\to\infty} \bigwedge_{i=1}^{n} p_i\right) = \textit{true}, \qquad (\exists i(p_i)) = \textit{true} \quad \Leftrightarrow \quad \left(\lim_{n\to\infty} \bigvee_{i=1}^{n} p_i\right) = \textit{true}.$$

Furthermore, the conjunction (disjunction) of limits coincides with the limit of the elementwise conjunction (disjunction), hence

$$\forall n_m \exists n_{m-1} \cdots \forall n_1 \Gamma_{n_m,\cdots,n_1} = \lim_{k_m} \lim_{k_{m-1}} \cdots \lim_{k_1} \bigwedge_{i_m=1}^{k_m} \bigvee_{i_{m-1}=1}^{k_{m-1}} \cdots \bigwedge_{i_1=1}^{k_1} \Gamma_{i_m,i_{m-1},\cdots,i_1}$$

and similarly for any other possible alternating quantifier form. Since the $\Gamma_{n_m,\cdots,n_1}$ in the alternating quantifier form at the left-hand side are General algorithms, the right-hand side yields a tower of algorithms of height m. Moreover, we obtain the required monotonic final limits.

Step III: We show that $\tilde{\Delta}_m = \Delta_m$. Let $m \in \mathbb{N}$ be the smallest number with Ξ being Δ_{m+1}. In the above steps, we have already seen that $m \leq \mathrm{SCI}(\Xi, \Omega)_{\mathrm{G}} \leq m+1$, and we next prove the following: If

$$\Xi(y) = \exists i \forall j(g_0(i,j,y)) = \forall n \exists m(g_1(n,m,y))$$

then $\Xi(y) = \lim_{k\to\infty} g(k,y)$ with a function g being easily derivable from g_0, g_1. The following construction is adopted from [63, Proofs of Theorems 1 and 3]. Fix y and define a function $h_0 : \mathbb{N} \to \mathcal{M}$ recursively as follows:

$i(1) := 1$, $j(1) := 1$, $h_0(1) := g_0(i(1), j(1), y)$.
`If` $h_0(l) =$ `true`
`then:` $i(l+1) := i(l)$, $j(l+1) := j(l) + 1$
`else:` $i(l+1) := i(l) + 1$, $j(l+1) := 1$.
$l := l + 1$.
$h_0(l) := g_0(i(l), j(l), y)$.

We observe that, if $\Xi(y) = \textit{true}$ then $h_0(l)$ converges as $l \to \infty$ with limit *true*. Otherwise, the limit does not exist or is *false*. The same construction applies to $\neg(\forall n \exists m(g_1(n,m,y))) = \exists n \forall m \neg(g_1(n,m,y))$ and yields a function h_1 which converges to *true* if and only if $\Xi(y) =$ *false*. Clearly, exactly one of the functions h_0, h_1 converges to *true*. Now we derive the desired g from h_0 and h_1 as follows:

$\alpha(1) = 0$.
`If` $h_{\alpha(k)}(k) =$ `true`
`then:` $\alpha(k+1) := \alpha(k)$
`else:` $\alpha(k+1) := 1 - \alpha(k)$.
$k := k + 1$.
`If` $\alpha(k) = 0$
`then:` $g(k,y) :=$ `true`
`else:` $g(k,y) :=$ `false`.

This provides $\Xi(y) = \lim_{k\to\infty} g(k,y)$.

Next, let g_0 and g_1 be of the form $g_s(i,j,y) = \lim_r f^s_{i,j,r}(y)$, $s \in \{0,1\}$. Fix y. Then for every pair (i,j) there is an $r(i,j)$ such that $f^s_{u,v,r}(y) = g_s(u,v,y)$ for all $u \leq i$, $v \leq j$, $s \in \{0,1\}$ and $r \geq r(i,j)$. Thus,

g is also of the form $g(k,y) = \lim_r f_{k,r}(y)$ with $f_{k,r}$ being defined by the above procedure applied to the functions $(i,j,y) \mapsto f^s_{i,j,k}(y)$ instead of $g_s(i,j,y)$ ($s \in \{0,1\}$).

Now we are left with iterating this argument: If both functions g_s ($s \in \{0,1\}$) are of the form $g_s(i,j,y) = \lim_{k_{m-1}} \lim_{k_{m-2}} \cdots \lim_{k_1} f^s_{i,j,k_{m-1},\cdots,k_1}(y)$ with certain General algorithms $f^s_{i,j,k_{m-1},\cdots,k_1}$, then also g is of the form

$$g(k,y) = \lim_{k_{m-1}} \lim_{k_{m-2}} \cdots \lim_{k_1} f_{k,k_{m-1},\cdots,k_1}(y)$$

with $f_{k,k_{m-1},\cdots,k_1}$ being defined by the same procedure as before applied to the functions $(i,j,y) \mapsto f^s_{i,j,k_{m-1},\cdots,k_1}(y)$ instead of $g_s(i,j,y)$ ($s \in \{0,1\}$). The resulting functions $y \mapsto f_{k,k_{m-1},\cdots,k_1}(y)$ are General algorithms for every k, since their evaluation requires only finitely many evaluations of the General algorithms $f^s_{i,j,k_{m-1},\cdots,k_1}$.

Step IV: It remains to show that $\tilde{\Sigma}_m \subset \Sigma_m$ and $\tilde{\Pi}_m \subset \Pi_m$. Suppose that $\Xi \in \tilde{\Sigma}_m (\in \tilde{\Pi}_m)$ then by considering the first $m-1$ limits there exists a family $\Xi_{n_m} \in \tilde{\Delta}_m = \Delta_m$ (this is also trivially true if $m = 1$) such that

$$\Xi(y) = \lim_{n_m \to \infty} \Xi_{n_m}(y)$$

with the final limit monotonic from below (above). But then we must have $\Xi(y) = \exists n_m \Xi_{n_m}(y)$ ($\Xi(y) = \forall n_m \Xi_{n_m}(y)$). But $\Xi_{n_m} \in \Sigma_m (\in \Pi_m)$ and we can collapse the double quantifier $\exists\exists$ ($\forall\forall$) to a single $\exists$($\forall$). $\square$

A.3. **The Baire hierarchy.** To end this appendix, we also make some remarks on the Baire hierarchy. The Baire hierarchy [84], which is closely related to the Borel hierarchy [84], in descriptive set theory, has similarities to the SCI hierarchy; however, it is fundamentally different. However, it is worth mentioning since the Baire hierarchy includes classes of functions obtained as limits of functions from lower levels in the hierarchy. Hence, the two hierarchies share some similarities.

Recall that given metrizable spaces X, Y and a continuous function $f : X \to Y$, we say that f is of Baire class 0. We define a function $g : X \to Y$ to be in Baire class 1 if there is a sequence of functions $\{g_n\}$, all of Baire class 0, such that $g(x) = \lim_{n\to\infty} g_n(x)$ for all $x \in X$. In general, for $1 < \rho < \omega_1$ we define a function $f : X \to Y$ to be of Baire class ρ if it is the pointwise limit of a sequence of functions $f_n : X \to Y$, where f_n is of Baire class $\rho_n < \rho$. In order to understand the similarities and differences between the two hierarchies, we provide a short discussion below.

Similarities between the SCI and Baire hierarchies. The main similarity between the hierarchies is the concept of pointwise limits. Indeed, for the integer values of the Baire classes, this number indeed resembles the SCI.

Differences between the SCI and Baire hierarchies. The differences between the hierarchies are due to the fact that they describe very different problems. This can be summed up as follows.

(i) (*Generality*). The SCI hierarchy is designed to be able to handle all types of computational problems, such as Smale's problem on iterative algorithms for polynomial root-finding, Doyle–McMullen towers, the insolvability of the quintic, etc. This is obviously not within the scope of the Baire hierarchy. However, this was never the intention of this hierarchy.

(ii) (*Refinements*). An important difference between the hierarchies is that the SCI hierarchy, when extra structure on $\mathcal{M}$ is available, allows for the refinements in terms of the Σ^α_k and Π^α_k classes. This type of refinement is not captured by the Baire hierarchy. However, that has never been the motivation.

(iii) (*Topology vs information*). A striking difference is that the Baire hierarchy is based on metrizable topologies, whereas the SCI hierarchy is based on the information Λ (see Definition 7.2) available to the algorithm. The computational spectral problem is a good example to illustrate the issue. Let $\Xi : \Omega \ni A \mapsto \mathrm{sp}(A) \in \mathcal{M}$ where Ω is the set of self-adjoint operators in $\mathcal{B}(l^2(\mathbb{N}))$ and $\mathcal{M}$ is the collection of non-empty compact subsets of $\mathbb{C}$ with the Hausdorff metric. If we equip Ω with the operator norm topology, then Ξ is Baire class 0. Yet, the SCI $= 2$ for Ξ. If one changes the metric on

Ω, the Baire class will change, yet the SCI remains unchanged. Also, as a side note, the algorithms used in this paper to show that the SCI $= 2$ are not continuous in any metrizable topology. Thus, there is no metric on Ω such that these become Baire class 0.

Finally, if we consider self-adjoint Schrödinger operators on $\mathrm{L}^2(\mathbb{R}^d)$ with bounded potential V such that $V \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$, then the SCI of the spectral map is 1 if we can access point samples of V. Also, if we equip this set of operators with the natural graph metric (equivalent to norm convergence in the bounded case) the spectral map is Baire class 0. However, if one changes Λ, such that we are given matrix elements of the operator with respect to some orthonormal basis of the domain, we may get that the SCI $= \infty$, as the matrix representation may not uniquely determine the spectrum. Thus, the SCI changes with Λ (see Definition 7.2) that determines which information is available, whereas the Baire class changes with the metric.

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