Numerical Analysis - Part II

Anders C. Hansen

Lecture 9

Partial differential equations of evolution

Solving the diffusion equation

We consider the solution of the *diffusion equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leq x \leq 1, \quad t \geq 0,$$

with *initial conditions* $u(x, 0) = u_0(x)$ for $t = 0$ and Dirichlet *boundary conditions* $u(0, t) = \phi_0(t)$ at $x = 0$ and $u(1, t) = \phi_1(t)$ at $x = 1$.

What if $-\infty < x < \infty$?

Fourier analysis of stability

Let us now assume a recurrence of the form

$$\sum_{k=r}^{s} a_k u_{m+k}^{n+1} = \sum_{k=r}^{s} b_k u_{m+k}^{n}, \qquad n \in \mathbb{Z}^+, \tag{1}$$

where m ranges over $\mathbb{Z}$. (Within our framework of discretizing PDEs of evolution, this corresponds to $-\infty < x < \infty$ in the undelying PDE and so there are no explicit boundary conditions, but the initial condition must be square-integrable in $(-\infty, \infty)$: this is known as a *Cauchy problem.*)

The coefficients a_k and b_k are independent of m, n, but typically depend upon μ. We investigate stability by *Fourier analysis.* [Note that it doesn't matter what is the underlying PDE: numerical stability is a feature of algebraic recurrences, not of PDEs!]

Fourier analysis of stability

Let $\mathbf{v} = (v_m)_{m\in\mathbb{Z}} \in \ell_2[\mathbb{Z}]$. Its *Fourier transform* is the function

$$\widehat{v}(\theta) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}m\theta} v_m, \qquad -\pi \le \theta \le \pi.$$

We equip sequences and functions with the norms

$$\|\mathbf{v}\| = \Big\{ \sum_{m\in\mathbb{Z}} |v_m|^2 \Big\}^{\frac{1}{2}} \qquad \text{and} \qquad \|\widehat{v}\|_* = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{v}(\theta)|^2 d\theta \right\}^{\frac{1}{2}}.$$

Amplification factor

For $\theta \in [-\pi, \pi]$, let $\widehat{u}^n(\theta) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}m\theta} u_m^n$ be the Fourier transform of the sequence $\mathbf{u}^n \in \ell_2[\mathbb{Z}]$. We multiply the discretized equations (1) by $\mathrm{e}^{-\mathrm{i}m\theta}$ and sum up for $m \in \mathbb{Z}$. Thus, the left-hand side yields

$$\begin{aligned}\sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\theta} \sum_{k=r}^{s} a_k u_{m+k}^{n+1} &= \sum_{k=r}^{s} a_k \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\theta} u_{m+k}^{n+1} \\ &= \sum_{k=r}^{s} a_k \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(m-k)\theta} u_m^{n+1} = \Big(\sum_{k=r}^{s} a_k \mathrm{e}^{\mathrm{i}k\theta}\Big)\widehat{u}^{n+1}(\theta).\end{aligned} \tag{2}$$

Similarly manipulating the right-hand side, we deduce that

$$\widehat{u}^{n+1}(\theta) = H(\theta)\widehat{u}^n(\theta)\,, \quad \text{where} \quad H(\theta) = \frac{\sum_{k=r}^{s} b_k \mathrm{e}^{\mathrm{i}k\theta}}{\sum_{k=r}^{s} a_k \mathrm{e}^{\mathrm{i}k\theta}}\,. \tag{3}$$

The function H is sometimes called the *amplification factor* of the recurrence (1)

Theorem 1

The method (1) is stable $\quad\Leftrightarrow\quad |H(\theta)| \leq 1$ *for all* $\theta \in [-\pi, \pi]$.

The advection and wave equations

Problem 2 (The advection equation)

A useful paradigm for hyperbolic PDEs is the *advection equation*

$$u_t = u_x, \qquad 0 \le x \le 1, \qquad t \ge 0, \tag{4}$$

where $u = u(x, t)$. It is given with the initial condition $u(x, 0) = \varphi(x)$, $x \in [0, 1]$ and (for simplicity) the boundary condition $u(1, t) = \varphi(t + 1)$. The exact solution of (4) is simply $u(x, t) = \varphi(x + t)$, a unilateral shift leftwards. This, however, does not mean that its numerical modelling is easy.

Instability and the advection equation

We commence by semidiscretizing $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{2h}[u_{m+1}(t) - u_{m-1}(t)]$, so coming to the ODE $u_m'(t) = \frac{1}{2h}[u_{m+1}(t) - u_{m-1}(t)]$. For the Euler method, the outcome is

$$u_m^{n+1} = u_m^n + \tfrac{1}{2}\mu(u_{m+1}^n - u_{m-1}^n), \qquad m = 0...M, \quad n \in \mathbb{Z}_+,$$

with $u_0^n = 0$ for all n. In matrix form this reads

$$\mathbf{u}^{n+1} = A\mathbf{u}^n, \qquad A = \begin{bmatrix} 1 & \frac{1}{2}\mu & & \\ -\frac{1}{2}\mu & 1 & \ddots & \\ & \ddots & \ddots & \frac{1}{2}\mu \\ & & -\frac{1}{2}\mu & 1 \end{bmatrix}.$$

The matrix A is normal, with the eigenvalues $\lambda_\ell = 1 + i\mu\cos\ell\pi h$ (see Example 2.15), so that $\|A\|^2 = 1 + \mu^2$, hence instability for any μ.

The wave equation

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in [0,1], \qquad t \geq 0,$$

given with initial (for u and u_t) and boundary conditions. The usual approximation looks as follows

$$u_m^{n+1} - 2u_m^n + u_m^{n-1} = \mu(u_{m+1}^n - 2u_m^n + u_{m-1}^n),$$

with the Courant number being now $\mu = k^2/h^2$.

Discretising the wave equation

To advance in time we have to pick up the numbers $u_m^1 = u(x_m, k)$ (of course they should depend on the initial derivative $u_t(x, 0)$. Euler's method provides the obvious choice $u(x_m, k) = u(x_m, 0) + ku_t(x_m, 0)$, but the following technique enjoys better accuracy. Specifically, we set u_m^1 to the right-hand side of the formula

$$\begin{aligned} u(x_m, k) &\approx u(x_m, 0) + ku_t(x_m, 0) + \tfrac{1}{2}k^2 u_{tt}(x_m, 0) \\ &= u(x_m, 0) + ku_t(x_m, 0) + \tfrac{1}{2}k^2 u_{xx}(x_m, 0) \\ &\approx u_m^0 + \tfrac{1}{2}\mu(u_{m-1}^0 - 2u_m^0 + u_{m+1}^0) + ku_t(x_m, 0)\,. \end{aligned}$$

Stability using Fourier analysis

The Fourier analysis (for Cauchy problem) provides

$$\widehat{u}^{n+1}(\theta) - 2\widehat{u}^{n}(\theta) + \widehat{u}^{n-1}(\theta) = -4\mu \sin^2 \tfrac{\theta}{2}\, \widehat{u}^{n}(\theta)\,,$$

with the characteristic equation $\lambda^2 - 2(1 - 2\mu \sin^2 \frac{\theta}{2})\lambda + 1 = 0$. The product of the roots is one, therefore stability (that requires the moduli of both λ to be at most one) is equivalent to the roots being complex conjugate, so we require

$$(1 - 2\mu \sin^2 \tfrac{\theta}{2})^2 \leq 1.$$

This condition is achieved if and only if $\mu = k^2/h^2 \leq 1$.

The diffusion equation in two space dimensions

The diffusion equation in two space dimensions

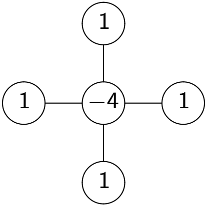
We are solving

$$\frac{\partial u}{\partial t} = \nabla^2 u, \qquad 0 \leq x, y \leq 1, \quad t \geq 0, \tag{5}$$

where $u = u(x, y, t)$, together with initial conditions at $t = 0$ and Dirichlet boundary conditions at $\partial\Omega$, where $\Omega = [0, 1]^2 \times [0, \infty)$. It is straightforward to generalize our derivation of numerical algorithms, e.g. by the method of lines.

Recall the five point formula

We have the *five-point method*



$$u_{i,j} = u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\ ,$$

discretising the two dimensional Laplacian.

Thus, let $u_{\ell,m}(t) \approx u(\ell h, mh, t)$, where $h = \Delta x = \Delta y$, and let $u^n_{\ell,m} \approx u_{\ell,m}(nk)$ where $k = \Delta t$. The five-point formula results in

$$u'_{\ell,m} = \tfrac{1}{h^2}(u_{\ell-1,m} + u_{\ell+1,m} + u_{\ell,m-1} + u_{\ell,m+1} - 4u_{\ell,m}),$$

or in the matrix form

$$\mathbf{u}' = \tfrac{1}{h^2} A_* \mathbf{u}, \qquad \mathbf{u} = (u_{\ell,m}) \in \mathbb{R}^N, \tag{6}$$

where A_* is the block TST matrix of the five-point scheme:

$$A_* = \begin{bmatrix} H & I & & \\ I & \ddots & \ddots & \\ & \ddots & \ddots & I \\ & & I & H \end{bmatrix}, \quad H = \begin{bmatrix} -4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -4 \end{bmatrix}.$$

The diffusion equation in two space dimensions

Thus, the Euler method yields

$$u_{\ell,m}^{n+1} = u_{\ell,m}^{n} + \mu(u_{\ell-1,m}^{n} + u_{\ell+1,m}^{n} + u_{\ell,m-1}^{n} + u_{\ell,m+1}^{n} - 4u_{\ell,m}^{n}), \quad (7)$$

or in the matrix form

$$\mathbf{u}^{n+1} = A\mathbf{u}^{n}, \qquad A = I + \mu A_*$$

where, as before, $\mu = \frac{k}{h^2} = \frac{\Delta t}{(\Delta x)^2}$. The local error is $\boldsymbol{\eta} = \mathcal{O}(k^2 + kh^2) = \mathcal{O}(h^4)$. To analyse stability, we notice that A is symmetric, hence normal, and its eigenvalues are related to those of A_* by the rule

$$\lambda_{k,\ell}(A) = 1 + \mu\lambda_{k,\ell}(A_*) \overset{\text{Prop. 1.12}}{=} 1 - 4\mu\left(\sin^2\frac{\pi k h}{2} + \sin^2\frac{\pi \ell h}{2}\right).$$

Consequently,

$$\sup_{h>0}\rho(A) = \max\{1, |1-8\mu|\}, \qquad \text{hence} \qquad \mu \le \tfrac{1}{4} \quad \Leftrightarrow \quad \text{stability.}$$

Fourier analysis in 2D

Fourier analysis generalizes to two dimensions: of course, we now need to extend the range of (x, y) in (5) from $0 \le x, y \le 1$ to $x, y \in \mathbb{R}$. A 2D Fourier transform reads

$$\widehat{u}(\theta, \psi) = \sum_{\ell, m \in \mathbb{Z}} u_{\ell,m} \mathrm{e}^{-\mathrm{i}(\ell\theta + m\psi)}$$

and all our results readily generalize.

Fourier analysis in 2D

In particular, the Fourier transform is an isometry from $\ell_2[\mathbb{Z}^2]$ to $L_2([-\pi,\pi]^2)$, i.e.

$$\Big(\sum_{\ell,m\in\mathbb{Z}} |u_{\ell,m}|^2\Big)^{1/2} =: \|\mathbf{u}\| = \|\widehat{u}\|_* := \Big(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\widehat{u}(\theta,\psi)|^2\, d\theta\, d\psi\Big)^{1/2},$$

and the method is stable iff $|H(\theta,\psi)| \le 1$ for all $\theta,\psi \in [-\pi,\pi]$. The proofs are an easy elaboration on the one-dimensional theory. Insofar as the Euler method (7) is concerned,

$$H(\theta,\psi) = 1+\mu\left(\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\psi}+\mathrm{e}^{\mathrm{i}\psi}-4\right) = 1-4\mu\big(\sin^2\tfrac{\theta}{2}+\sin^2\tfrac{\psi}{2}\big),$$

and we again deduce stability if and only if $\mu \le \frac{1}{4}$.

Parseval's identity

Lemma 3 (Parseval's identity)

For any $\mathbf{v} \in \ell_2[\mathbb{Z}]$, *we have* $\|\mathbf{v}\| = \|\widehat{v}\|_*$.

Proof. By definition,

$$\begin{aligned}
\|\widehat{v}\|_*^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\sum_{m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}m\theta} v_m\Big|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} v_m \bar{v}_k \mathrm{e}^{-\mathrm{i}(m-k)\theta} d\theta \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} v_m \bar{v}_k \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(m-k)\theta} d\theta \overset{(*)}{=} \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} v_m \bar{v}_k \delta_{m-k} = \|\mathbf{v}\|^2,
\end{aligned}$$

where equality $(*)$ is due to the fact that

$$\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}\ell\theta} d\theta = \begin{cases} 2\pi, & \ell = 0, \\ 0, & \ell \in \mathbb{Z}\setminus\{0\}, \end{cases} \qquad \square$$

The implication of the lemma is that the Fourier transform is an *isometry* of the Euclidean norm. This is an important reason underlying its many applications in mathematics and beyond.

Crank-Nicolson for 2D

Applying the trapezoidal rule to our semi-dicretization (6) we obtain the two-dimensional Crank-Nicolson method:

$$(I - \tfrac{1}{2}\mu A_*)\,\mathbf{u}^{n+1} = (I + \tfrac{1}{2}\mu A_*)\,\mathbf{u}^{n}\,, \tag{8}$$

in which we move from the n-th to the $(n+1)$-st level by solving the system of linear equations $B\mathbf{u}^{n+1} = C\mathbf{u}^n$, or $\mathbf{u}^{n+1} = B^{-1}C\mathbf{u}^n$. For stability, similarly to the one-dimensional case, the eigenvalue analysis implies that $A = B^{-1}C$ is normal and shares the same eigenvectors with B and C, hence

$$\lambda(A) = \frac{\lambda(C)}{\lambda(B)} = \frac{1 + \frac{1}{2}\mu\lambda(A_*)}{1 - \frac{1}{2}\mu\lambda(A_*)} \quad \Rightarrow \quad |\lambda(A)| < 1 \text{ as } \lambda(A_*) < 0$$

and the method is stable for all μ. The same result can be obtained through the Fourier analysis.

Splitting

We would like to find a fast solver to the system (8). The matrix $B = I - \frac{1}{2}\mu A_*$ has a structure similar to that of A_*, where

$$A_* = \begin{bmatrix} H & I & & \\ I & \ddots & \ddots & \\ & \ddots & \ddots & I \\ & & I & H \end{bmatrix}, \quad H = \begin{bmatrix} -4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -4 \end{bmatrix}.$$

so we may apply the Hockney method.

Observation 4 (Special structure of 5-point equations)

We wish to motivate and introduce a family of efficient solution methods for the 5-point equations: the *fast Poisson solvers*. Thus, suppose that we are solving $\nabla^2 u = f$ in a square $m \times m$ grid with the 5-point formula (all this can be generalized a great deal, e.g. to the nine-point formula). Let the grid be enumerated in *natural ordering*, i.e. by columns. Thus, the linear system $A\mathbf{u} = \mathbf{b}$ can be written explicitly in the block form

$$\underbrace{\begin{bmatrix} B & I & & \\ I & B & \ddots & \\ & \ddots & \ddots & I \\ & & I & B \end{bmatrix}}_{A} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_m \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{bmatrix}, \qquad B = \begin{bmatrix} -4 & 1 & & \\ 1 & -4 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -4 \end{bmatrix}_{m \times m},$$

where $\mathbf{u}_k, \mathbf{b}_k \in \mathbb{R}^m$ are portions of $\mathbf{u}$ and $\mathbf{b}$, respectively, and B is a TST-matrix which means *tridiagonal, symmetric* and *Toeplitz* (i.e., constant along diagonals).

Special structure of 5-point equations

Observation 5 (Special structure of 5-point equations)

By Exercise 4, its eigenvalues and orthonormal eigenvectors are given as

$$B\mathbf{q}_\ell = \lambda_\ell \mathbf{q}_\ell, \qquad \lambda_\ell = -4 + 2\cos\tfrac{\ell\pi}{m+1},$$

$$\mathbf{q}_\ell = \gamma_m \big(\sin\tfrac{j\ell\pi}{m+1}\big)_{j=1}^{m}, \qquad \ell = 1..m,$$

where $\gamma_m = \sqrt{\frac{2}{m+1}}$ is the normalization factor. Hence $B = QDQ^{-1} = QDQ$, where $D = \operatorname{diag}(\lambda_\ell)$ and $Q = Q^T = (q_{j\ell})$. Note that all $m \times m$ TST matrices share the same full set of eigenvectors, hence they all commute!

The Hockney method

Set $\mathbf{v}_k = Q\mathbf{u}_k$, $\mathbf{c}_k = Q\mathbf{b}_k$, therefore our system becomes

$$\begin{bmatrix} D & I & & \\ I & D & \ddots & \\ & \ddots & \ddots & I \\ & & I & D \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_m \end{bmatrix}.$$

Let us by this stage reorder the grid *by rows, instead of by columns.*. In other words, we permute $\mathbf{v} \mapsto \widehat{\mathbf{v}} = P\mathbf{v}$, $\mathbf{c} \mapsto \widehat{\mathbf{c}} = P\mathbf{c}$, so that the portion $\widehat{\mathbf{c}}_1$ is made out of the first components of the portions $\mathbf{c}_1, \ldots, \mathbf{c}_m$, the portion $\widehat{\mathbf{c}}_2$ out of the second components and so on.

The Hockney method

This results in new system

$$\begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_m \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_2 \\ \vdots \\ \widehat{\mathbf{v}}_m \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{c}}_1 \\ \widehat{\mathbf{c}}_2 \\ \vdots \\ \widehat{\mathbf{c}}_m \end{bmatrix}, \quad \Lambda_k = \begin{bmatrix} \lambda_k & 1 & & \\ 1 & \lambda_k & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & \lambda_k \end{bmatrix}_{m \times m},$$

where $k = 1...m$.

The Hockney method

These are m *uncoupled* systems, $\Lambda_k \widehat{\mathbf{v}}_k = \widehat{\mathbf{c}}_k$ for $k = 1...m$. Being *tridiagonal*, each such system can be solved fast, at the cost of $\mathcal{O}(m)$. Thus, the steps of the algorithm and their computational cost are as follows.

1. Form the products $\mathbf{c}_k = Q\mathbf{b}_k$, $k = 1...m$ . . . . $\mathcal{O}(m^3)$
2. Solve $m \times m$ tridiagonal systems $\Lambda_k \widehat{\mathbf{v}}_k = \widehat{\mathbf{c}}_k$, $k = 1...m$ . . . . . . $\mathcal{O}(m^2)$
3. Form the products $\mathbf{u}_k = Q\mathbf{v}_k$, $k = 1...m$ . . . . $\mathcal{O}(m^3)$

Splitting

However, since the method (8) has a local truncation error $\mathcal{O}(k^3 + kh^2)$, we don't need an exact solution of the system: it would be enough to have one within the error.
Let us employ the notation

$$\Delta_x^2 u_{\ell,m} = u_{\ell-1,m} - 2u_{\ell,m} + u_{\ell+1,m}, \qquad \Delta_y^2 u_{\ell,m} = u_{\ell,m-1} - 2u_{\ell,m} + u_{\ell,m+1} \,.$$

Then the Crank-Nicolson method calculates $\mathbf{u}^{n+1}$ by solving the system

$$\left[I - \tfrac{1}{2}\mu(\Delta_x^2 + \Delta_y^2)\right] u_{\ell,m}^{n+1} = \left[I + \tfrac{1}{2}\mu(\Delta_x^2 + \Delta_y^2)\right] u_{\ell,m}^{n} \,, \quad \ell, m = 1...M. \tag{9}$$

Splitting

The local error is however preserved if we replace this formula by the difference equation

$$\left[I - \tfrac{1}{2}\mu\Delta_x^2\right]\left[I - \tfrac{1}{2}\mu\Delta_y^2\right] u_{\ell,m}^{n+1} = \left[I + \tfrac{1}{2}\mu\Delta_x^2\right]\left[I + \tfrac{1}{2}\mu\Delta_y^2\right] u_{\ell,m}^{n}\,, \quad (10)$$

which is called the split version of Crank-Nicolson. Indeed, the difference between two schemes is equal to

$$\begin{aligned} &\tfrac{1}{4}\mu^2\Delta_x^2\Delta_y^2(u_{\ell,m}^{n+1} - u_{\ell,m}^{n}) = \tfrac{k^2}{4}\tfrac{1}{h^2}\Delta_x^2\tfrac{1}{h^2}\Delta_y^2\Big(k\tfrac{\partial}{\partial t}u_{\ell,m}^{n} + \mathcal{O}(k^2)\Big) \\ &= \tfrac{k^3}{4}\Big(\tfrac{\partial^2}{\partial x^2}\tfrac{\partial^2}{\partial y^2}\tfrac{\partial}{\partial t}u_{\ell,m}^{n} + \mathcal{O}(k+h^2)\Big) = \mathcal{O}(k^3 + kh^2)\,, \end{aligned} \quad (11)$$

the same magnitude as of the local error.

Splitting

In the matrix form, (10) is equivalent to splitting the matrix A_* into the sum of two matrices A_x and A_y as

$$A_* = A_x + A_y,$$

$$A_x = \begin{bmatrix} -2I & I & & \\ I & \ddots & \ddots & \\ & \ddots & \ddots & I \\ & & I & -2I \end{bmatrix}, \; A_y = \begin{bmatrix} H & & & \\ & H & & \\ & & \ddots & \\ & & & H \end{bmatrix}, \; H = \begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$$

and solving the uncoupled system

$$\left[I - \tfrac{1}{2}\mu A_x\right]\left[I - \tfrac{1}{2}\mu A_y\right]\mathbf{u}^{n+1} = \left[I + \tfrac{1}{2}\mu A_x\right]\left[I + \tfrac{1}{2}\mu A_y\right]\mathbf{u}^{n}.$$

as

$$B_x \mathbf{u}^{n+1/2} = C_x C_y \mathbf{u}^n, \qquad B_y \mathbf{u}^{n+1} = \mathbf{u}^{n+1/2}.$$

Splitting

The matrix

$$B_y = I - \frac{1}{2}\mu A_y$$

is block diagonal, and solving $B_y\mathbf{u} = \mathbf{v}$ is just solving one and the same tridiagonal system $B\mathbf{u}_i = \mathbf{v}_i$ with different right-hand sides. Matrix $B_x = I - \frac{1}{2}\mu A_x$ is of the same form up to a permutation (reodering of the grid), so solving $B_x\mathbf{v} = \mathbf{b}$ is again a fast procedure.

The general diffusion equation

Consider the general diffusion equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla^\top \left(a(x,y)\nabla u\right) + f(x,y) \\ &= \frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(a(x,y)\frac{\partial u}{\partial y}\right) + f(x,y),\end{aligned} \tag{12}$$

where $a(x,y) > \alpha > 0$ and $f(x,y)$ are given, together with initial conditions on $[0,1]^2$ and Dirichlet boundary conditions along $\partial[0,1]^2 \times [0,\infty)$. Replace each space derivative by *central differences* at midpoints,

$$\frac{\mathrm{d}g(\xi)}{\mathrm{d}\xi} \approx \frac{g(\xi+\frac{1}{2}h) - g(\xi-\frac{1}{2}h)}{h},$$

resulting in the ODE system

$$\begin{aligned} u'_{\ell,m} \quad = \quad & \frac{1}{h^2}\Big[a_{\ell-\frac{1}{2},m}u_{\ell-1,m} + a_{\ell+\frac{1}{2},m}u_{\ell+1,m} + a_{\ell,m-\frac{1}{2}}u_{\ell,m-1} + a_{\ell,m+\frac{1}{2}}u_{\ell,m+1} \\ & - \left(a_{\ell-\frac{1}{2},m} + a_{\ell+\frac{1}{2},m} + a_{\ell,m-\frac{1}{2}} + a_{\ell,m+\frac{1}{2}}\right)u_{\ell,m}\Big] + f_{\ell,m}. \end{aligned} \tag{13}$$

The general diffusion equation

Assuming zero boundary conditions and $f \equiv 0$, we have a system $\mathbf{u}' = A\mathbf{u}$, and we may solve it again by Crank–Nicolson, and apply the split

$$A = A_x + A_y.$$

Here, A_x and A_y are again constructed from the contribution of discretizations in the x- and y-directions respectively, namely A_x includes all the $a_{\ell\pm\frac{1}{2},m}$ terms, and A_y consists of the remaining $a_{\ell,m\pm\frac{1}{2}}$ components. Arguments similar to what we used in moving from (9) to (10) justify the use of the split version in this general case as well.