

Mathematical Tripod Part II: Michaelmas Term 2025

Numerical Analysis – Lecture 2

Finite-difference discretization of $\nabla^2 u = f$ replaces the PDE by a large system of linear equations. In the sequel we pay special attention to the *five-point formula*, which results in the approximation

$$h^2 \nabla^2 u(x, y) \approx u(x-h, y) + u(x+h, y) + u(x, y-h) + u(x, y+h) - 4u(x, y). \quad (1.5)$$

For the sake of simplicity, we restrict our attention to the important case of Ω being a *unit square*, where $h = \frac{1}{m+1}$ for some positive integer m . Thus, we estimate the m^2 unknown function values $u(ih, jh)_{i,j=1}^m$ (where $(ih, jh) \in \Omega$) by letting the right-hand side of (1.5) equal $h^2 f(ih, jh)$ at each value of i and j . This yields an $n \times n$ system of linear equations with $n = m^2$ unknowns $u_{i,j}$:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh). \quad (1.6)$$

(Note that when i or j is equal to 1 or m , then the values $u_{0,j}$, $u_{i,0}$ or $u_{i,m+1}$, $u_{m+1,j}$ are known boundary values and they should be moved to the right-hand side, thus leaving fewer unknowns on the left.) Having ordered grid points, we can write (1.6) as a linear system, say

$$A\mathbf{u} = \mathbf{b}.$$

Our present concern is to prove that, as $h \rightarrow 0$, the numerical solution (1.6) tends to the exact solution of the Poisson equation $\nabla^2 u = f$ (with appropriate Dirichlet boundary conditions).

Example 1.8 (Natural ordering) The way the matrix A of this system looks depends of course on the way how the grid points (ih, jh) are being assembled in the one-dimensional array. In the *natural ordering*, when the grid points are arranged by columns, A is the following block tridiagonal matrix:

$$A = \begin{bmatrix} B & I & & & \\ I & B & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & B & I \\ & & & I & B \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix}.$$

Before heading on let us prove the following simple but useful theorem whose importance will become apparent in the course of the lecture.

Theorem 1.9 (Gershgorin theorem) All eigenvalues of an $n \times n$ matrix A are contained in the union of the Gershgorin discs in the complex plane:

$$\sigma(A) \subset \bigcup_{i=1}^n \Gamma_i, \quad \Gamma_i := \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i\}, \quad r_i := \sum_{j \neq i} |a_{ij}|.$$

Proof. For any matrix A , if $A\mathbf{x} = \lambda\mathbf{x}$ and $|x_i| = \max |x_j|$, then the i th equation of the relation $A\mathbf{x} = \lambda\mathbf{x}$ gives

$$|\lambda - a_{ii}| \cdot |x_i| = \left| \sum_{j \neq i} a_{ij} x_j \right| \leq \sum_{j \neq i} |a_{ij}| |x_j| \leq |x_i| \sum_{j \neq i} |a_{ij}| =: |x_i| r_i,$$

and after dividing by $|x_i|$ we obtain $|\lambda - a_{ii}| \leq r_i$. So, for any eigenvalue λ of A , the inequality $|\lambda - a_{ii}| \leq r_i$ is valid for at least one value of i , hence the theorem. \square

Lemma 1.10 For any ordering of the grid points, the matrix A of the system (1.6) is symmetric and negative definite.

Proof. Equation (1.6) implies that if $a_{ij} \neq 0$ for $i \neq j$, then the i -th and j -th points of the grid (ph, qh) , are nearest neighbours. Hence $a_{ij} \neq 0$ implies $a_{ij} = a_{ji} = 1$, which proves the symmetry of A . Therefore A has real eigenvalues and eigenvectors.

It remains to prove that all the eigenvalues are negative. The arguments are parallel to the proof of Gershgorin theorem. Let $Ax = \lambda x$, and let i be an integer such that $|x_i| = \max |x_j|$. With such an i we address the following identity (which is a reordering of the equation $(Ax)_i = \lambda x_i$):

$$\underbrace{|(\lambda - a_{ii}) x_i|}_{|\lambda + 4| |x_i|} = \underbrace{\left| \sum_{j \neq i}^n a_{ij} x_j \right|}_{\leq 4 |x_i|}. \quad (1.7)$$

Here $a_{ii} = -4$ and $a_{ij} \in \{0, 1\}$ for $j \neq i$, with at most four nonzero elements on the right-hand side. It is seen that the case $\lambda > 0$ is impossible. Assuming $\lambda = 0$, we obtain $|x_j| = |x_i|$ whenever $a_{ij} = 1$, so we can alter the value of i in (1.7) to any of such j and repeat the same arguments. Thus, the modulus of every component of x would be $|x_i|$, but then the equations (1.7) that occur at the boundary of the grid and have fewer than four off-diagonal terms (see (1.6)) could not be true. Hence, $\lambda = 0$ is impossible too, hence $\lambda < 0$ which proves that A is negative definite. \square

Proposition 1.11 *The eigenvalues of the matrix A are*

$$\lambda_{k,\ell} = -4 \left(\sin^2 \frac{k\pi h}{2} + \sin^2 \frac{\ell\pi h}{2} \right), \quad h = \frac{1}{m+1}, \quad k, \ell = 1 \dots m.$$

Proof. Let us show that, for every pair (k, ℓ) , the vectors

$$v = (v_{i,j}), \quad v_{i,j} = \sin ix \sin jy, \quad \text{where } x = k\pi h, \quad y = \ell\pi h,$$

are the eigenvectors of A . Indeed, for $i, j = 1 \dots m$, we have

$$\begin{aligned} (Av)_{i,j} &= \sin(jy) [\sin(ix - x) - 2\sin(ix) + \sin(ix + x)] \\ &\quad + \sin(ix) [\sin(jy - y) - 2\sin(jy) + \sin(jy + y)] \\ &= \sin(jy) \sin(ix) [2\cos x - 2] + \sin(ix) \sin(jy) [2\cos y - 2] = \lambda v_{i,j}. \end{aligned}$$

Note that the terms $u_{i\pm 1,j}$, $u_{i,j\pm 1}$ do not appear in (1.6) for $i, j = 1$ or $i, j = m$, respectively, therefore (for such i, j) we should have dropped the corresponding components from above equation, but they are equal to zero because $\sin(i-1)x = 0$ for $i = 1$, while $\sin(i+1)x = 0$ for $i = m$, since $x = \frac{k\pi}{m+1}$. Thus, the eigenvalues are

$$\lambda_{k,\ell} = [2\cos x - 2] + [2\cos y - 2] = -4 \left(\sin^2 \frac{x}{2} + \sin^2 \frac{y}{2} \right) = -4 \left(\sin^2 \frac{k\pi h}{2} + \sin^2 \frac{\ell\pi h}{2} \right). \quad \square$$

Remark 1.12 As a matter of independent mathematical interest, note that for $1 \leq k, \ell \ll m$ we have $\sin x \approx x$, hence the eigenvalues for the discretized Laplacian ∇_h^2 are

$$\frac{\lambda_{k,\ell}}{h^2} \approx -\frac{4}{h^2} \left[\frac{k^2 \pi^2 h^2}{4} + \frac{\ell^2 \pi^2 h^2}{4} \right] = -(k^2 + \ell^2) \pi^2.$$

Now, recall (e.g. from the solution of the Poisson equation in a square by separation of variables in Maths Methods) that the *exact* eigenvalues of ∇^2 (in the unit square) are $-(k^2 + \ell^2) \pi^2$, $k, \ell \in \mathbb{N}$, with the corresponding eigenfunctions $V_{k,\ell}(x, y) = \sin k\pi x \sin \ell\pi y$. So, the eigenvectors of the discretized ∇_h^2 are the values of $V_{k,\ell}(x, y)$ on the grid-points, and the eigenvalues of ∇_h^2 approximate those for continuous case.