Mathematical Tripos Part II: Michaelmas Term 2020

Numerical Analysis – Lecture 8

Problem 2.18 (The advection equation) A useful paradigm for hyperbolic PDEs is the *advection equation*

$$u_t = u_x, \qquad 0 \le x \le 1, \qquad t \ge 0, \tag{2.6}$$

where $u = u(x,t)$. It is given with the initial condition $u(x,0) = \varphi(x)$, $x \in [0,1]$ and (for simplicity) the boundary condition $u(1,t) = \varphi(t+1)$. The exact solution of (2.6) is simply $u(x,t) = \varphi(x+t)$, a unilateral shift leftwards. This, however, does not mean that its numerical modelling is easy.

Example 2.19 (Instability) We commence by semidiscretizing $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{2h}\left[u_{m+1}(t) - u_{m-1}(t)\right]$, so coming to the ODE $u_m'(t) = \frac{1}{2h}\left[u_{m+1}(t) - u_{m-1}(t)\right]$. For the Euler method, the outcome is

$$u_m^{n+1} = u_m^n + \tfrac{1}{2}\mu(u_{m+1}^n - u_{m-1}^n), \qquad m = 0...M, \quad n \in \mathbb{Z}_+,$$

with $u_0^n = 0$ for all n. In matrix form this reads

$$\boldsymbol{u}^{n+1} = A\boldsymbol{u}^n, \qquad A = \begin{bmatrix} 1 & \frac{1}{2}\mu & & \\ -\frac{1}{2}\mu & 1 & \ddots & \\ & \ddots & \ddots & \frac{1}{2}\mu \\ & & -\frac{1}{2}\mu & 1 \end{bmatrix}.$$

The matrix A is normal, with the eigenvalues $\lambda_\ell = 1 + i\mu\cos\ell\pi h$ (see Example 2.15), so that $\|A\|^2 = 1+\mu^2$, hence instability for any μ.

Method 2.20 (Upwind method) If we semidiscretize $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{h}\left[u_{m+1}(t) - u_m(t)\right]$, and solve the ODE again by Euler's method, then the result is

$$u_m^{n+1} = u_m^n + \mu(u_{m+1}^n - u_m^n), \qquad m = 0...M, \quad n \in \mathbb{Z}_+ \tag{2.7}$$

The local error is $\mathcal{O}(k^2 + kh)$ which is $\mathcal{O}(h^2)$ for a fixed μ, hence convergence if the method is stable.

The eigenvalue analysis of stability does not apply here, since the matrix A in $\boldsymbol{u}^{n+1} = A\boldsymbol{u}^n$ is no longer normal (see Example 2.16), so we do it directly (as in Lecture 5). We let the boundary condition at $x = 1$ be zero and define $\|\boldsymbol{u}^n\| = \max_m |u_m^n|$. It follows at once from (2.7) that

$$\|\boldsymbol{u}^{n+1}\| = \max_m |u_m^{n+1}| \le \max_m\{|1-\mu|\,|u_m^n| + \mu\,|u_{m+1}^n|\} \le (|1-\mu| + \mu)\|\boldsymbol{u}^n\|, \qquad n \in \mathbb{Z}_+.$$

Therefore, $\mu \in (0,1]$ means that $\|\boldsymbol{u}^{n+1}\| \le \|\boldsymbol{u}^n\| \le \cdots \le \|\boldsymbol{u}^0\|$, hence stability.

Method 2.21 (The leapfrog method) We semidicretize (2.6) as $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{2h}\left[u_{m+1}(t) - u_{m-1}(t)\right]$, but now solve the ODE with the second-order *midpoint rule*

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_{n-1} + 2k\boldsymbol{f}(t_n, \boldsymbol{y}_n), \qquad n \in \mathbb{Z}_+\,.$$

The outcome is the two-step *leapfrog* method

$$u_m^{n+1} = \mu\,(u_{m+1}^n - u_{m-1}^n) + u_m^{n-1}. \tag{2.8}$$

The error is now $\mathcal{O}(k^3 + kh^2) = \mathcal{O}(h^3)$. We analyse stability by the Fourier technique, assuming that we are solving a Cauchy problem. Thus, proceeding as before,

$$\widehat{u}^{n+1}(\theta) = \mu\left(e^{i\theta} - e^{-i\theta}\right)\widehat{u}^n(\theta) + \widehat{u}^{n-1}(\theta) \tag{2.9}$$

whence

$$\widehat{u}^{n+1}(\theta) - 2i\mu\sin\theta\,\widehat{u}^n(\theta) - \widehat{u}^{n-1}(\theta) = 0, \qquad n \in \mathbb{Z}_+\,,$$

and our goal is to determine values of μ such that $|\widehat{u}^n(\theta)|$ is uniformly bounded for all n, θ. This is a difference equation $w_{n+1} + bw_n + cw_{n-1} = 0$ with the general solution $w_n = c_1\lambda_1^n + c_2\lambda_2^n$, where λ_1, λ_2 are the roots of the characteristic equation $\lambda^2 + b\lambda + c = 0$, and c_1, c_2 are constants, dependent on the initial values w_0 and w_1. If $\lambda_1 = \lambda_2$, then solution is $w_n = (c_1 + c_2 n)\lambda^n$. In our case, we obtain

$$\lambda_{1,2}(\theta) = i\mu\sin\theta \pm \sqrt{1 - \mu^2\sin^2\theta}\,.$$

Stability is equivalent to $|\lambda_{1,2}(\theta)| \le 1$ for all θ and this is true if and only if $\mu \le 1$.

Problem 2.22 (Stability in the presence of boundaries) It is easy to extend Fourier analysis for the Euler method $u_m^{n+1} = u_m^n + \mu(u_{m+1}^n - u_m^n)$, with the initial condition $u(x,0) = \phi(x)$, $x \in [0,1)$, and zero boundary condition along $x = 1$. Consider the Cauchy problem for the advection equation with the initial condition $u(x,0) = \phi(x)$ for $x \in [0,1)$, and $u(x,0) = 0$ otherwise (it isn't differentiable, but this is not much of a problem). Solving the Cauchy problem with Euler, we recover $\boldsymbol{u}^n$ that is identical to the solution obtained from the zero boundary condition. This justifies using Fourier analysis for the problem with a boundary, and we obtain

$$\widehat{u}^{n+1}(\theta) = H(\theta)\,\widehat{u}^n(\theta)\,, \qquad H(\theta) = (1-\mu) + \mu e^{\mathrm{i}\theta}\,,$$

so that $\max|H(\theta)| = |1-\mu| + |\mu|$, hence stability if and only if $\mu \le 1$.

Unfortunately, this is no longer true for leapfrog. Closer examination reveals that we cannot use leapfrog at $m = 0$, since u_{-1}^n is unknown. The naive remedy, setting $u_{-1}^n = 0$, leads to instability, which propagates from the boundary inwards. We can recover stability letting, for example, $u_0^{n+1} = u_1^n$ (the proof is *very* difficult).

Problem 2.23 (The wave equation) Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,, \qquad x \in [0,1], \qquad t \ge 0,$$

given with initial (for u and u_t) and boundary conditions. The usual approximation looks as follows

$$u_m^{n+1} - 2u_m^n + u_m^{n-1} = \mu(u_{m+1}^n - 2u_m^n + u_{m-1}^n)\,,$$

with the Courant number being now $\mu = k^2/h^2$.

To advance in time we have to pick up the numbers $u_m^1 = u(x_m, k)$ (of course they should depend on the initial derivative $u_t(x,0)$. Euler's method provides the obvious choice $u(x_m,k) = u(x_m,0) + ku_t(x_m,0)$, but the following technique enjoys better accuracy. Specifically, we set u_m^1 to the right-hand side of the formula

$$\begin{aligned} u(x_m,k) &\approx u(x_m,0) + ku_t(x_m,0) + \tfrac{1}{2}k^2 u_{tt}(x_m,0) \\ &= u(x_m,0) + ku_t(x_m,0) + \tfrac{1}{2}k^2 u_{xx}(x_m,0) \\ &\approx u_m^0 + \tfrac{1}{2}\mu(u_{m-1}^0 - 2u_m^0 + u_{m+1}^0) + ku_t(x_m,0)\,. \end{aligned}$$

The Fourier analysis (for Cauchy problem) provides

$$\widehat{u}^{n+1}(\theta) - 2\widehat{u}^n(\theta) + \widehat{u}^{n-1}(\theta) = -4\mu\sin^2\tfrac{\theta}{2}\,\widehat{u}^n(\theta)\,,$$

with the characteristic equation $\lambda^2 - 2(1 - 2\mu\sin^2\frac{\theta}{2})\lambda + 1 = 0$. The product of the roots is one, therefore stability (that requires the moduli of both λ to be at most one) is equivalent to the roots being complex conjugate, so we require

$$(1 - 2\mu\sin^2\tfrac{\theta}{2})^2 \le 1.$$

This condition is achieved if and only if $\mu = k^2/h^2 \le 1$.