Mathematical Tripos Part II: Michaelmas Term 2020

Numerical Analysis – Lecture 9

Problem 2.25 (The diffusion equation in two space dimensions) We are solving

$$\frac{\partial u}{\partial t} = \nabla^2 u, \qquad 0 \le x, y \le 1, \quad t \ge 0, \tag{2.11}$$

where $u = u(x, y, t)$, together with initial conditions at $t = 0$ and Dirichlet boundary conditions at $\partial\Omega$, where $\Omega = [0,1]^2 \times [0,\infty)$. It is straightforward to generalize our derivation of numerical algorithms, e.g. by the method of lines. Thus, let $u_{\ell,m}(t) \approx u(\ell h, mh, t)$, where $h = \Delta x = \Delta y$, and let $u^n_{\ell,m} \approx u_{\ell,m}(nk)$ where $k = \Delta t$. The five-point formula results in

$$u'_{\ell,m} = \tfrac{1}{h^2}(u_{\ell-1,m} + u_{\ell+1,m} + u_{\ell,m-1} + u_{\ell,m+1} - 4u_{\ell,m}),$$

or in the matrix form

$$\boldsymbol{u}' = \tfrac{1}{h^2} A_* \boldsymbol{u}, \qquad \boldsymbol{u} = (u_{\ell,m}) \in \mathbb{R}^N , \tag{2.12}$$

where A_* is the block TST matrix of the five-point scheme:

$$A_* = \begin{bmatrix} H & I & & \\ I & \ddots & \ddots & \\ & \ddots & \ddots & I \\ & & I & H \end{bmatrix}, \quad H = \begin{bmatrix} -4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -4 \end{bmatrix}.$$

Thus, the Euler method yields

$$u^{n+1}_{\ell,m} = u^n_{\ell,m} + \mu(u^n_{\ell-1,m} + u^n_{\ell+1,m} + u^n_{\ell,m-1} + u^n_{\ell,m+1} - 4u^n_{\ell,m}), \tag{2.13}$$

or in the matrix form

$$\boldsymbol{u}^{n+1} = A\boldsymbol{u}^n, \qquad A = I + \mu A_*$$

where, as before, $\mu = \frac{k}{h^2} = \frac{\Delta t}{(\Delta x)^2}$. The local error is $\boldsymbol{\eta} = \mathcal{O}(k^2 + kh^2) = \mathcal{O}(h^4)$. To analyse stability, we notice that A is symmetric, hence normal, and its eigenvalues are related to those of A_* by the rule

$$\lambda_{k,\ell}(A) = 1 + \mu\lambda_{k,\ell}(A_*) \overset{\text{Prop. 1.12}}{=} 1 - 4\mu\left(\sin^2\tfrac{\pi k h}{2} + \sin^2\tfrac{\pi \ell h}{2}\right).$$

Consequently,

$$\sup_{h>0} \rho(A) = \max\{1, |1 - 8\mu|\}, \qquad \text{hence} \qquad \mu \le \tfrac{1}{4} \quad \Leftrightarrow \quad \text{stability.}$$

Method 2.26 (Fourier analysis) Fourier analysis generalizes to two dimensions: of course, we now need to extend the range of (x, y) in (2.11) from $0 \le x, y \le 1$ to $x, y \in \mathbb{R}$. A 2D Fourier transform reads

$$\widehat{u}(\theta, \psi) = \sum_{\ell,m\in\mathbb{Z}} u_{\ell,m} \mathrm{e}^{-\mathrm{i}(\ell\theta + m\psi)}$$

and all our results readily generalize. In particular, the Fourier transform is an isometry from $\ell_2[\mathbb{Z}^2]$ to $L_2([-\pi,\pi]^2)$, i.e.

$$\Big(\sum_{\ell,m\in\mathbb{Z}} |u_{\ell,m}|^2\Big)^{1/2} =: \|\boldsymbol{u}\| = \|\widehat{u}\|_* := \Big(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\widehat{u}(\theta,\psi)|^2 \, d\theta \, d\psi\Big)^{1/2},$$

and the method is stable iff $|H(\theta, \psi)| \le 1$ for all $\theta, \psi \in [-\pi, \pi]$. The proofs are an easy elaboration on the one-dimensional theory. Insofar as the Euler method (2.13) is concerned,

$$H(\theta,\psi) = 1 + \mu\left(\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\psi} + \mathrm{e}^{\mathrm{i}\psi} - 4\right) = 1 - 4\mu\big(\sin^2\tfrac{\theta}{2} + \sin^2\tfrac{\psi}{2}\big),$$

and we again deduce stability if and only if $\mu \le \frac{1}{4}$.

Method 2.27 (Crank-Nicolson for 2D) Applying the trapezoidal rule to our semi-dicretization (2.12) we obtain the two-dimensional Crank-Nicolson method:

$$(I - \tfrac{1}{2}\mu A_*)\, \boldsymbol{u}^{n+1} = (I + \tfrac{1}{2}\mu A_*)\, \boldsymbol{u}^n \,, \tag{2.14}$$

in which we move from the n-th to the $(n+1)$-st level by solving the system of linear equations $B\boldsymbol{u}^{n+1} = C\boldsymbol{u}^n$, or $\boldsymbol{u}^{n+1} = B^{-1}C\boldsymbol{u}^n$. For stability, similarly to the one-dimensional case, the eigenvalue analysis implies that $A = B^{-1}C$ is normal and shares the same eigenvectors with B and C, hence

$$\lambda(A) = \frac{\lambda(C)}{\lambda(B)} = \frac{1 + \frac{1}{2}\mu\lambda(A_*)}{1 - \frac{1}{2}\mu\lambda(A_*)} \quad \Rightarrow \quad |\lambda(A)| < 1 \text{ as } \lambda(A_*) < 0$$

and the method is stable for all μ. The same result can be obtained through the Fourier analysis.

Technique 2.28 (Splitting) We would like to find a fast solver to the system (2.14). The matrix $B = I - \frac{1}{2}\mu A_*$ has a structure similar to that of A_*, so we may apply the Hockney method. However, since the method (2.14) has a local truncation error $\mathcal{O}(k^3 + kh^2)$, we don't need an exact solution of the system: it would be enough to have one within the error.

Let us employ the notation

$$\Delta_x^2 u_{\ell,m} = u_{\ell-1,m} - 2u_{\ell,m} + u_{\ell+1,m}, \qquad \Delta_y^2 u_{\ell,m} = u_{\ell,m-1} - 2u_{\ell,m} + u_{\ell,m+1} \,.$$

Then the Crank-Nicolson method calculates $\boldsymbol{u}^{n+1}$ by solving the system

$$\left[I - \tfrac{1}{2}\mu(\Delta_x^2 + \Delta_y^2)\right] u_{\ell,m}^{n+1} = \left[I + \tfrac{1}{2}\mu(\Delta_x^2 + \Delta_y^2)\right] u_{\ell,m}^n \,, \qquad \ell, m = 1...M. \tag{2.15}$$

The local error is however preserved if we replace this formula by the difference equation

$$\left[I - \tfrac{1}{2}\mu\Delta_x^2\right]\left[I - \tfrac{1}{2}\mu\Delta_y^2\right] u_{\ell,m}^{n+1} = \left[I + \tfrac{1}{2}\mu\Delta_x^2\right]\left[I + \tfrac{1}{2}\mu\Delta_y^2\right] u_{\ell,m}^n \,, \tag{2.16}$$

which is called the split version of Crank-Nicolson. Indeed, the difference between two schemes is equal to

$$\begin{aligned} \tfrac{1}{4}\mu^2\Delta_x^2\Delta_y^2(u_{\ell,m}^{n+1} - u_{\ell,m}^n) &= \tfrac{k^2}{4}\tfrac{1}{h^2}\Delta_x^2\tfrac{1}{h^2}\Delta_y^2\Big(k\tfrac{\partial}{\partial t}u_{\ell,m}^n + \mathcal{O}(k^2)\Big) \\ &= \tfrac{k^3}{4}\Big(\tfrac{\partial^2}{\partial x^2}\tfrac{\partial^2}{\partial y^2}\tfrac{\partial}{\partial t}u_{\ell,m}^n + \mathcal{O}(k + h^2)\Big) = \mathcal{O}(k^3 + kh^2)\,, \end{aligned}$$

the same magnitude as of the local error. In the matrix form, (2.16) is equivalent to splitting the matrix A_* into the sum of two matrices A_x and A_y as

$$A_* = A_x + A_y, \qquad A_x = \begin{bmatrix} -2I & I & & \\ I & \ddots & \ddots & \\ & \ddots & \ddots & I \\ & & I & -2I \end{bmatrix}, \quad A_y = \begin{bmatrix} H & & & \\ & H & & \\ & & \ddots & \\ & & & H \end{bmatrix}, \quad H = \begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$$

and solving the uncoupled system

$$\left[I - \tfrac{1}{2}\mu A_x\right]\left[I - \tfrac{1}{2}\mu A_y\right] \boldsymbol{u}^{n+1} = \left[I + \tfrac{1}{2}\mu A_x\right]\left[I + \tfrac{1}{2}\mu A_y\right] \boldsymbol{u}^n \,.$$

as

$$B_x\boldsymbol{u}^{n+1/2} = C_xC_y\boldsymbol{u}^n, \qquad B_y\boldsymbol{u}^{n+1} = \boldsymbol{u}^{n+1/2}.$$

Matrix $B_y = I - \frac{1}{2}\mu A_y$ is block diagonal, and solving $B_y\boldsymbol{u} = \boldsymbol{v}$ is just solving one and the same tridiagonal system $B\boldsymbol{u}_i = \boldsymbol{v}_i$ with different right-hand sides. Matrix $B_x = I - \frac{1}{2}\mu A_x$ is of the same form up to a permutation (reodering of the grid), so solving $B_x\boldsymbol{v} = \boldsymbol{b}$ is again a fast procedure.