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Mathematical Tripos Part II: Michaelmas Term 2020

Numerical Analysis – Lecture 11

3 Spectral Methods

Discussion 3.1 (Large matrices versus small matrices) Finite difference schemes rest upon the replacement of derivatives by a linear combination of function values. This leads to the solution of a system of algebraic equations, which on the one hand tends to be large (due to the slow convergence properties of the approximation) but on the other hand is highly structured and sparse, leading itself to effective algorithms for its solution. We will get to know some of these algorithms in Section 4.

However, an enticing alternative to this strategy are methods that produce small matrices in the first place. Although, these matrices will usually not be sparse anymore, the much smaller the size of the matrices renders its solution affordable. The key point for such approximations are better convergence properties requiring much smaller number of parameters.

Problem 3.2 (Fourier approximation of functions) We consider the *truncated Fourier approximation* of a function f on the interval $[-1, 1]$:

$$f(x) \approx \phi_N(x) = \sum_{n=-N/2+1}^{N/2} \widehat{f}_n e^{i\pi nx}, \quad x \in [-1, 1], \tag{3.1}$$

where here and elsewhere in this section $N \geq 2$ is an even integer and

$$\widehat{f}_n = \frac{1}{2}\int_{-1}^{1} f(t)e^{-i\pi nt}\, dt, \quad n \in \mathbb{Z}$$

are the (Fourier) coefficients of this approximation. We want to analyse the approximation properties of (3.1).

Theorem 3.3 (The de la Valleé Poussin theorem) *If the function f is Riemann integrable and $\widehat{f}_n = \mathcal{O}(n^{-1})$ for $|n| \gg 1$, then $\phi_N(x) = f(x) + \mathcal{O}(N^{-1})$ as $N \to \infty$ for every point $x \in (-1, 1)$ where f is Lipschitz.*

Remark 3.4 (The Gibbs effect at the end points) Note that if f is smoothly differentiable then, integrating by parts,

$$\widehat{f}_n = \frac{(-1)^{n+1}}{2\pi in}[f(1) - f(-1)] + \frac{1}{\pi in}\widehat{f'_n} = \mathcal{O}(n^{-1}) \text{ for } |n| \gg 1.$$

Since such an f is Lipschitz on $(-1, 1)$, we deduce from Theorem 3.3 that ϕ_N converges to f there with speed $\mathcal{O}(N^{-1})$. However, convergence with speed $\mathcal{O}(N^{-1})$ is very slow and moreover, we cannot guarantee convergence at the endpoints -1 and 1. In fact, it is possible to show that

$$\phi_N(\pm 1) \to \frac{1}{2}[f(-1) + f(1)] \text{ as } n \to \infty$$

and hence, unless f is periodic we fail to converge.

Method 3.5 (Fourier approximation for periodic functions) Suppose f is an analytic function in $[-1, 1]$, that can be extended analytically to a closed complex domain Ω. In addition let f be periodic with period 2. In particular, $f^{(m)}(-1) = f^{(m)}(1)$ for all $m \in \mathbb{Z}_+$. Then, by multiple integration by parts, we get

$$\widehat{f}_n = \frac{1}{\pi in}\widehat{f'_n} = \frac{1}{(\pi in)^2}\widehat{f''_n} = \frac{1}{(\pi in)^3}\widehat{f'''_n} = \dots.$$

Thus, we have

$$\widehat{f}_n = \frac{1}{(\pi i n)^m} \widehat{f_n^{(m)}}, \quad m = 0, 1, \dots. \tag{3.2}$$

But, how large is $|\widehat{f_n^{(m)}}|$? To answer this question we use Cauchy's theorem of complex analysis, which states that

$$f^{(m)}(x) = \frac{m!}{2\pi i} \int_\gamma \frac{f(z)\, dz}{(z-x)^{m+1}}, \quad x \in [-1, 1],$$

where γ is the positively oriented boundary of Ω. Therefore, with $\alpha^{-1} > 0$ being the minimal distance between γ and $[-1, 1]$ and $M = \max\{|f(z)| : z \in \gamma\} < \infty$, it follows that

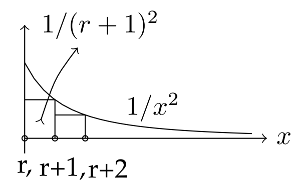
$$|f^{(m)}(x)| \le \frac{m!}{2\pi} \int_\gamma \frac{|f(z)|\, |dz|}{|z-x|^{m+1}} \le \frac{M \text{ length } \gamma}{2\pi} m!\, \alpha^{m+1},$$

and hence, we can bound $|\widehat{f_n^{(m)}}| \le c\, m!\, \alpha^{m+1}$ for some $c > 0$. Now, using (3.2) and the above upper bound,

$$\begin{aligned} |\phi_N(x) - f(x)| &= \Big| \sum_{n=-N/2+1}^{N/2} \widehat{f}_n e^{i\pi n x} - \sum_{n=-\infty}^{\infty} \widehat{f}_n e^{i\pi n x} \Big| \\ &\le \sum_{|n| \ge N/2} |\widehat{f}_n| = \sum_{|n|\ge N/2} \frac{|\widehat{f_n^{(m)}}|}{|\pi n|^m} \le \frac{c m! \alpha^{m+1}}{\pi^m} \sum_{n=N/2}^{+\infty} \frac{1}{n^m}. \end{aligned}$$

Using, that for any $r \in \mathbb{N}$, and $m > 1$

$$\sum_{n=r+1}^{+\infty} \frac{1}{n^m} \le \int_r^\infty \frac{dt}{t^m} = \frac{1}{m-1} r^{-m+1},$$



we deduce that

$$|\phi_N(x) - f(x)| \le c' m! \left(\frac{\alpha}{\pi N}\right)^{m-1}, \quad m \ge 2.$$

Finally, we have a competition between $(\alpha/(\pi N))^{m-1}$ and $m!$ for large m. Because of Stirling's formula

$$m! \approx \sqrt{2\pi}\, m^{m+1/2} e^{-m}$$

we have

$$m! \Big(\frac{\alpha}{\pi N}\Big)^{m-1} \approx \sqrt{2\pi m}\, \frac{m}{e} \Big(\frac{\alpha m}{\pi e N}\Big)^{m-1}$$

which becomes very small for large N. Hence, $|\phi_N - f| = \mathcal{O}(N^{-p})$ for any $p \in \mathbb{N}$ and we deduce that the Fourier approximation of an analytic periodic function is of infinite order.

Definition 3.6 (Convergence at spectral speed) An N-term approximation ϕ_N of a function f converges to f at *spectral* speed if $\|\phi_N - f\|$ decays faster than $\mathcal{O}(N^{-p})$ for any $p = 1, 2, \dots$.

Remark 3.7 It is possible to prove that there exist constants $c_1, w > 0$ such that $\|\phi_N - f\| \le c_1 e^{-wN}$ for all $N \in \mathbb{N}$ uniformly in $[-1, 1]$. Thus, convergence is at least at an exponential rate.