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Mathematical Tripos Part II: Michaelmas Term 2020

Numerical Analysis – Lecture 12

Method 3.8 (The algebra of Fourier expansions) *Let $\mathcal{A}$ be the set of all functions $f : [-1,1] \to C$, which are analytic in $[-1,1]$, periodic with period 2, and that can be extended analytically into the complex plane.* Then $\mathcal{A}$ is a linear space, i.e., $f, g \in \mathcal{A}$ and $\alpha \in \mathbb{C}$ then $f+g \in \mathcal{A}$ and $af \in \mathcal{A}$. In particular, with f and g expressed in its Fourier series, i.e.,

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}_n e^{i\pi n x}, \quad g(x) = \sum_{n=-\infty}^{\infty} \widehat{g}_n e^{i\pi n x}$$

we have

$$f(x) + g(x) = \sum_{n=-\infty}^{\infty} (\widehat{f}_n + \widehat{g}_n) e^{i\pi n x}, \quad \alpha f(x) = \sum_{n=-\infty}^{\infty} \alpha \widehat{f}_n e^{i\pi n x} \tag{3.3}$$

and

$$f(x) \cdot g(x) = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} \widehat{f}_{n-m} \widehat{g}_m \right) e^{i\pi n x} = \sum_{n=-\infty}^{\infty} \left(\widehat{f} * \widehat{g}\right)_n e^{i\pi n x}, \tag{3.4}$$

where $*$ denotes the convolution operator, hence $\widehat{(f \cdot g)}_n = (\widehat{f} * \widehat{g})_n$. Moreover, if $f \in \mathcal{A}$ then $f' \in \mathcal{A}$ and

$$f'(x) = i\pi \sum_{n=-\infty}^{\infty} n \cdot \widehat{f}_n e^{i\pi n x}. \tag{3.5}$$

Since $\{\widehat{f}_n\}$ decays faster than $\mathcal{O}(n^{-p})$ for any $p \in \mathbb{N}$, this provides that all derivatives of f have rapidly convergent Fourier expansions.

Example 3.9 (Application to differential equations) Consider the two-point boundary value problem: $y = y(x)$, $-1 \le x \le 1$, solves

$$y'' + a(x)y' + b(x)y = f(x), \quad y(-1) = y(1), \tag{3.6}$$

where $a, b, f \in \mathcal{A}$ and we seek a *periodic solution* $y \in \mathcal{A}$ for (3.6). Substituting y, a, b and f by their Fourier series and using (3.3)-(3.5) we obtain an infinite dimensional system of linear equations for the Fourier coefficients $\widehat{y}_n$:

$$-\pi^2 n^2 \widehat{y}_n + i\pi \sum_{m=-\infty}^{\infty} m \widehat{a}_{n-m} \widehat{y}_m + \sum_{m=-\infty}^{\infty} \widehat{b}_{n-m} \widehat{y}_m = \widehat{f}_n, \quad n \in \mathbb{Z}. \tag{3.7}$$

Since $a, b, f \in \mathcal{A}$, their Fourier coefficients decrease rapidly, like $\mathcal{O}(n^{-p})$ for every $p \in \mathbb{N}$. Hence, we can truncate (3.7) into the N-dimensional system

$$-\pi^2 n^2 \widehat{y}_n + i\pi \sum_{m=-N/2+1}^{N/2} m \widehat{a}_{n-m} \widehat{y}_m + \sum_{m=-N/2+1}^{N/2} \widehat{b}_{n-m} \widehat{y}_m = \widehat{f}_n, \qquad n = -N/2+1, \ldots, N/2. \tag{3.8}$$

Remark 3.10 The matrix of (3.8) is in general dense, but our theory predicts that fairly small values of N, hence very small matrices, are sufficient for high accuracy. For instance: choosing $a(x) = f(x) = \cos \pi x$, $b(x) = \sin 2\pi x$ (which incidentally even leads to a sparse matrix) we get

$N = 16$	error of size 10^{-10}
$N = 22$	error of size 10^{-15} (which is already hitting the accuracy of computer arithmetic)

Method 3.11 (Computation of Fourier coefficients (DFT)) We have to compute

$$\widehat{f}_n = \frac{1}{2}\int_{-1}^{1} f(t)e^{-i\pi nt}\,dt, \quad n \in \mathbb{Z}. \tag{3.9}$$

For this, suppose we wish to compute the integral on $[-1,1]$ of a function $h \in \mathcal{A}$ by means of the Riemann sums on the uniform partition

$$\int_{-1}^{1} h(t)\,dt \approx \frac{2}{N}\sum_{k=-N/2+1}^{N/2} h\left(\frac{2k}{N}\right). \tag{3.10}$$

This is known as a *rectangle rule*. We want to know how good this approximation is. As in Definition 1.18, let $\omega_N = e^{2\pi i/N}$. Then we have

$$\frac{2}{N}\sum_{k=-N/2+1}^{N/2} h\left(\frac{2k}{N}\right) \;=\; \frac{2}{N}\sum_{k=-N/2+1}^{N/2}\sum_{n=-\infty}^{\infty}\widehat{h}_n e^{2\pi ink/N} = \frac{2}{N}\sum_{n=-\infty}^{\infty}\widehat{h}_n \sum_{k=-N/2+1}^{N/2}\omega_N^{nk}.$$

Since $\omega_N^N = 1$ we have

$$\sum_{k=-N/2+1}^{N/2}\omega_N^{nk} = \omega_N^{-n(N/2-1)}\sum_{k=0}^{N-1}\omega_N^{nk} = \begin{cases} N, & n \equiv 0 \pmod N, \\ 0, & n \not\equiv 0 \pmod N, \end{cases}$$

and we deduce that

$$\frac{2}{N}\sum_{k=-N/2+1}^{N/2} h\left(\frac{2k}{N}\right) = 2\sum_{r=-\infty}^{\infty}\widehat{h}_{Nr}.$$

Hence, the error committed by the Riemann approximation is

$$\begin{aligned} e_N(h) \;&:=\; \frac{2}{N}\sum_{k=-N/2+1}^{N/2} h\left(\frac{2k}{N}\right) - \int_{-1}^{1} h(t)\,dt \;=\; 2\sum_{r=-\infty}^{\infty}\widehat{h}_{Nr} - 2\widehat{h}_0 \\ &=\; 2\sum_{r=1}^{\infty}\left(\widehat{h}_{Nr} + \widehat{h}_{-Nr}\right). \end{aligned}$$

Since $h \in \mathcal{A}$, its Fourier coefficients decay at spectral rate, namely $\widehat{h}_{Nr} = \mathcal{O}((Nr)^{-p})$, and hence the error of the Riemann sums approximation (3.10) decays spectrally as a function of N,

$$e_N(h) = \mathcal{O}(N^{-p}) \quad \forall p \in \mathbb{N}.$$

Going back to the computation of the Fourier coefficients (3.9), we see that we may compute the integral of $h(x) = \frac{1}{2}f(x)e^{-i\pi nx}$ by means of the Riemann sums, and this gives a spectral method for calculating the Fourier coefficients of f:

$$\widehat{f}_n \approx \frac{1}{N}\sum_{k=-N/2+1}^{N/2} f\left(\frac{2k}{N}\right)\omega_N^{-nk}, \qquad n = -N/2+1, \ldots, N/2. \tag{3.11}$$

Remark 3.12 One can recognise that formula (3.11) is the *discrete Fourier transform (DFT)* of the sequence $(y_k) = \left(f(\frac{2k}{N})\right)$, see previous definition, hence not only have we a spectral rate of convergence, but also a fast algorithm (FFT) of computing the Fourier coefficients.

Revision 3.13 (The fast Fourier transform (FFT)) The *fast Fourier transform (FFT)* is a computational algorithm, which computes the leading N Fourier coefficients of a function in just $\mathcal{O}(N\log_2 N)$ operations (cf. Algorithm 1.19). We assume that N is a power of 2, i.e. $N = 2m = 2^p$, and for $\boldsymbol{y} \in \Pi_{2m}$, denote by

$$\boldsymbol{y}^{(\mathrm{E})} = \{y_{2j}\}_{j\in\mathbb{Z}} \qquad \text{and} \qquad \boldsymbol{y}^{(\mathrm{O})} = \{y_{2j+1}\}_{j\in\mathbb{Z}}$$

the even and odd portions of $\boldsymbol{y}$, respectively. Note that $\boldsymbol{y}^{(\mathrm{E})}, \boldsymbol{y}^{(\mathrm{O})} \in \Pi_m$. To execute FFT, we start from vectors of unit length and in each s-th stage, $s = 1...p$, assemble 2^{p-s} vectors of length 2^s from vectors of length 2^{s-1} with

$$x_\ell = x_\ell^{(\mathrm{E})} + \omega_{2^s}^\ell x_\ell^{(\mathrm{O})}, \qquad \ell = 0, \ldots, 2^{s-1}-1. \tag{3.12}$$

Therefore, it costs just s products to evaluate the first half of $\boldsymbol{x}$, provided that $\boldsymbol{x}^{(\mathrm{E})}$ and $\boldsymbol{x}^{(\mathrm{O})}$ are known. It actually costs nothing to evaluate the second half, since

$$x_{2^{s-1}+\ell} = x_\ell^{(\mathrm{E})} - \omega_{2^s}^\ell x_\ell^{(\mathrm{O})}, \qquad \ell = 0, \ldots, 2^{s-1}-1.$$

Altogether, the cost of FFT is $p2^{p-1} = \frac{1}{2} N \log_2 N$ products.