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Mathematical Tripos Part II: Michaelmas Term 2025

Numerical Analysis – Lecture 13

Problem 3.13 (The Poisson equation) We consider the *Poisson equation*

$$\nabla^2 u = f, \quad -1 \le x, y \le 1, \tag{3.11}$$

where f is analytic and obeys the periodic boundary conditions

$$f(-1, y) = f(1, y), \quad -1 \le y \le 1, \qquad f(x, -1) = f(x, 1), \quad -1 \le x \le 1.$$

Moreover, we add to (3.11) the following *periodic boundary conditions*

$$\begin{aligned} u(-1, y) &= u(1, y), \quad u_x(-1, y) = u_x(1, y), \quad -1 \le y \le 1 \\ u(x, -1) &= u(x, 1), \quad u_y(x, -1) = u_y(x, 1), \quad -1 \le x \le 1. \end{aligned} \tag{3.12}$$

With these boundary conditions alone, a solution of (3.11) is only defined up to an additive constant. Hence, we add a *normalisation condition* to fix the constant:

$$\int_{-1}^{1} \int_{-1}^{1} u(x, y)\, dx\, dy = 0. \tag{3.13}$$

We have the spectrally convergent Fourier expansion

$$f(x, y) = \sum_{k,l=-\infty}^{\infty} \widehat{f}_{k,\ell} e^{i\pi(kx+\ell y)}$$

and seek the Fourier expansion of u

$$u(x, y) = \sum_{k,\ell=-\infty}^{\infty} \widehat{u}_{k,\ell} e^{i\pi(kx+\ell y)}.$$

Since

$$0 = \int_{-1}^{1} \int_{-1}^{1} u(x, y)\, dx\, dy = \sum_{k,\ell=-\infty}^{\infty} \widehat{u}_{k,\ell} \int_{-1}^{1} \int_{-1}^{1} e^{i\pi(kx+\ell y)}\, dx\, dy = \widehat{u}_{0,0},$$

and

$$\nabla^2 u(x, y) = -\pi^2 \sum_{k,\ell=-\infty}^{\infty} (k^2 + \ell^2) \widehat{u}_{k,\ell} e^{i\pi(kx+\ell y)},$$

together with (3.11), we have

$$\begin{cases} \widehat{u}_{k,\ell} = -\dfrac{1}{(k^2+\ell^2)\pi^2} \widehat{f}_{k,\ell}, & k, \ell \in \mathbb{Z},\ (k, \ell) \ne (0, 0) \\ \widehat{u}_{0,0} = 0. \end{cases}$$

Remark 3.14 Applying a spectral method to the Poisson equation is not representative for its application to other PDEs. The reason is the special structure of the Poisson equation. In fact, $\phi_{k,\ell} = e^{i\pi(kx+\ell y)}$ are the eigenfunctions of the Laplace operator with

$$\nabla^2 \phi_{k,\ell} = -\pi^2 (k^2 + \ell^2) \phi_{k,\ell},$$

and they obey periodic boundary conditions.

Problem 3.15 (General second-order linear elliptic PDE) We consider the more general second-order linear elliptic PDE

$$\nabla^\top (a\nabla u) = f, \quad -1 \le x, y \le 1,$$

with $a(x,y) > 0$, and a and f periodic. We again impose the periodic boundary conditions (3.12) and the normalisation condition (3.13). We rewrite

$$\nabla^\top (a\nabla u) = \frac{\partial}{\partial x}(a u_x) + \frac{\partial}{\partial y}(a u_y) = f\,,$$

and use the Fourier expansions

$$g(x,y) = \sum_{k,\ell\in\mathbb{Z}} \widehat{g}_{k,\ell}\phi_{k,\ell}(x,y), \qquad h(x,y) = \sum_{m,n\in\mathbb{Z}} \widehat{h}_{m,n}\phi_{m,n}(x,y),$$

together with the bivariate versions of (3.4)-(3.5)

$$\widehat{(g\cdot h)}_{k,\ell} = \sum_{m,n\in\mathbb{Z}} \widehat{g}_{k-m,\ell-n}\widehat{h}_{m,n}, \qquad \widehat{(g_x)}_{k,\ell} = i\pi k\,\widehat{g}_{k,\ell}\,, \qquad \widehat{(g_y)}_{k,\ell} = i\pi \ell\,\widehat{g}_{k,\ell}\,,$$

$$\widehat{(h_x)}_{m,n} = i\pi m\,\widehat{h}_{m,n}\,, \qquad \widehat{(h_y)}_{m,n} = i\pi n\,\widehat{h}_{m,n}\,.$$

This gives

$$-\pi^2 \sum_{k,\ell\in\mathbb{Z}} \sum_{m,n\in\mathbb{Z}} (km+\ell n)\,\widehat{a}_{k-m,\ell-n}\widehat{u}_{m,n}\phi_{k,\ell}(x,y) = \sum_{k,\ell\in\mathbb{Z}} \widehat{f}_{k,\ell}\phi_{k,\ell}(x,y)\,.$$

In the next steps, we truncate the expansions to $-N/2+1 \le k,\ell,m,n \le N/2$ and impose the normalisation condition $\widehat{u}_{0,0} = 0$. This results in a system of N^2-1 linear algebraic equations in the unknowns $\widehat{u}_{m,n}$, where $m,n = -N/2+1...N/2$, and $(m,n)\neq(0,0)$:

$$\sum_{m,n=-N/2+1}^{N/2} (km+\ell n)\,\widehat{a}_{k-m,\ell-n}\,\widehat{u}_{m,n} = -\frac{1}{\pi^2}\,\widehat{f}_{k,\ell}\,, \qquad k,\ell = -N/2+1...N/2\,.$$

Discussion 3.16 (Analyticity and periodicity) The fast convergence of spectral methods rests on two properties of the underlying problem: analyticity and periodicity. If one is not satisfied the rate of convergence in general drops to polynomial. However, to a certain extent, we can relax these two assumptions while still retaining the substantive advantages of Fourier expansions.

- *Relaxing analyticity*: In general, the speed of convergence of the truncated Fourier series of a function f depends on the smoothness of the function. In fact, the smoother the function the faster the truncated series converges, i.e., for $f\in C^p(-1,1)$ we receive an $\mathcal{O}(N^{-p})$ order of convergence.

  Spectral convergence can be recovered, once analyticity is replaced by the requirement that $f\in C^\infty(-1,1)$, i.e., $f^{(m)}(x)$ exists for all $x\in(-1,1)$ and $m = 0,1,2,\ldots$. Consider, for instance, $f(x) = e^{-1/(1-x^2)}$. Then, $f\in C^\infty(-1,1)$ but cannot be extended analytically because of essential singularities at ±1. Nevertheless, one can show that $|\widehat{f}_n| \sim \mathcal{O}(e^{-cn^\alpha})$, where $c>0$ and $\alpha\approx 0.44$. While this is slower than exponential convergence in the analytic case (cf. Remark 3.7), it is still faster than $\mathcal{O}(n^{-m})$ for any integer m and hence, we have spectral convergence.

- *Relaxing periodicity*: Disappointingly, periodicity is necessary for spectral convergence. Once this condition is dropped, we are back to the setting of Theorem 3.3, i.e., Fourier series converge as $\mathcal{O}(N^{-1})$ unless $f(-1) = f(1)$. One way around this is to change our set of basis functions, e.g., to Chebyshev polynomials.