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Mathematical Tripos Part II: Michaelmas Term 2024

Numerical Analysis – Lecture 14

Revision 3.17 (Chebyshev polynomials) The Chebyshev polynomial of degree n is defined as

$$T_n(x) := \cos n \arccos x, \quad x \in [-1, 1],$$

or, in a more instructive form,

$$T_n(x) := \cos n\theta, \quad x = \cos\theta, \quad \theta \in [0, \pi]. \tag{3.14}$$

1) The sequence (T_n) obeys the three-term recurrence relation

$$\begin{aligned} &T_0(x) \equiv 1, \quad T_1(x) = x, \\ &T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \end{aligned}$$

in particular, T_n is indeed an algebraic polynomial of degree n, with the leading coefficient 2^{n-1}. (The recurrence is due to the equality $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$ via substitution $x = \cos\theta$, expressions for T_0 and T_1 are straightforward.)

2) Also, (T_n) form a sequence of orthogonal polynomials with respect to the inner product $(f, g)_w := \int_{-1}^1 f(x)g(x)w(x)dx$, with the weight function $w(x) := (1 - x^2)^{-1/2}$. Namely, we have

$$(T_n, T_m)_w = \int_{-1}^1 T_m(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}} = \int_0^\pi \cos m\theta \cos n\theta\, d\theta = \begin{cases} \pi, & m = n = 0\,, \\ \frac{\pi}{2}, & m = n \geq 1\,, \\ 0, & m \neq n\,. \end{cases} \tag{3.15}$$

Method 3.18 (Chebyshev expansion) Since $(T_n)_{n=0}^\infty$ form an orthogonal sequence, a function f such that $\int_{-1}^1 |f(x)|^2 w(x)\,dx < \infty$ can be expanded in the series

$$f(x) = \sum_{n=0}^\infty \breve{f}_n T_n(x),$$

with the Chebyshev coefficients $\breve{f}_n$. Making inner product of both sides with T_n and using orthogonality yields

$$(f, T_n)_w = \breve{f}_n (T_n, T_n)_w \quad \Rightarrow \quad \breve{f}_n = \frac{(f, T_n)_w}{(T_n, T_n)_w} = \frac{c_n}{\pi} \int_{-1}^1 f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}}\,, \tag{3.16}$$

where $c_0 = 1$ and $c_n = 2$ for $n \geq 1$.

Connection to the Fourier expansion. Letting $x = \cos\theta$ and $g(\theta) = f(\cos\theta)$, we obtain

$$\int_{-1}^1 f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}} = \int_0^\pi f(\cos\theta)T_n(\cos\theta)\,d\theta = \frac{1}{2}\int_{-\pi}^\pi g(\theta)\cos n\theta\,d\theta\,. \tag{3.17}$$

Given that $\cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta})$, and using the Fourier expansion of the 2π-periodic function g,

$$g(\theta) = \sum_{n\in\mathbb{Z}} \widehat{g}_n \mathrm{e}^{in\theta}, \quad \text{where} \quad \widehat{g}_n = \frac{1}{2\pi}\int_{-\pi}^\pi g(t)e^{-int}\,dt, \quad n \in \mathbb{Z}\,,$$

we continue (3.17) as

$$\int_{-1}^1 f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}(\widehat{g}_{-n} + \widehat{g}_n)\,,$$

and from (3.16) we deduce that

$$\breve{f}_n = \begin{cases} \widehat{g}_0, & n = 0\,, \\ \widehat{g}_{-n} + \widehat{g}_n, & n \geq 1\,. \end{cases}$$

Discussion 3.19 (Properties of the Chebyshev expansion) As we have seen, for a general integrable function f, the computation of its Chebyshev expansion is equivalent to the Fourier expansion of the function $g(\theta) = f(\cos\theta)$. Since the latter is periodic with period 2π, we can use a discrete Fourier transform (DFT) to compute the Chebyshev coefficients $\breve{f}_n$. [Actually, based on this connection, one can perform a direct fast Chebyshev transform].

Also, if f can be analytically extended from $[-1, 1]$ (to the so-called Bernstein ellipse), then $\breve{f}_n$ decays spectrally fast for $n \gg 1$ (with the rate depending on the size of the ellipse). Hence, the Chebyshev expansion inherits the rapid convergence of spectral methods without assuming that f is periodic.

Method 3.20 (The algebra of Chebyshev expansions) Let $\mathcal{B}$ be the set of analytic functions in $[-1, 1]$ that can be extended analytically into the complex plane. We identify each such function with its Chebyshev expansion. Like the set $\mathcal{A}$, the set $\mathcal{B}$ is a linear space and is closed under multiplication. In particular, we have

$$\begin{aligned} T_m(x)T_n(x) &= \cos(m\theta)\cos(n\theta) \\ &= \tfrac{1}{2}\left[\cos((m-n)\theta) + \cos((m+n)\theta)\right] \\ &= \tfrac{1}{2}\left[T_{|m-n|}(x) + T_{m+n}(x)\right] \end{aligned}$$

and hence,

$$\begin{aligned} f(x)g(x) &= \sum_{m=0}^{\infty} \breve{f}_m T_m(x) \cdot \sum_{n=0}^{\infty} \breve{g}_n T_n(x) = \frac{1}{2} \sum_{m,n=0}^{\infty} \breve{f}_m \breve{g}_n \left[T_{|m-n|}(x) + T_{m+n}(x)\right] \\ &= \frac{1}{2} \sum_{m,n=0}^{\infty} \breve{f}_m \Big(\breve{g}_{|m-n|} + \breve{g}_{m+n}\Big) T_n(x). \end{aligned}$$

Lemma 3.21 (Derivatives of Chebyshev polynomials) *We can express derivatives T'_n in terms of (T_k) as follows,*

$$T'_{2n}(x) = (2n) \cdot 2 \sum_{k=1}^{n} T_{2k-1}(x), \tag{3.18}$$

$$T'_{2n+1}(x) = (2n+1)\left[T_0(x) + 2\sum_{k=1}^{n} T_{2k}(x)\right]. \tag{3.19}$$

Proof. From (3.14), we deduce

$$T_m(x) = \cos m\theta \quad \Rightarrow \quad T'_m(x) = \frac{m \sin m\theta}{\sin\theta} \qquad x = \cos\theta\,.$$

So, for $m = 2n$, (3.18) follows from the identity $\frac{\sin 2n\theta}{\sin\theta} = 2\sum_{k=1}^{n} \cos(2k-1)\theta$, which is verified as

$$2\sin\theta \sum_{k=1}^{n} \cos(2k-1)\theta = \sum_{k=1}^{n} 2\cos(2k-1)\theta \sin\theta = \sum_{k=1}^{n} \left[\sin 2k\theta - \sin(2k-1)\theta\right] = \sin 2n\theta.$$

For $m = 2n+1$, (3.19) turns into identity $\frac{\sin(2n+1)\theta}{\sin\theta} = 1 + 2\sum_{k=1}^{n} \cos 2k\theta$, and that follows from

$$\sin\theta\Big(1 + 2\sum_{k=1}^{n} \cos 2k\theta\Big) = \sin\theta + \sum_{k=1}^{n} \left[\sin(2k+1)\theta - \sin(2k-1)\theta\right] = \sin(2n+1)\theta.$$

□

Remark 3.22 (Application to PDEs) With Lemma 3.21 all derivatives of u can be expressed in an explicit form as a Chebyshev expansion (cf. Exercise 19 on Example Sheets). For the computation of the Chebyshev coefficients the function f has to be sampled at the so-called Chebyshev points $\cos(2\pi k/N)$, $k = -N/2+1, \ldots, N/2$. This results into a grid, which is denser towards the edges. For elliptic problems this is not problematic, however for initial value PDEs such grids can cause numerical instabilities.