Dr A. Hansen

Mathematical Tripos Part II: Michaelmas Term 2020

Numerical Analysis – Lecture 15

Remark 3.23 (Chebyshev expansion for the derivatives) For an analytic function u, the coefficients $\breve{u}_n^{(k)}$ of the Chebyshev expansion for its derivatives are given by the following recursion,

$$\breve{u}_n^{(k)} = c_n \sum_{\substack{m=n+1 \\ n+m \text{ odd}}}^{\infty} m\, \breve{u}_m^{(k-1)}, \quad \forall\, k \geq 1,$$

where $c_0 = 1$ and $c_n = 2$ for $n \geq 1$. This can be derived from Lemma 3.21 (the case $m = 1$ is the topic of Ex. 19 on the Example Sheets).

Method 3.24 (The spectral method for evolutionary PDEs) We consider the problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \mathcal{L}u(x,t), & x \in [-1,1], \quad t \geq 0\,, \\ u(x,0) = g(x), & x \in [-1,1], \end{cases} \tag{3.20}$$

with appropriate boundary conditions on $\{-1,1\} \times \mathbb{R}_+$ and where $\mathcal{L}$ is a linear operator (acting on x), e.g., a differential operator. We want to solve this problem by the method of lines (semi-discretization), using a spectral method for the approximation of u and its derivatives in the spatial variable x. Then, in a general spectral method, we seek solutions $u_N(x,t)$ with

$$u_N(x,t) = \sum_{\#\{n\}=N} c_n(t)\, \varphi_n(x), \tag{3.21}$$

where $c_n(t)$ are expansion coefficients and φ_n are basis functions chosen according to the specific structure of (3.20). For example, we may take

1) the *Fourier expansion* with $c_n(t) = \widehat{u}_n(t)$, $\varphi_n(x) = e^{i\pi nx}$ for periodic boundary conditions,

2) a polynomial expansion such as the *Chebyshev expansion* with $c_n(t) = \breve{u}_n(t)$, $\varphi_n(x) = T_n(x)$ for other boundary conditions.

The spectral approximation in space (3.21) results into a $N \times N$ system of ODEs for the expansion coefficients $\{c_n(t)\}$:

$$\boldsymbol{c}' = B\boldsymbol{c}\,, \tag{3.22}$$

where $B \in \mathbb{R}^{N\times N}$, and $\boldsymbol{c} = \{c_n(t)\} \in \mathbb{R}^N$. We can solve it with standard ODE solvers (Euler, Crank-Nikolson, etc.) which as we have seen are approximations to the matrix exponent in the exact solution $\boldsymbol{c}(t) = \mathrm{e}^{tB}\boldsymbol{c}(0)$.

Example 3.25 (The diffusion equation) Consider the diffusion equation for a function $u = u(x,t)$,

$$\begin{cases} u_t = u_{xx}, & (x,t) \in [-1,1] \times \mathbb{R}_+\,, \\ u(x,0) = g(x), & x \in [-1,1]\,. \end{cases} \tag{3.23}$$

with the periodic boundary conditions $u(-1,t) = u(1,t)$, $u_x(-1,t) = u_x(1,t)$, and standard normalisation $\int_{-1}^{1} u(x,t)\, dx = 0$, both imposed for all values $t \geq 0$.

For each t, we approximate $u(x,t)$ by its N-th order partial Fourier sum in x,

$$u(x,t) \approx u_N(x,t) \;=\; \sum_{n\in\Gamma_N} \widehat{u}_n(t)\, e^{i\pi nx}\,, \qquad \Gamma_N := \{-N/2{+}1, ..., N/2\}\,.$$

Then, from (3.23), we see that each coefficient $\widehat{u}_n$ fulfills the ODE

$$\widehat{u}_n'(t) = -\pi^2 n^2 \widehat{u}_n(t)\,. \qquad n \in \Gamma_N \tag{3.24}$$

Its exact solution is $\widehat{u}_n(t) = \mathrm{e}^{-\pi^2 n^2 t}\, \widehat{g}_n$ for $n \neq 0$ and we set $\widehat{u}_0(t) = 0$ due to the normalisation condition, so that

$$u_N(x,t) = \sum_{n\in\Gamma_N} \widehat{g}_n\, \mathrm{e}^{-\pi^2 n^2 t}\, \mathrm{e}^{i\pi nx}\,,$$

which is the exact solution truncated to N terms.

Here, we were able to find the exact solution without solving ODE numerically due to the special structure of the Laplacian. However, for more general PDE we will need a numerical method, and thus the issue of stability arises, so we consider this issue on that simplified example.

Analysis 3.26 (Stability analysis) The system (3.24) has the form

$$\widehat{\boldsymbol{u}}' = B\widehat{\boldsymbol{u}}\,, \qquad B = \operatorname{diag}\{-\pi^2 n^2\}\,, \quad n \in \Gamma_N\,,$$

and we note that (a) all the eigenvalues of B are negative, and that (b) they consist of the eigenvalues $\lambda_n^{(2)}$ of the second order differentiation operator, with $\max|\lambda_n^{(2)}| = (\frac{N}{2})^2$.

If we approximate this system with the Euler method:

$$\widehat{\boldsymbol{u}}^{k+1} = (I + \tau B)\widehat{\boldsymbol{u}}^k, \qquad \tau := \Delta t,$$

then we see that, for stability condition $\|I+\tau B\| \le 1$, we need to scale teh time step $\tau = \Delta t \sim N^{-2}$.

Note that, for the Crank-Nikolson scheme, since the spectrum of B is negative, we get stability for any time step $\tau > 0$.

For general linear operator $\mathcal{L}$ in (3.20) with constant coefficients, the matix B is again diagonal (hence normal), and provided that it spectrum is negative, for stability we must scale the time step $\tau \sim N^{-m}$, where m is the maximal order of differentiation.

The scaling $\tau \sim N^{-2}$ may seem similar to the scaling $k \sim h^2$ in difference methods which we viewed as a disadvantage, however in spectral methods we can take N, the order of partial Fourier or Chebyshev sums to achieve a good appoximation, rather small. (We may still need to choose τ small enough to get a desired accuracy.)

Example 3.27 (The diffusion equation with non-constant coefficient) We want to solve the diffusion equation with a non-constant coefficient $a(x) > 0$ for a function $u = u(x,t)$

$$\begin{cases} u_t = (a(x)u_x)_x, & (x,t) \in [-1,1] \times \mathbb{R}_+\,, \\ u(x,0) = g(x), & x \in [-1,1]\,, \end{cases} \tag{3.25}$$

with boundary and normalization conditions as before. Approximating u by its partial Fourier sum results in the following system of ODEs for the coefficients $\widehat{u}_n$

$$\widehat{u}_n'(t) = -\pi^2 \sum_{m\in\Gamma_N} mn\,\widehat{a}_{n-m}\,\widehat{u}_m(t), \qquad n \in \Gamma_N\,.$$

For the discretization in time we may apply the Euler method, this gives

$$\widehat{u}_n^{k+1} = \widehat{u}_n^k - \tau\,\pi^2 \sum_{m\in\Gamma_N} mn\,\widehat{a}_{n-m}\,\widehat{u}_m^k\,, \qquad \tau = \Delta t\,,$$

or in the vector form

$$\widehat{\boldsymbol{u}}^{k+1} = (I + \tau B)\widehat{\boldsymbol{u}}^k,$$

where $B = (b_{m,n}) = (-\pi^2 mn\,\widehat{a}_{n-m})$. For stability of Euler method, we again need $\|I + \tau B\| \le 1$, but analysis here is less straightforward.

Remark 3.28 (Chebyshev methods for evolutionary problems) In general, the boundary conditions for the considered PDEs have to be implemented in the Chebyshev expansion. If the boundary conditions are to be imposed exactly, either the basis functions have to be slightly modified, e.g., to $T_n(x) - 1$ instead of $T_n(x)$ for the boundary condition $u(1) = 0$, or we get additional conditions on the expansion coefficients $\breve{u}_n$ (cf. Exercise 20 from the Example Sheets). While the exact imposition is in general not a problem for the numerical treatment of elliptic PDEs, as soon as the boundary conditions depend on time we may run into serious stability issues. One way around this is the use of penalty methods in which the boundary conditions is added to the scheme later as a penalty term.