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Mathematical Tripos Part II: Michaelmas Term 2025

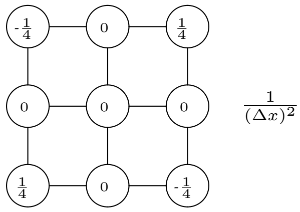
Numerical Analysis – Examples' Sheet 1

1. The Laplace operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is approximated by the nine-point formula

$$h^2\nabla^2 u(ih, jh) \approx -\frac{10}{3}u_{i,j} + \frac{2}{3}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) + \frac{1}{6}\left(u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1}\right),$$

where $u_{i,j} \approx u(ih, jh)$. Find the error of this approximation when u is any infinitely-differentiable function. Show that the error is smaller if u happens to satisfy Laplace's equation $\nabla^2 u = 0$.

2. Determine the order (in the form $\mathcal{O}((\Delta x)^p)$) of the finite difference approximation to $\partial^2/\partial x\partial y$ given by the computational stencil



3. Let $M \geq 2$ and $N \geq 2$ be integers and let $u \in \mathbb{R}^{M\times N}$ have the components $u_{m,n}$, $1 \leq m \leq M$, $1 \leq n \leq N$, where two subscripts occur because we associate the components with the interior points of a rectangular grid. Further, let $u_{m,n}$ be zero on the boundary of the grid, which means $u_{m,n} = 0$ if either $m \in \{0, M+1\}$ or $n \in \{0, N+1\}$ Thus, for any real constants α, β and γ, we can define a linear transformation A from $\mathbb{R}^{M\times N}$ to $\mathbb{R}^{M\times N}$ by the equations

$$\begin{aligned}(A\boldsymbol{u})_{m,n} &= \alpha u_{m,n} + \beta\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1}\right)\\ &\quad + \gamma\left(u_{m-1,n-1} + u_{m+1,n-1} + u_{m-1,n+1} + u_{m+1,n+1}\right), \quad 1 \leq m \leq M, 1 \leq n \leq N.\end{aligned}$$

We now let the components of $\boldsymbol{u}$ have the special form $u_{m,n} = \sin(\frac{mk\pi}{M+1})\sin(\frac{n\ell\pi}{N+1})$, where k and ℓ are fixed integers. Prove that $\boldsymbol{u}$ is an eigenvector of A and find its eigenvalue. Hence deduce that, if α, β and γ provide the nine-point formula of Exercise 1, and if M and N are large, then the least modulus of an eigenvalue is approximately $4\sin^2(\frac{\pi}{2(M+1)}) + 4\sin^2(\frac{\pi}{2(N+1)})$.

4. Verify that the $n \times n$ tridiagonal matrix

$$A = \begin{bmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \beta & \alpha & \beta & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \beta & \alpha & \beta \\ 0 & \cdots & 0 & \beta & \alpha \end{bmatrix}$$

has the eigenvalues $\lambda_k = \alpha + 2\beta\cos\frac{k\pi}{n+1}$, $k = 1, \ldots, n$. Hence, deduce $\rho(A) = |\alpha| + 2|\beta|\cos\frac{\pi}{n+1}$.
[*Hint: Show that* $v \in \mathbb{R}^n$ *with the components* $v_i = \sin ix$, $i = 1, \ldots, n$, *where* $x = \frac{\pi k}{n+1}$, *satisfies the eigenvalue equation* $A\boldsymbol{v} = \lambda_k\boldsymbol{v}$.]

5. Let A be the $m^2 \times m^2$ matrix that occurs in the five-point difference method for Laplace's equation on a square grid. By applying the orthogonal similarity transformation of Hockney's method, find a tridiagonal matrix, say T, that is similar to A, and derive expressions for each element of T. Hence, deduce the eigenvalues of T. Verify that they agree with the eigenvalues of Proposition 1.12

6. Let

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

$=$

2	0	6	-2	6	0	6	2

By applying the FFT algorithm, calculate $x_\ell = \sum_{j=0}^{7} w_8^{j\ell} y_j$ for $\ell = 0, 2, 4, 6$, where $w_8 = \exp\frac{2\pi i}{8}$. Check your results by direct calculation. [*Hint: Because all values of* ℓ *are even, you can omit some parts of the usual FFT algorithm.*]

7. Let $u(x,t) : \mathbb{R}^2 \to \mathbb{R}^2$ be an infinitely-differentiable solution of the convection-diffusion equation $u_t = u_{xx} - bu_x$, where the subscripts denote partial derivatives and where b is a positive constant, and let $u(x,0)$ for $0 \le x \le 1$ and $u(0,t), u(1,t)$ for $t > 0$ be given. A difference method sets $h = \Delta x = \frac{1}{M+1}$ and $k = \Delta t = \frac{T}{N}$, where M and N are positive integers and T is a fixed bound on t. Then it calculates the estimates $u_m^n \approx u(mh, nk)$, $1 \le m \le M$, $1 \le n \le N$, by applying the formula

$$u_m^{n+1} = u_m^n + \mu \left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right) - \frac{1}{2}(\Delta x) b\mu \left(u_{m+1}^n - u_{m-1}^n\right),$$

where $\mu = \frac{\Delta t}{(\Delta x)^2}$, the values of u_m^n being to set to $u(mh, nk)$ when (mh, nk) is on the boundary. Show that, subject to μ being constant, the local truncation error of the formula is $\mathcal{O}(h^4)$.

Let $e(h,k)$ be the greatest of the errors $|u(mh,nk) - u_m^n|$, $1 \le m \le M$, $1 \le n \le N$. Prove convergence from first principles: if $h \to 0$ and $\mu \le \frac{1}{2}$ is constant, then $e(h,k)$ also tends to zero. [*Hint: Relate the maximum error at each time level to the maximum error at the previous time level.*]

8. Let $v(x,y)$ be a solution of Laplace's equation $v_{xx} + v_{yy} = 0$ on the unit square $0 \le x, y \le 1$, and let $u(x,y,t)$ solve the diffusion equation $u_t = u_{xx} + u_{yy}$, where the subscripts denote partial derivatives. Further, let u satisfy the boundary conditions $u(\xi, \eta, t) = v(\xi, \eta)$ at all points (ξ, η) on the boundary of the unit square for all $t \ge 0$, Prove that, if u and v are sufficiently differentiable, then the integral

$$\phi(t) = \int_0^1 \int_0^1 [u(x,y,t) - v(x,y)]^2 \, dx \, dy, \quad t \ge 0,$$

has the property $\phi'(t) \le 0$. Then prove that $\phi(t)$ tends to zero as $t \to \infty$. [*Hint: In the first part, try to replace u_{xx} and u_{yy} when they occur by $u_{xx} - v_{xx}$ and $u_{yy} - v_{yy}$ respectively.*]

9. Let $u(x,t) : \mathbb{R}^2 \to \mathbb{R}^2$ be a sufficiently differentiable function that satisfies the diffusion equation $u_t = u_{xx}$, and let θ be a positive constant. Using the notation $u_m^n \approx u(mh, nk)$, where $\mu = \frac{k}{h^2} = \frac{\Delta t}{(\Delta x)^2}$ is constant, we consider the implicit finite-difference scheme

$$u_m^{n+1} - \frac{1}{2}(\mu - \theta)\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right) = u_m^n + \frac{1}{2}(\mu + \theta)\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right).$$

Show that its local error is $\mathcal{O}(h^4)$, unless $\theta = \frac{1}{6}$ (the Crandall method), which makes the local error of order $\mathcal{O}(h^6)$. Is it possible for the order to be even higher?

10. The Crank-Nicolson formula is applied to the diffusion equation $u_t = u_{xx}$ on a rectangular mesh (mh, nk), $0 \le m \le M+1$, $n \ge 0$, where $h = \Delta x = \frac{1}{M+1}$. We assume zero boundary conditions $u(0,t) = u(1,t) = 0$ for all $t \ge 0$. Prove that the estimates $u_m^n \approx u(mh, nk)$ satisfy the equation

$$\sum_{m=1}^{M} \left[\left(u_m^{n+1}\right)^2 - \left(u_m^n\right)^2\right] = -\frac{1}{2}\frac{\Delta t}{(\Delta x)^2} \sum_{m=1}^{M+1} \left[\left(u_m^{n+1} - u_{m-1}^{n+1}\right) + \left(u_m^n - u_{m-1}^n\right)\right]^2, \quad n = 0, 1, 2, \ldots.$$

Because the right hand side is nonpositive, it follows that $\sum_{m=1}^{M} (u_m^n)^2$ is a monotonically decreasing function of n. We see that this property is analogous to part of Exercise 8 if $v \equiv 0$ there. [*Hint: Substitute the value of $u_m^{n+1} - u_m^n$ that is given by the Crank-Nicolson formula into the elementary equation*

$$\sum_{m=1}^{M} \left[\left(u_m^{n+1}\right)^2 - \left(u_m^n\right)^2\right] = \sum_{m=1}^{M} \left(u_m^{n+1} - u_m^n\right)\left(u_m^{n+1} + u_m^n\right)$$

and use $u_{m+1}^n - 2u_m^n + u_{m-1}^n = (u_{m+1}^n - u_m^n) - (u_m^n - u_{m-1}^n)$. It is also helpful occasionally to change the index m of the summation by one.]