

## Mathematical Tripas Part II: Michaelmas Term 2024

## Numerical Analysis – Examples' Sheet 4

30. Let  $A_n$  be an  $n \times n$  TST matrix such that  $a_{i,i} = \alpha$  and  $a_{i,i+1} = a_{i+1,i} = \beta$ . Show that the Jacobi iteration for solving  $Ax = b$  converges if  $2|\beta| < |\alpha|$ . Moreover, prove that if convergence is required for all  $A_n$  with  $n \geq 1$  then this inequality is necessary as well as sufficient.
31. Let  $A$  be an  $n \times n$  TST matrix with  $a_{k,k} = \alpha$  and  $a_{k,k+1} = a_{k+1,k} = \beta$ . Verify that  $\alpha \geq 2|\beta| > 0$  implies that the matrix is positive definite. Now, we precondition the conjugate gradient method for  $Ax = b$  with the Toeplitz lower-triangular bidiagonal matrix  $Q$ ,

$$q_{k,k} = \gamma, \quad q_{k,k-1} = \delta, \quad q_{k,\ell} = 0 \quad \text{otherwise.}$$

Determine real numbers  $\gamma$  and  $\delta$  such that  $QQ^T$  differs from  $A$  in just the  $(1, 1)$  coordinate. Prove that with this choice of  $\gamma$  and  $\delta$  the preconditioned conjugate gradient method converges in just two iterations.

32. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}, \quad S = \begin{pmatrix} A_1 & O \\ O & A_3 \end{pmatrix},$$

where  $A_1, A_3$  are symmetric  $n \times n$  matrices and the rank of the  $n \times n$  matrix  $A_2$  is  $r \leq n - 1$ . (This is the case for example when  $A$  is a band matrix with the bandwidth  $2r + 1$ .) We further stipulate that the  $(2n) \times (2n)$  matrix  $A$  is positive definite (hence so are  $A_1$  and  $A_3$ ). Let  $A_1 = Q_1 Q_1^T$ ,  $A_3 = Q_3 Q_3^T$  be Cholesky factorizations and assume that the preconditioner  $Q$  is the lower-triangular Cholesky factor of  $S$  (hence  $QQ^T = S$ ).

- (a) Prove that

$$B = Q^{-1} A Q^{-T} = \begin{pmatrix} I & F \\ F^T & I \end{pmatrix}, \quad \text{where } F = Q_1^{-1} A_2 Q_3^{-T}.$$

- (b) Assuming that the eigenvalues of  $B$  are  $\lambda_1, \dots, \lambda_{2n}$ , while the eigenvalues of  $FF^T$  are  $\mu_1, \dots, \mu_n \geq 0$ , prove that, without loss of generality,

$$\lambda_k = 1 - \sqrt{\mu_k}, \quad \lambda_{n+k} = 1 + \sqrt{\mu_k}, \quad k = 1, 2, \dots, n.$$

[Hint: For  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , we have  $\det M = \det(A - BD^{-1}C) \det(D)$ .]

- (c) Prove that the rank of  $FF^T$  is at most  $r$ , thereby deducing that  $B$  has at most  $2r + 1$  distinct eigenvalues. What does this tell you about the number of steps before the preconditioned conjugate gradient method terminates in exact arithmetic?

33. Let  $A$  be the  $3 \times 3$  matrix

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where  $\lambda$  is real and nonzero. Find an explicit expression for  $A^k$ ,  $k = 1, 2, 3, \dots$

The sequence  $\mathbf{x}^{(k+1)}$ ,  $k = 0, 1, 2, \dots$ , is generated by the power method  $\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)} / \|A\mathbf{x}^{(k)}\|$ , where  $\mathbf{x}^{(0)}$  is a nonzero vector in  $\mathbb{R}^3$ . Deduce from your expression for  $A^k$  that the second and third components of  $\mathbf{x}^{(k+1)}$  tend to zero as  $k \rightarrow \infty$ . Further, show that this remark implies  $A\mathbf{x}^{(k+1)} - \lambda\mathbf{x}^{(k+1)} \rightarrow \mathbf{0}$ , so the power method tends to provide a solution to the eigenvalue equation.

34. Let  $A$  be a symmetric  $2 \times 2$  matrix with distinct eigenvalues and normalized eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . Given  $\mathbf{x}^{(0)} \in \mathbb{R}^2$ , the sequence  $\mathbf{x}^{(k+1)}$ ,  $k = 0, 1, 2, \dots$ , is generated in the following way. The *Rayleigh quotient*  $\lambda_k = \mathbf{x}^{(k)T} A \mathbf{x}^{(k)} / \|\mathbf{x}^{(k)}\|^2$  is taken as an estimate for an eigenvalue of  $A$ , the vector norm being Euclidean. Then, inverse iteration gives

$$\mathbf{y} = (A - \lambda_k I)^{-1} \mathbf{x}^{(k)}, \quad \text{and we set } \mathbf{x}^{(k+1)} = \mathbf{y} / \|\mathbf{y}\|.$$

Show that, if  $\mathbf{x}^{(k)} = (\mathbf{v}_1 + \epsilon_k \mathbf{v}_2) / (1 + \epsilon_k^2)^{1/2}$ , where  $|\epsilon_k|$  is small, then  $|\epsilon_{k+1}|$  is of magnitude  $|\epsilon_k|^3$ . In other words, the method enjoys a *third order* rate of convergence.

35. The symmetric matrix

$$A = \begin{pmatrix} 9 & -8 & 2 \\ -8 & 9 & -2 \\ 2 & -2 & 10 \end{pmatrix} \quad \text{has the eigenvector } \mathbf{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Calculate an orthogonal matrix  $\Omega$  by a Householder transformation such that  $\Omega \mathbf{v}$  is a multiple of the first coordinate vector  $\mathbf{e}_1$ . Then, form the product  $\Omega^T A \Omega$ . You should find that this matrix is suitable for deflation. Hence, identify all the eigenvalues and eigenvectors of  $A$ .

36. Show that the vectors  $\mathbf{x}$ ,  $A\mathbf{x}$  and  $A^2\mathbf{x}$  are linearly dependent in the case

$$A = \begin{pmatrix} 4 & 5 & 2 & 0 \\ -26 & -14 & 1 & 4 \\ -2 & 2 & 3 & 1 \\ -43 & -8 & 13 & 9 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 5 \end{pmatrix}.$$

Hence, calculate two of the eigenvalues of  $A$ . Obtain by deflation a  $2 \times 2$  matrix whose eigenvalues are the remaining eigenvalues of  $A$ . Then, find the other eigenvalues of  $A$ .

37. Use Householder transformations to generate a tridiagonal matrix that is similar to the matrix

$$A = \begin{pmatrix} 9 & -1 & 2 & 2 \\ -1 & 3 & 4 & 2 \\ 2 & 4 & 14 & -3 \\ 2 & 2 & -3 & 4 \end{pmatrix}.$$

Your final matrix should be symmetric and should have the same trace as  $A$ .

38. Let  $A$  be an  $n \times n$  symmetric tridiagonal matrix that is not deflatable (i.e., all the elements of  $A$  that are adjacent to the diagonal are nonzero). Prove that  $A$  has  $n$  distinct eigenvalues. Prove also that, if  $A$  has a zero eigenvalue and a single iteration of the QR algorithm is applied to  $A$ , then the resultant tridiagonal matrix is deflatable. [Hint: In the first part show that for each eigenvalue  $\lambda$  there is a unique solution to  $A\mathbf{w} = \lambda\mathbf{w}$ . In the second part deduce that a diagonal element of  $R$  is zero.]
39. Let  $A$  be a  $2 \times 2$  symmetric matrix whose trace does not vanish, let  $A_0 = A$ , and let the sequence of matrices  $\{A_k : k = 1, 2, \dots\}$  be calculated by applying the QR algorithm to  $A_0$  (without any origin shifts). Express the matrix element  $(A_{k+1})_{1,1}$  in terms of the elements of  $A_k$ . Show that, except in the special case when  $A$  is already diagonal, the sequence  $\{(A_k)_{1,1} : k = 0, 1, \dots\}$  converges monotonically to the eigenvalue of  $A$  of larger modulus. [Hint: The sign of this eigenvalue is the same as the sign of the trace of  $A$ . Also, for any symmetric matrix  $B$ , we have  $B_{1,1} = \mathbf{e}_1^T B \mathbf{e}_1$  and  $\lambda_{\min} \|\mathbf{x}\|^2 \leq \mathbf{x}^T B \mathbf{x} \leq \lambda_{\max} \|\mathbf{x}\|^2$ .]
40. Apply a single step of the QR method to the matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 1 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix},$$

where  $\epsilon > 0$ . You should find that the  $(2, 3)$  element of the new matrix is  $\mathcal{O}(\epsilon^3)$  and that the new matrix has exactly the same trace as  $A$ .

41. (For those who like analysis). Let  $A$  be a real  $4 \times 4$  upper Hessenberg matrix whose eigenvalues all have nonzero imaginary parts, where the moduli of the two complex pairs of eigenvalues are different. Prove that, if the matrices  $A_k$ ,  $k = 0, 1, 2, \dots$ , are calculated from  $A$  by the QR algorithm, then the subdiagonal elements  $(A_k)_{2,1}$  and  $(A_k)_{4,3}$  stay bounded away from zero, but  $(A_k)_{3,2}$  converges to zero as  $k \rightarrow \infty$ .