Weighted frames of exponentials and stable recovery of multidimensional functions from nonuniform Fourier samples

Ben Adcock[*] Milana Gataric[†] Anders C. Hansen [‡]

May 13, 2014

Abstract

In this paper, we consider the problem of recovering a compactly-supported multivariate function from a collection of pointwise samples of its Fourier transform taken nonuniformly. We do this by using the concept of weighted Fourier frames. A seminal result of Beurling shows that sample points give rise to a classical Fourier frame provided they are relatively separated and of sufficient density. However, this result does not allow for arbitrary clustering of sample points, as is often the case in practice. Whilst keeping the density condition sharp and dimension independent, our first result removes the separation condition and shows that density alone suffices. However, this result does not lead to estimates for the frame bounds. A known result of Gröchenig provides explicit estimates, but only subject to a density condition that deteriorates linearly with dimension. In our second result we improve these bounds by reducing this dimension dependence. In particular, we provide explicit frame bounds which are dimensionless for functions having compact support contained in a sphere. Next, we demonstrate how our two main results give new insight into a reconstruction algorithm – based on the existing generalized sampling framework – that allows for stable and quasi-optimal reconstruction in any particular basis from a finite collection of samples. Finally, we construct sufficiently dense sampling schemes that are often used in practice – jittered, radial and spiral sampling schemes – and provide several examples illustrating the effectiveness of our approach when tested on these schemes.

1 Introduction

The recovery of a compactly-supported function from pointwise measurements of its Fourier transform – or equivalently, the recovery of a band-limited function from its direct samples – has been the subject of comprehensive research during the past century, driven by numerous practical applications ranging from Magnetic Resonance Imaging (MRI) to Computed Tomography (CT), geophysical imaging, seismology and microscopy. In many of these applications, the case when the data is acquired nonuniformly is of particular interest. For instance, MR scanners often use spiral sampling geometries for fast data acquisition. Such sampling geometries are often preferable because of the higher resolution obtained in the Fourier domain and the lower magnetic gradients required to scan along such trajectories. Another important example is radial (also known as polar) sampling of the Fourier transform, which is used in MRI, as well as in applications where the Radon transform is involved in the sampling process; CT, for instance. For examples of different sampling patterns used in applications see Figure 1. Spurred by its practical importance, the past decades have witnessed the development of an extensive mathematical theory of nonuniform sampling, as evidenced by a vast body of literature.

[*]Department of Mathematics, Purdue University, USA
[†]CCA, Centre for Mathematical Sciences, University of Cambridge, UK
[‡]DAMTP, Centre for Mathematical Sciences, University of Cambridge, UK

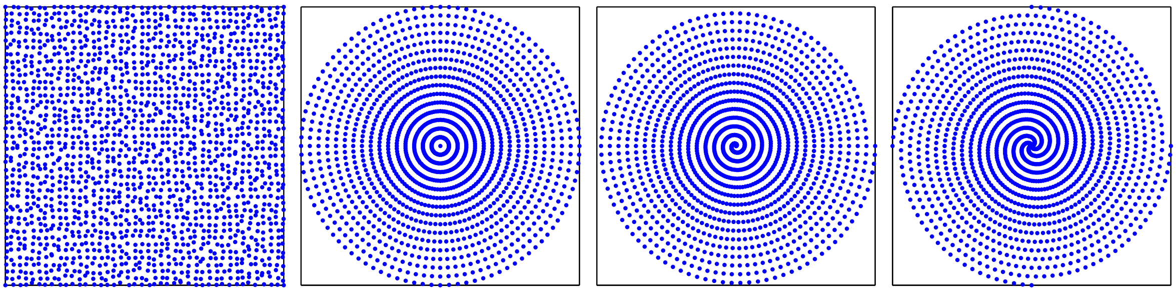

Figure 1: Different sampling schemes: (i) *jittered sampling scheme*, a standard model when the measurements are not taken exactly on a uniform grid, often used MRI, seismology and geophysics [8, 37], (ii) *polar sampling scheme* used in computed tomography [20], (iii) *spiral* and (iv) *interleaving spiral* used in MRI [16]. All of them satisfy an appropriate (K, δ_{E°)-density condition (see Definition 3.4), for $E = [-1, 1]^2$, $\delta_{E^\circ} < 0.25$ and $K = 4$.

An inexhaustive list includes the books of Marvasti [37], Benedetto and Ferreira [12], Young [50], Seip [44] and others, as well as many excellent articles; see [8, 10, 11, 21, 22, 28, 45] and references therein.

In the case of Cartesian sampling, the celebrated Nyquist–Shannon theorem [47] guarantees a full reconstruction of a compactly-supported signal from its Fourier measurements, provided that the samples are taken equidistantly at a sufficiently large rate, equal to or exceeding the so-called Nyquist rate. In other words, the samples must be taken uniformly and densely enough. Nonuniform sampling is typically studied within the context of so-called Fourier frames. The theory of Fourier frames was developed by Duffin and Shaeffer [17], more than half a century ago, and its roots can be traced back to earlier works of Paley and Wiener [39] and Levinson [36]. In one dimension, there exists a near-complete characterization of Fourier frames in terms of the density of underlying samples, due primarily to Beurling [13], Landau [35], Jaffard [32] and Seip [43]. However, in higher dimensions, the situation becomes considerably more complicated [11, 38]. Nevertheless, Beurling's seminal paper [13] (see also [14]) provides a sharp sufficient condition for sampling points in multiple dimensions to give rise to a Fourier frame. For a more detailed review on the theory of Fourier frames and nonuniform sampling, see [11, 15].

1.1 Main results

A limitation of the results mentioned above is that they require a minimal separation between the sampling points. In particular, clustering of sampling points deteriorates the associated frame bounds, which leads to numerical instability. The main contribution of the first part of this paper removes the minimal separation restriction whilst keeping the sharpness of the result. Through the use of a weighted Fourier frame approach, based on Gröchenig's earlier work (see below), we adapt Beurling's result to allow for arbitrary clustering of sampling points. Specifically, we prove the following:

Theorem 1.1. *Let* $\mathrm{H} = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \mathrm{supp}(f) \subseteq E\}$*, where* $E \subseteq \mathbb{R}^d$ *is compact, convex and symmetric. If a countable set* $\Omega \subseteq \hat{\mathbb{R}}^d$ *has density* $\delta_{E^\circ} < 1/4$ *(see Definition 2.1) then there exist weights* $\mu_\omega > 0$ *such that* $\{\sqrt{\mu_\omega} e_\omega\}_{\omega\in\Omega}$ *is a weighted Fourier frame for* H*, where* $e_\omega(x) = \mathrm{e}^{\mathrm{i}2\pi\omega\cdot x}\mathbf{1}_E(x)$*. In other words, there exist constants* $A, B > 0$ *such that*

$$\forall f \in \mathrm{H}\backslash\{0\}, \quad A\|f\|^2 \leq \sum_{\omega\in\Omega} \mu_\omega |\hat{f}(\omega)|^2 \leq B\|f\|^2.$$

In particular, it suffices to choose the weights $\{\mu_\omega\}_{\omega\in\Omega}$ *as the measures of Voronoi regions (see Definition (2.4)) with respect to the* $|\cdot|_{E^\circ}$ *norm (see (2.3) and (2.4)).*

The 1/4 density condition given here is sharp: if a countable set Ω does not satisfy the required density condition, then the associated family of weighted exponentials $\{\sqrt{\mu_\omega} e_\omega\}_{\omega\in\Omega}$ does not have to give a weighted Fourier frame with the weights chosen as in Theorem 1.1.

This result has both theoretical and practical significance. First, it is interesting to address the issue of arbitrary clustering, since it is natural to anticipate that adding more sampling points should not impair the recovery of a function. Second, this scenario often arises in applications. For example, consider Fourier measurements acquired on a polar sampling scheme. By increasing the number of radial lines along which samples are acquired, the sampling points cluster at low frequencies, which deteriorates the frame bounds of the corresponding Fourier frame. On the other hand, if we weight those points according to their relative densities, the resulting *weighted* Fourier frame has controllable frame bounds.

Weighted Fourier frames, which we also refer to as weighted frames of exponentials, were studied by Gröchenig [25], and later also by Gabardo [24]. In [25], Gröchenig presents a sufficient density condition in order for a family of exponentials to constitute a weighted Fourier frame, and provides explicit frame bounds. This density condition is sharp in dimension $d = 1$, but fails to be sharp in higher dimensions, with the estimate on the density deteriorating linearly, and the estimates on the frame bounds, exponentially in d. The multidimensional result has been improved in [9], but under the assumption that the sampling set consists of a sequence of uniformly distributed independent random variables. In this setting, Bass and Grochenig provide rather probabilistic estimates.

Our work focuses on deterministic statements and provides two improvements of Gröchenig's result from [25]. First, as discussed above, in Theorem 1.1 we provide a density condition which is both sharp *and* dimensionless. Unfortunately, however, this condition does not give rise to explicit frame bounds. Therefore, in our second result we present explicit frame bounds under a less stringent density condition than previously known:

Theorem 1.2. *Let* $\mathrm{H} = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \mathrm{supp}(f) \subseteq E\}$, *where* $E \subseteq \mathbb{R}^d$ *is compact. Suppose that* $|\cdot|_*$ *is an arbitrary norm on* $\mathbb{R}^d$ *and* $c^* > 0$ *is the smallest constant for which* $|\cdot| \le c^* |\cdot|_*$, *where* $|\cdot|$ *denotes the Euclidean norm. Let* $\Omega \subseteq \mathbb{R}^d$ *be* δ_**-dense* (*see Definition 2.1*) *with*

$$\delta_* < \frac{\log(2)}{2\pi m_E c^*}, \tag{1.1}$$

where $m_E = \sup_{x\in E} |x|$. *Then* $\{\sqrt{\mu_\omega} e_\omega\}_{\omega\in\Omega}$ *is a weighted Fourier frame for* H *with the weights defined as the measures of Voronoi regions with respect to norm* $|\cdot|_*$. *The weighted Fourier frame bounds* $A, B > 0$ *satisfy*

$$\sqrt{A} \ge 2 - \exp(2\pi m_E \delta_* c^*), \qquad \sqrt{B} \le \exp(2\pi m_E \delta_* c^*) < 2.$$

Taking $|\cdot|_* = |\cdot|$ for simplicity, where $|\cdot|$ is the Euclidean norm, we see that the key estimate (1.1), which is a refinement of Gröchenig's, deteriorates with dimension only for certain function supports E. Specifically, it depends on the radius of the largest sphere in which E is contained. In particular, (1.1) is dimensionless when a function has a compact support contained in the unit Euclidean ball $\mathcal{B}_1$. In this case, Theorem 1.1 gives the sharp sufficient condition $\delta < 0.25$ (where δ corresponds to the Euclidean norm) but without explicit frame bounds. On the other hand, Theorem 1.2 provides explicit frame bounds under the slightly stronger, but dimension independent, condition $\delta < \frac{\log(2)}{2\pi} \approx 0.11$.

We note at this stage that, whilst Gröchenig was arguably the first to rigorously study weighted Fourier frames in sampling, the use of weights is commonplace in MRI reconstructions, where they are often referred to as 'density compensation factors' (see [16, 46] and references therein). However, such approaches are often heuristic. Building on Gröchenig's earlier work, our results provide further mathematical sampling theory for their use.

In practice, one only has access to a finite number of samples. In the final part of this paper, we consider a reconstruction algorithm for this problem, based on the generalized sampling (GS) framework introduced in [3] (see also [2, 4, 5, 6]). In particular, in Section 3, we give the third main result of this paper, Theorem 3.5, which shows that stable, quasi-optimal reconstruction is possible in any subspace $\mathrm{T} \subseteq \mathrm{H}$ provided the samples satisfy the same density conditions as in Theorems 1.1 and 1.2, and additionally, provided the samples possess a sufficiently large bandwidth, in a sense we define later. Hence, we extend the analysis of the framework considered in [1] – so-called nonuniform generalized sampling (NUGS) – to the multidimensional setting. An important issue in higher dimensions is that of efficient implementation. As has been already noted in [1], this is not the case with wavelet reconstruction bases, since then the NUGS reconstruction can be computed efficiently by using nonuniform fast Fourier transforms (NUFFTs) [23].

We also remark that our analog recovery model is the same as that used with great success in the recent work of Guerquin-Kern, Haberlin, Pruessmann and Unser [29] on iterative, wavelet-based reconstructions for MRI. Moreover, the popular iterative reconstruction algorithm of Sutton, Noll and Fessler [46] for non-Cartesian MRI is a special case of NUGS based on a digital signal model. The results we prove in this paper provide theoretical foundations for the success of those algorithms. Moreover, our results also improve existing bounds for the well-known ACT (Adaptive weights, Conjugate gradients, Toeplitz) algorithm in nonuniform sampling [21, 22, 27, 28], which can also be viewed as a particular case of NUGS. For further discussion, see §3.2 of this paper.

The remainder of this paper is organized as follows. In §2 we consider weighted Fourier frames and the proofs of Theorems 1.1 and 1.2. We discuss the NUGS framework in §3, and show stable and accurate recovery by using the results from §2. Next in §4 we construct several popular sampling schemes so that they satisfy appropriate density conditions. Finally, we illustrate our theoretical results in §5 with some numerical experiments.

2 Weighted frames of exponentials

2.1 Background material and preliminaries

Let

$$\mathrm{H} = \left\{ f \in \mathrm{L}^2(\mathbb{R}^d) : \operatorname{supp}(f) \subseteq E \right\}$$

be the Hilbert space of square-integrable functions supported on a compact set $E \subseteq \mathbb{R}^d$, with the standard L^2-norm $\|\cdot\|$ and L^2-inner product $\langle \cdot, \cdot \rangle$. The d-dimensional Euclidean vector space is denoted by $\mathbb{R}^d$, and, following a standard convention, $\hat{\mathbb{R}}^d$ is used whenever $\mathbb{R}^d$ is considered as a frequency domain. For $f \in \mathrm{H}$, the Fourier transform is defined by

$$\hat{f}(\omega) = \int_E f(x) \mathrm{e}^{-\mathrm{i}2\pi\omega\cdot x} \, \mathrm{d}x, \quad \omega \in \hat{\mathbb{R}}^d,$$

where $\cdot$ stands for Euclidean inner product. We also use the following notation

$$e_\omega(x) = \mathrm{e}^{\mathrm{i}2\pi\omega\cdot x} \mathbf{1}_E(x), \tag{2.1}$$

where $\mathbf{1}_E$ is the indicator function of the set E. Note that $\hat{f}(\omega) = \langle f, e_\omega \rangle$.

Let $|\cdot|_*$ denote an arbitrary norm on $\mathbb{R}^d$. Note that for every such norm the set $\{x \in \mathbb{R}^d : |x|_* \leq 1\}$ is convex, compact and symmetric. Moreover, all norms on a finite dimensional space are equivalent to the Euclidean norm, which we denote simply by $|\cdot|$. Hence, by $c_*, c^* > 0$, we denote the sharp constants for which

$$\forall x \in \mathbb{R}^d, \quad c_*|x|_* \leq |x| \leq c^*|x|_*. \tag{2.2}$$

Conversely, if $E \subseteq \mathbb{R}^d$ is a compact, convex and symmetric set, the function $|\cdot|_E : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\forall x \in \mathbb{R}^d, \quad |x|_E = \inf\{a > 0 : x \in aE\}, \tag{2.3}$$

is a norm on $\mathbb{R}^d$ [11]. Here, E is the unit ball with the respect to the norm $|\cdot|_E$, i.e.

$$E = \{x \in \mathbb{R}^d : |x|_E \leq 1\}.$$

Also, for such set $E \subseteq \mathbb{R}^d$, its polar set is defined as

$$E^\circ = \{\hat{y} \in \hat{\mathbb{R}}^d : \forall x \in E,\ x \cdot \hat{y} \leq 1\}. \tag{2.4}$$

Note that E° is itself a convex, compact and symmetric set in $\hat{\mathbb{R}}^d$, which is the unit ball with respect to the norm $|\cdot|_{E^\circ}$. Also observe that, if E is the unit ball in the Euclidean norm, which we denote by $\mathcal{B}_1$, then $\mathcal{B}_1 = \mathcal{B}_1^\circ$ and $|\cdot|_{\mathcal{B}_1} = |\cdot|_{\mathcal{B}_1^\circ} = |\cdot|$.

Throughout the paper, we denote ℓ^p norm by $|\cdot|_p$, i.e. for $x \in \mathbb{R}^d$, $|x|_p = \left(\sum_{j=1}^d |x_j|^p\right)^{1/p}$. Hence $|\cdot|_2 = |\cdot|_{\mathcal{B}_1} = |\cdot|$. Also, we recall the well-know inequality

$$\forall x \in \mathbb{R}^d, \quad |x|_q \leq |x|_r \leq d^{1/r-1/q}|x|_q, \quad q > r > 0. \tag{2.5}$$

Now, let $\Omega \subseteq \hat{\mathbb{R}}^d$ be a countable set of sampling points, which we also refer to as a *sampling scheme*. The set Ω is said to be *separated* if there exists a constant $\eta > 0$ such that

$$\forall \omega, \lambda \in \Omega, \quad \omega \neq \lambda, \quad |\omega - \lambda| \geq \eta,$$

and it is *relatively separated* if it is a finite union of separated sets. It is clear that, if Ω is separated in Euclidean metric then it is separated in any metric on $\hat{\mathbb{R}}^d$ and vice-versa.

Next, we introduce the crucial notion of *density* of a countable set $\Omega \subseteq \hat{\mathbb{R}}^d$. This definition originates in Beurling's work [13] and it is used frequently in multidimensional nonuniform sampling literature.

Definition 2.1. *Let Ω be a sampling scheme contained in a closed, simply connected set $Y \subseteq \hat{\mathbb{R}}^d$ with 0 in its interior. Let $|\cdot|_*$ be an arbitrary norm on $\mathbb{R}^d$, and let $\delta_* \in (0, 1/4)$. We say that Ω is δ_*-dense in the domain Y if*

$$\delta_* = \sup_{\hat{y} \in Y} \inf_{\omega \in \Omega} |\omega - \hat{y}|_*.$$

If $|\cdot|_* = |\cdot|_E$ for a compact, convex and symmetric set E, then we write δ_E. Also, to emphasise the sampling scheme, where necessary we use notation $\delta_*(\Omega)$.

Note that the δ_*-density condition from the Definition 2.1 is equivalent to the δ_*-*covering condition*: there exists $\delta_* \in (0, 1/4)$ such that for all $\rho \geq \delta_*$ it holds that

$$Y \subseteq \bigcup_{\omega \in \Omega} \left\{x \in \mathbb{R}^d : |x - \omega|_* \leq \rho\right\}.$$

Before we define weighted frames, let us discuss the classical frames of exponentials. A countable family of functions $\{e_\omega\}_{\omega \in \Omega} \subseteq \mathrm{H}$ is said to be a *Fourier frame* for H if there exist constants $A, B > 0$ such that

$$\forall f \in \mathrm{H}\backslash\{0\}, \quad A\|f\|^2 \leq \sum_{\omega \in \Omega} |\hat{f}(\omega)|^2 \leq B\|f\|^2. \tag{2.6}$$

The constants A and B are called *upper and lower frame bounds*, respectively. If $\{e_\omega\}_{\omega \in \Omega}$ is the frame, then the *frame operator* $\mathcal{S} : \mathrm{H} \to \mathrm{H}$ is defined by

$$\forall f \in \mathrm{H}, \quad \mathcal{S} : f \mapsto \mathcal{S}f = \sum_{\omega \in \Omega} \hat{f}(\omega) e_\omega. \tag{2.7}$$

Since the inequality (2.6) holds, the frame operator $\mathcal{S}$ is a topological isomorphism with the inverse $\mathcal{S}^{-1} : \mathrm{H} \to \mathrm{H}$, and also

$$\forall f \in \mathrm{H}, \quad f = \sum_{\omega \in \Omega} \langle \mathcal{S}^{-1} f, e_\omega \rangle e_\omega. \tag{2.8}$$

Formula (2.8), with the appropriately truncated sum, is sometimes used for signal reconstruction [11]. However, for the types of sets Ω considered in practice, finding the inverse frame operator $\mathcal{S}^{-1}$ is often a nontrivial task. Typically, this renders such an approach infeasible in more than one dimension.

If the relation (2.6) holds with $A = B$, the family $\{e_\omega\}_{\omega \in \Omega}$ is called a *tight frame*, and if $A = B = 1$, this family forms an orthonormal basis for H. In these cases, the relation (2.6) is known as (generalized) Parseval's equality. Also, then the frame operator becomes $\mathcal{S} = A\mathcal{I}$, where $\mathcal{I}$ is the identity operator on H, and the formula (2.8) represents the Fourier series of f. Moreover, the appropriately truncated Fourier series converges to f on H. This leads to a considerably simpler framework in the case when the samples are acquired uniformly, corresponding to an orthonormal basis or a tight frame for H.

In [13], Beurling provides a sufficient density condition for a nonuniform set of sampling points to give a Fourier frame for H consisting of functions supported on the unit sphere in the Euclidean norm. In what follows, we use a variation of Beurling's result given by Benedetto & Wu in [11] (see also the work by Olevskii & Ulanovskii [38]) which is a generalization to arbitrary convex, compact and symmetric domains:

Theorem 2.2. *Let $E \subseteq \mathbb{R}^d$ be compact, convex and symmetric set. If $\Omega \subseteq \hat{\mathbb{R}}^d$ is relatively separated and δ_{E°-dense in the domain $Y = \hat{\mathbb{R}}^d$ with $\delta_{E^\circ} < 1/4$, then $\{e_\omega\}_{\omega \in \Omega}$ is a Fourier frame for* H.

Beurling [13] also shows that this result is sharp in the sense that there exists a countable set with the density $\delta_{E^\circ} = 1/4$, where E is the unit ball in the Euclidean metric, which does not satisfy the lower frame condition in (2.6) (see also [38, Prop. 4.1]).

Now we define weighted frames of exponentials:

Definition 2.3. *A countable family of functions $\{\sqrt{\mu_\omega} e_\omega\}_{\omega \in \Omega}$ is a weighted Fourier frame for* H*, with weights $\{\mu_\omega\}_{\omega \in \Omega}$, $\mu_\omega > 0$, if there exist constants $A, B > 0$ such that*

$$\forall f \in \mathrm{H} \backslash \{0\}, \quad A\|f\|^2 \leq \sum_{\omega \in \Omega} \mu_\omega |\hat{f}(\omega)|^2 \leq B\|f\|^2. \tag{2.9}$$

For a weighted Fourier frame $\{\sqrt{\mu_\omega} e_\omega\}_{\omega \in \Omega}$ let us introduce the *weighted Fourier frame operator*:

$$\mathcal{S} : \mathrm{H} \to \mathrm{H}, \; f \mapsto \mathcal{S}f = \sum_{\omega \in \Omega} \mu_\omega \hat{f}(\omega) e_\omega. \tag{2.10}$$

In view of the relation (2.9), we conclude that the weighted frame operator satisfies

$$\forall f \in \mathrm{H} \backslash \{0\}, \quad A\|f\|^2 \leq \langle \mathcal{S}f, f \rangle \leq B\|f\|^2.$$

As discussed, the use of weights is to compensate for arbitrary clustering in Ω. In order to define appropriate weights $\{\mu_\omega\}_{\omega \in \Omega}$ corresponding to the sampling scheme Ω, in this paper, we use measures of Voronoi regions. This is a standard practice in nonuniform sampling [8, 41].

Definition 2.4. *Let Ω be a set of distinct points in $Y \subseteq \hat{\mathbb{R}}^d$ and let $|\cdot|_*$ be an arbitrary norm on $\mathbb{R}^d$. The Voronoi region at $\omega \in \Omega$, with respect to the norm $|\cdot|_*$ and in the domain Y, is given by*

$$V_\omega^* = \{\hat{y} \in Y : \forall \lambda \in \Omega, \; \lambda \neq \omega, \; |\omega - \hat{y}|_* \leq |\lambda - \hat{y}|_*\}.$$

The Lebesgue measure of the Voronoi region V_ω^* we denote as

$$\mathrm{meas}\,(V_\omega^*) = \int_Y \mathbf{1}_{V_\omega^*}(\hat{y})\,\mathrm{d}\hat{y}.$$

In [25], Gröchenig provides explicit frame bounds for weighted Fourier frames, provided the sample points Ω are sufficiently dense. In one dimension, the condition on the density is sharp, i.e sampling points with density such that $\delta < 1/4$ give rise to a weighted Fourier frame, but sets of points with lower density (i.e. bigger delta) do not necessarily yield a weighted Fourier frame. However, the sharpness of the result is lost in higher dimensions.

Here we state Gröchenig's multidimensional result [27, Prop. 7.3], which is a more recent reformulation of [25, Thm. 5]:

Theorem 2.5. *Let* $\mathrm{H} = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \mathrm{supp}(f) \subseteq E\}$, *where* $E = [-1,1]^d$. *If* $\Omega \subseteq \hat{\mathbb{R}}^d$ *is a* $\delta_{\mathcal{B}_1}$*-dense set of distinct points such that*

$$\delta_{\mathcal{B}_1} < \frac{\log(2)}{2\pi d}, \tag{2.11}$$

then $\{\sqrt{\mu_\omega}e_\omega\}_{\omega\in\Omega}$ *is a weighted Fourier frame for* H, *where the weights are defined as measures of the Voronoi regions of the points* Ω *with respect to Euclidean norm. The weighted frame bounds* $A, B > 0$ *satisfy*

$$\sqrt{A} \geq 2 - \mathrm{e}^{2\pi\delta_{\mathcal{B}_1}d}, \qquad \sqrt{B} \leq \mathrm{e}^{2\pi\delta_{\mathcal{B}_1}d} < 2.$$

Note that the bound (2.11) deteriorates linearly with the dimension d. Also, E can be any rectangular domain of the form $\prod_{i=1}^d[-s_i, s_i]$, since $\mathrm{supp}(f) \subseteq \prod_{i=1}^d[-s_i, s_i]$ implies that $\tilde{f}(x) = f(x_1/s_1, \ldots, x_d/s_d)$ has support in $[-1,1]^d$. Hence, the result is stated for $E = [-1,1]^d$ without loss of generality [27]. Moreover note that, E may also be any compact set that is a subset of $[-1,1]^d$ such as any ℓ^p unit ball, $p > 0$, for example.

2.2 Weighted Fourier frames with explicit frame bounds and the proof of Theorem 1.2

Much like Beurling's result, Theorem 2.2, it is expected that the density condition for weighted Fourier frames given in Theorem 2.5 does not depend on dimension. Unfortunately, Gröchenig's estimates deteriorate linearly with the dimension d, and thus cease to be sharp. Therefore, in Theorem 1.2 we provide an modification of Gröchenig's result by presenting explicit bounds with slower, and sometimes no deterioration with respect to dimension.

The estimates in Theorem 1.2 are presented in terms of the following quantity

$$m_E = \sup_{x\in E}|x|, \tag{2.12}$$

where $E \subseteq \mathbb{R}^d$ and $|\cdot|$ is Euclidean norm. Note that $m_{\mathcal{B}_1} = 1$ and therefore it is independent of dimension for spheres. Moreover, if E is the ℓ^p unit ball, i.e. $E = \{x : \mathbb{R}^d : |x|_p \leq 1\}$, $p > 0$, then

$$m_E = \max\{1, d^{1/2-1/p}\}, \tag{2.13}$$

due to inequality (2.5).

Let us recall here the multinomial formula. For any $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^d$, we have

$$\sum_{|\alpha|_1=k} \frac{k!}{\alpha!}x^\alpha = (x_1 + \cdots + x_d)^k, \tag{2.14}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha|_1 = |\alpha_1| + \ldots + |\alpha_d|$, $\alpha! = \prod_{j=1}^d \alpha_j!$ and $x^\alpha = \prod_{j=1}^d x_j^{\alpha_j}$. Regarding the multi-index notation, in what follows, we also use the derivative operator defined as

$$D^\alpha = \frac{\partial^{|\alpha|_1}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}.$$

Now we are ready to give prove our main result for weighted Fourier frames with explicit bounds, namely Theorem 1.2.

Proof of Theorem 1.2. The proof is set up in the same manner as the proof of Gröchenig's original result, Theorem 2.5. For a function $f \in \mathrm{H}\backslash\{0\}$, define

$$\chi(\hat{y}) = \sum_{\omega \in \Omega} \hat{f}(\omega) \mathbf{1}_{V_\omega^*}(\hat{y}), \quad \hat{y} \in \hat{\mathbb{R}}^d.$$

Since the sets V_ω^*, $\omega \in \Omega$, make disjoint partition of $\hat{\mathbb{R}}^d$, it holds that

$$\|\chi\| = \sqrt{\sum_{\omega \in \Omega} \mu_\omega |\hat{f}(\omega)|^2},$$

where $\mu_\omega = \mathrm{meas}(V_\omega^*)$. Note that

$$\|f\| - \|\hat{f} - \chi\| \le \|\chi\| \le \|\hat{f} - \chi\| + \|f\|. \tag{2.15}$$

Hence, we aim to estimate $\|\hat{f} - \chi\|$. Again, by using properties of Voronoi regions, it is possible to conclude

$$\|\hat{f} - \chi\| = \sqrt{\sum_{\omega \in \Omega} \int_{V_\omega^*} |\hat{f}(\hat{y}) - \hat{f}(\omega)|^2 \, \mathrm{d}\hat{y}}.$$

In order to estimate $|\hat{f}(\hat{y}) - \hat{f}(\omega)|^2$, for all $\omega \in \Omega$ and all $\hat{y} \in V_\omega^*$, Taylor's expansion of $\hat{f}$ – which is an entire function – is used. Therefore, by the Cauchy–Schwarz inequality we get

$$\begin{aligned} |\hat{f}(\hat{y}) - \hat{f}(\omega)|^2 &\le \left(\sum_{\alpha \neq 0} \frac{|(\hat{y} - \omega)^\alpha|}{\alpha!} |D^\alpha \hat{f}(\hat{y})| \right)^2 \\ &\le \sum_{\alpha \neq 0} \frac{c^{|\alpha|_1} (\hat{y} - \omega)^{2\alpha}}{\alpha!} \sum_{\alpha \neq 0} \frac{c^{-|\alpha|_1}}{\alpha!} |D^\alpha \hat{f}(\hat{y})|^2, \end{aligned} \tag{2.16}$$

for some constant $c > 0$ to be determined later. The inequality (2.16) is where this proof starts to differ from Gröchenig's original proof. For the first term in (2.16), by the multinomial formula (2.14) we get

$$\sum_{\alpha \neq 0} \frac{c^{|\alpha|_1} (\hat{y} - \omega)^{2\alpha}}{\alpha!} = \sum_{k=0}^\infty \frac{c^k}{k!} \sum_{|\alpha|_1 = k} \frac{k!}{\alpha!} (\hat{y} - \omega)^{2\alpha} - 1 = \sum_{k=0}^\infty \frac{c^k}{k!} |\hat{y} - \omega|^{2k} - 1 \le \exp(c(\delta_* c^*)^2) - 1,$$

where in the final inequality δ_*-density of the set Ω is used:

$$\forall \omega \in \Omega, \quad \forall \hat{y} \in V_\omega^*, \quad |\hat{y} - \omega| \le \delta_* c^*.$$

Now consider the other term in (2.16). If we integrate over the Voronoi region V_ω^* and sum over $\omega \in \Omega$ then

$$\begin{aligned} \sum_{\alpha \neq 0} \frac{c^{-|\alpha|_1}}{\alpha!} \sum_{\omega \in \Omega} \int_{V_\omega^*} |D^\alpha \hat{f}(\hat{y})|^2 \, \mathrm{d}\hat{y} &= \sum_{k=1}^\infty \frac{c^{-k}}{k!} \sum_{|\alpha|_1 = k} \frac{k!}{\alpha!} \|D^\alpha \hat{f}\|^2 \\ &= \sum_{k=1}^\infty \frac{c^{-k}}{k!} \int_E \sum_{|\alpha|_1 = k} \frac{k!}{\alpha!} (2\pi x)^{2\alpha} |f(x)|^2 \, \mathrm{d}x, \end{aligned}$$

since by Parseval's identity

$$\|D^\alpha \hat{f}\|^2 = \|\hat{F}\|^2 = \|F\|^2 = \int_E (2\pi x)^{2\alpha} |f(x)|^2 \, \mathrm{d}x,$$

where $F(x) = (-\mathrm{i}2\pi x)^\alpha f(x)$. Hence, again by the multinomial formula (2.14), we obtain

$$\sum_{\alpha \neq 0} \frac{c^{-|\alpha|_1}}{\alpha!} \sum_{\omega\in\Omega} \int_{V_\omega^*} |D^\alpha \hat{f}(\hat{y})|^2 \, \mathrm{d}\hat{y} = \sum_{k=1}^{\infty} \frac{c^{-k}(2\pi m_E)^{2k}}{k!} \|f\|^2 = \big(\exp((2\pi m_E)^2/c) - 1\big) \, \|f\|^2.$$

Therefore, from (2.16), we get

$$\|\hat{f} - \chi\|^2 \leq \big(\exp(c(\delta_* c^*)^2) - 1\big) \big(\exp((2\pi m_E)^2/c) - 1\big) \, \|f\|^2.$$

If we equate the two terms, then we set $c = 2\pi m_E/(\delta_* c^*)$ to get

$$\|\hat{f} - \chi\| \leq (\exp(2\pi m_E \delta_* c^*) - 1) \, \|f\|.$$

Thus (2.15) now gives

$$\sqrt{B} \leq \exp(2\pi m_E \delta_* c^*), \quad \sqrt{A} \geq 2 - \exp(2\pi m_E \delta_* c^*),$$

with the condition that

$$\delta_* < \frac{\log(2)}{2\pi m_E c^*},$$

as required. □

To illustrate this result, let $E = \{x \in \mathbb{R}^d : |x|_p \leq 1\}$, $p > 0$, and let $|\cdot|_*$ be the ℓ^q norm, $q \geq 1$. Then, the density condition (1.1) becomes

$$\delta_q < \frac{\log(2)}{2\pi \max\{1, d^{1/2-1/p}\} \max\{1, d^{1/2-1/q}\}}, \tag{2.17}$$

due to (2.5) and (2.13). This bound attains minimum for $p = q = \infty$, when it deteriorates linearly with the dimension d. However, in all other cases the deterioration of the bound on density, and also, the deterioration of weighted frame bounds estimations, is slower with the dimension. Moreover, they are independent of dimension whenever $p \leq 2$ and $q \leq 2$.

To compare this theorem with Gröchenig result given in Theorem 1.2, we set $p = \infty$ and $q = 2$ in (2.17). The bound (2.17) gives $\delta_2 < \frac{\log(2)}{2\pi\sqrt{d}}$, whereas (2.11) gives $\delta_2 < \frac{\log(2)}{2\pi d}$. Hence Theorem 1.2 leads to an improvement by a factor of $\sqrt{d}$ and no deterioration in the constant $\frac{\log(2)}{2\pi}$.

2.3 Sharp sufficient condition for weighted Fourier frames and the proof of Theorem 1.1

The relative separation of a sampling set Ω is necessary and sufficient for the existence of an upper frame bound [50, Thm. 2.17], see also [32]. However, if we introduce appropriate weights $\{\mu_\omega\}_{\omega\in\Omega}$ to compensate for the clustering of the sampling points Ω, and consider $\{\sqrt{\mu_\omega} e_\omega\}_{\omega\in\Omega}$ instead of $\{e_\omega\}_{\omega\in\Omega}$, then this condition ceases to be necessary, as it is evident from Grochenig's Theorem 2.5 and the improved result given in Theorem 1.2. On the other hand, in order to have a lower weighted frame bound, the condition on density from Theorem 1.2 is still far from being sharp. To mitigate this, we next establish Theorem 1.1.

Without imposing restrictions such as separation, Theorem 1.1 gives sufficient condition on a density of set of points to yield a weighted Fourier frame, which is dimension independent. Therefore, in all dimensions, once this density condition is fulfilled, the sampling points are allowed to cluster arbitrarily, as long as the appropriate weights are used. Moreover, this result is sharp, as can be seen by following the same arguments which lead to the sharpness of Beurling's result (Theorem 2.2).

Note that, in this theorem, as well as in Beurling's Theorem 2.2, it is required that E is compact, convex and symmetric. This is due to a characterization of Paley-Wiener spaces that holds for such sets E (see [38, Thm. A]).

In order to prove Theorem 1.1, we need the following lemma.

Lemma 2.6. *If Ω is a sequence with the density $\delta_{E^\circ}(\Omega) < 1/4$ in $\hat{\mathbb{R}}^d$, then there exists a subsequence $\tilde{\Omega} \subseteq \Omega$ which is $\eta-$separated for some $\eta > 0$, and also has density $\delta_{E^\circ}(\tilde{\Omega}) < 1/4$ in $\hat{\mathbb{R}}^d$.*

Proof. To begin with, we introduce some notation. For the set E, we define $E(0,1) = S$, $E(0,r) = rE$ and $E(x,r) = x + rE$. Here, for δ_{E°, we simply write δ.

Let us choose $\epsilon > 0$ such that $\delta + \epsilon/2 < 1/4$ and set $\delta_1 = \delta + \epsilon$. Now define $\tilde{\Omega}$ inductively as follows. For arbitrary picked point $\omega_0 \in \Omega$, set $\tilde{\omega}_0 = \omega_0$. Given $\tilde{\omega}_0, \dots, \tilde{\omega}_N$, define $\tilde{\omega}_{N+1}$ by

$$\tilde{\omega}_{N+1} \in \Omega \cap E^\circ(x, \delta),$$

where

$$x \in \partial G = \partial \left(\bigcup_{\tilde{\omega}_n \in \tilde{\Omega}_N} E^\circ(\tilde{\omega}_n, \delta_1) \right) \quad \text{and} \quad \tilde{\Omega}_N = \{\tilde{\omega}_n\}_{n=0}^N.$$

Here, we picked any $x \in \partial G$ and then, for that x, any $\tilde{\omega}_{N+1} \in \Omega \cap E^\circ(x,\delta)$. Finally, we let $\tilde{\Omega} = \{\tilde{\omega}_n\}_{n=0}^\infty$.

Note that for any $x \in \hat{\mathbb{R}}^d$ there must exist a point $\omega \in \Omega$ in the set $E^\circ(x,\delta)$ such that x is covered by $E^\circ(\omega,\delta)$, since Ω is $\delta-$dense in the norm $|\cdot|_{E^\circ}$ and $\hat{\mathbb{R}}^d$ can be covered by the sets $E^\circ(\omega,\delta)$, $\omega \in \Omega$. Moreover, for every $x \in \partial G$, every $\omega \in \Omega \cap E^\circ(x,\delta)$ must be different than any other $\omega \in \tilde{\Omega}_N$, since $\delta < \delta_1$. Also, note that for every such $\omega \in \Omega \cap E^\circ(x,\delta)$ it holds that

$$\epsilon = \delta_1 - \delta \le \inf_{\tilde{\omega}_n \in \tilde{\Omega}_N} |\omega - \tilde{\omega}_n|_{E^\circ} \le \delta_1 + \delta = 2\delta + \epsilon.$$

Therefore if we choose $\tilde{\omega}_{N+1}$ from $\Omega \cap E^\circ(x,\delta)$ arbitrarily, and continue the procedure until $G = \hat{\mathbb{R}}^d$, by the construction, $\tilde{\Omega}$ is $\tilde{\delta}$-dense in the norm $|\cdot|_{E^\circ}$ where $\tilde{\delta} = (2\delta + \epsilon)/2 < 1/4$. Also, it is ϵ-separated in the norm $|\cdot|_{E^\circ}$, and therefore, it is η-separated in the Euclidean norm for some $\eta(\epsilon) > 0$. ☐

Proof of Theorem 1.1. First of all, Gröchenig's result gives the explicit upper bound B (Theorem 2.5). Note that the upper bound B in Theorem 1.2 improves the dimension dependence of the bound given by the original Gröchenig's result.

For the lower bound, we note that if Ω is separated, then everything follows easily. If we denote the volume, i.e. the measure of the ball $\mathcal{B}_{\eta/2}$ by $\text{meas}(\mathcal{B}_{\eta/2})$, since Ω is separated with the separation $\eta > 0$, we get

$$\sum_{\omega \in \Omega} \mu_\omega |\hat{f}(\omega)|^2 \ge \text{meas}(\mathcal{B}_{\eta/2}) \sum_{\omega \in \Omega} |\hat{f}(\omega)|^2 \ge \text{meas}(\mathcal{B}_{\eta/2}) A' \|f\|^2,$$

where $A' > 0$ comes from application of Theorem 2.2. Thus we take $A = \text{meas}(\mathcal{B}_{\eta/2})A' \sim (\eta/2)^d A'$.

However, if Ω is not separated, we proceed as follows. By Lemma 2.6, we know that there exists a subsequence $\tilde{\Omega} \subseteq \Omega$ with density $\delta_{E^\circ}(\tilde{\Omega}) < 1/4$ and separation $\eta > 0$. Let $\epsilon < \eta/2$. Then

$$\sum_{\omega \in \Omega} \mu_\omega |\hat{f}(\omega)|^2 \ge \sum_{\tilde{\omega} \in \tilde{\Omega}} \sum_{\omega \in \mathcal{B}_\epsilon(\tilde{\omega}) \cap \Omega} \mu_\omega |\hat{f}(\omega)|^2.$$

Since $\hat{f}$ is continuous function, from the Extreme value theorem, for each $\tilde{\omega}$, we know there is a point $z_{\tilde{\omega}} \in \overline{\mathcal{B}_\epsilon(\tilde{\omega})} = \mathcal{B}_\epsilon(\tilde{\omega})$, such that

$$\forall \omega \in \mathcal{B}_\epsilon(\tilde{\omega}), \quad |\hat{f}(\omega)| \geq |\hat{f}(z_{\tilde{\omega}})|.$$

Since also $\mu_\omega = \text{meas}\left(V_\omega^{E^\circ}\right)$ and the sets $V_\omega^{E^\circ}$ are disjoint, we get

$$\sum_{\omega \in \Omega} \mu_\omega |\hat{f}(\omega)|^2 \geq \sum_{\tilde{\omega} \in \tilde{\Omega}} \left(|\hat{f}(z_{\tilde{\omega}})|^2 \sum_{\omega \in \mathcal{B}_\epsilon(\tilde{\omega}) \cap \Omega} \mu_\omega \right) = \sum_{\tilde{\omega} \in \tilde{\Omega}} \left(|\hat{f}(\tilde{\omega})|^2 \text{meas} \left(\bigcup_{\omega \in \mathcal{B}_\epsilon(\tilde{\omega}) \cap \Omega} V_\omega^{E^\circ} \right) \right).$$

Now we claim the following:

$$\bigcup_{\omega \in \mathcal{B}_\epsilon(\tilde{\omega}) \cap \Omega} V_\omega^{E^\circ} \supseteq \mathcal{B}_\rho(\tilde{\omega}), \quad \rho = \frac{\epsilon}{2} \frac{c_1}{c_2}.$$

where $0 < c_1 \leq c_2 < \infty$ are the constants such that

$$\forall x \in \mathbb{R}^d, \quad c_1 \, |x|_{E^\circ} \leq |x| \leq c_2 \, |x|_{E^\circ} \, . \tag{2.18}$$

To see this, let $|\hat{y} - \tilde{\omega}| \leq \frac{\epsilon}{2} \frac{c_1}{c_2}$. Since $\hat{y} \in V_\omega^{E^\circ}$ for some $\omega \in \Omega$, we have $|\hat{y} - \omega|_{E^\circ} \leq |\hat{y} - \tilde{\omega}|_{E^\circ}$. Therefore

$$|\hat{y} - \omega| \leq \frac{c_2}{c_1} |\hat{y} - \tilde{\omega}| \leq \frac{\epsilon}{2},$$

and hence

$$|\omega - \tilde{\omega}| \leq |\hat{y} - \omega| + |\hat{y} - \tilde{\omega}| \leq \epsilon.$$

Thus $\omega \in \mathcal{B}_\epsilon(\tilde{\omega}) \cap \Omega$ as required. Therefore, we get

$$\sum_{\omega \in \Omega} \mu_\omega |\hat{f}(\omega)|^2 \geq \text{meas}(\mathcal{B}_\rho) \sum_{\bar{\omega} \in \bar{\Omega}} |\hat{f}(\bar{\omega})|^2,$$

where $\bar{\Omega} = \{z_{\tilde{\omega}} \ : \ \tilde{\omega} \in \tilde{\Omega}\}$. To complete the proof, we only need to show that the set $\bar{\Omega}$ is separated and sufficiently dense, so that we can apply the Theorem 2.2. Consider $\bar{\omega}_1$ and $\bar{\omega}_2$. Then we clearly have

$$|\bar{\omega}_1 - \bar{\omega}_2| \geq \eta - 2\epsilon > 0,$$

since $\tilde{\Omega}$ is separated with the separation η and the $\bar{\omega}$'s lie in the ϵ-cover of this set. Moreover, it is straightforward to see that

$$\delta_{E^\circ}(\bar{\Omega}) \leq \delta_{E^\circ}(\tilde{\Omega}) + \frac{\epsilon}{c_1}.$$

Thus, since $\delta_{E^\circ}(\tilde{\Omega}) < 1/4$, we have the same for $\bar{\Omega}$ for sufficiently small $\epsilon > 0$. We set $A = \text{meas}(\mathcal{B}_\rho) A'$, where $A' > 0$ is as in Theorem 2.2, and finish the proof. $\square$

Remark 2.7 From the proof of Theorem 1.1, we can conclude the following. If Ω has density $\delta_{E^\circ} < 1/4$, it yields a weighted Fourier frame with the lower weighted Fourier frame bound of the form

$$A = \text{meas}(\mathcal{B}_\rho) A', \quad \rho = \frac{\epsilon}{2} \frac{c_1}{c_2},$$

where $A' > 0$ is the lower Fourier frame bound from Beurling's result (Theorem 2.2), the constants $c_1, c_2 > 0$ relate $|\cdot|_{E^\circ}$ to the Euclidean norm by (2.18) and the constant $\epsilon > 0$ is such that $\epsilon < \eta/2$, where $\eta > 0$ is the separation of a subsequence $\tilde{\Omega} \subseteq \Omega$ with the same density $\delta_{E^\circ} < 1/4$. However, this does not in general have an explicit estimate of A since we typically do not know an explicit estimate of A'. On the other hand, the upper weighted Fourier frame bound B is explicitly estimated by Theorem 1.2.

To end this section, in order to illustrate differences between classical and weighted Fourier frames, as well as different uses of previously given results, let us consider the following two-dimensional example.

Example 2.8 Let $E = \mathcal{B}_1 \subseteq \mathbb{R}^2$ and let

$$\Lambda_1 = \tfrac{1}{8}\mathbb{Z}^2, \qquad \Lambda_2 = \left\{ \left(\frac{1}{n}, \frac{1}{m} \right) : (n,m) \in \mathbb{Z}^2, \min\{|n|, |m|\} > 8 \right\}.$$

Note that, for such E, $E^\circ = \mathcal{B}_1$ and therefore the E°-norm is the Euclidean norm $|\cdot|$.

The set of points Λ_1 is separated with the density

$$\delta_{\mathcal{B}_1}(\Lambda_1) = \frac{\sqrt{2}}{16} \approx 0.0884 < \frac{1}{4}.$$

Therefore, by Theorem 2.2, we conclude the family of functions $\{e_\lambda\}_{\lambda\in\Lambda_1}$ is a frame for $\mathrm{L}^2(\mathcal{B}_1)$. However, if we now consider the set

$$\Omega = \Lambda_1 \cup \Lambda_2,$$

for which $\delta_{\mathcal{B}_1}(\Omega) = \delta_{\mathcal{B}_1}(\Lambda_1) = \sqrt{2}/16$, Theorem 2.2 can not be used since Ω has infinitely many accumulation points at

$$\{0\} \cup \left\{ \left(\frac{1}{n}, 0 \right) : n \in \mathbb{Z}, |n| > 8 \right\} \cup \left\{ \left((0, \frac{1}{m} \right) : m \in \mathbb{Z}, |m| > 8 \right\},$$

and therefore it is not separated. Moreover, it can be verified that the family $\{e_\omega\}_{\omega\in\Omega}$ fails in satisfying the right inequality of (2.6). To see this, we first note that

$$\int_{\mathcal{B}_1} \mathrm{e}^{-2\pi\mathrm{i}\omega\cdot x}\,\mathrm{d}x = \frac{J_1(2\pi|\omega|)}{|\omega|},$$

where J_1 is the Bessel function of the first kind and order 1. Therefore, there exists $c > 0$ such that

$$c \leq \left| \int_{\mathcal{B}_1} \mathrm{e}^{-2\pi\mathrm{i}\left(\frac{1}{n}x_1 + \frac{1}{m}x_2\right)}\,\mathrm{d}x_1\,\mathrm{d}x_2 \right|^2 \leq \pi^2, \tag{2.19}$$

for all $(n,m) \in \mathbb{Z}^2$ such that $\sqrt{1/n^2 + 1/m^2} < a j'_{1,1}/(2\pi) \approx 0.6098$, where a is some fixed constant from the interval $(0,1)$ and $j'_{1,1}$ is the first positive zero of the function J_1. Hence, it is enough to take the function $g(x) = \mathbf{1}_{\mathcal{B}_1}(x)$ for which $\|g\|^2 = \pi$, whereas $\sum_{\omega\in\Omega} |\hat{g}(\omega)|^2$ is unbounded. Thus, we conclude that the set Ω does not give a Fourier frame.

On the other hand, if, for the same set of points $\Omega = \Lambda_1 \cup \Lambda_2$, we consider the weighted family $\{\sqrt{\mu_\omega}e_\omega\}_{\omega\in\Omega}$ with the weights defined as Voronoi regions in ℓ^2-norm, this particular function g satisfies the relation (2.9) with some $0 < A, B < \infty$. This can be easily proved by using the inequalities (2.19), and the fact that

$$\sum_{n=9}^{\infty}\sum_{m=9}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \left(\frac{1}{m-1} - \frac{1}{m+1} \right) = \left(\frac{17}{72} \right)^2.$$

which implies that the sum of Voronoi regions corresponding to the points Λ_2 converges. Moreover, since $\delta_{\mathcal{B}_1}(\Omega) = \sqrt{2}/16$, by Theorem 1.1 we conclude that Ω gives rise to a weighted Fourier frame.

Also, note that, in order to verify that Ω forms a weighted Fourier frame, Gröchenig's original result could not be used since

$$\delta_{\mathcal{B}_1}(\Omega) = \frac{\sqrt{2}}{16} > \frac{\log(2)}{4\pi} \approx 0.0552.$$

However, since in this case $m_E = 1$ and $c^* = 1$ and since

$$\delta_{\mathcal{B}_1}(\Omega) = \frac{\sqrt{2}}{16} < \frac{\log(2)}{2\pi} \approx 0.1103,$$

we are able to use Theorem 1.2 to conclude that Ω generates a weighted Fourier frame with the weighted Fourier frame bounds $\sqrt{A} \geq 0.2574$ and $\sqrt{B} \leq 1.7426$.

3 Multidimensional function recovery

Having provided guarantees for samples to give rise to weighted Fourier frames, we now consider the question of function recovery. To do so, we shall use the generalized sampling approach for nonuniform samples (NUGS) from [1]. As in [1], let $\Omega \subseteq \hat{\mathbb{R}}^d$ be a countable set of distinct frequencies, i.e. the sampling scheme, not necessarily taken on the Cartesian grid, and let $\mathrm{T} \subseteq \mathrm{H}$ be a finite-dimensional subspace; the so-called *reconstruction space*. Given that data $\{\hat{f}(\omega)\}_{\omega\in\Omega}$ of an unknown function $f \in \mathrm{H}$, NUGS provides a reconstruction, i.e. a mapping $F : f \mapsto \tilde{f}$ depending on the given samples only, that satisfies the following two properties:

(i) F is *quasi-optimal*: there exists a constant $\mu = \mu(F) \ll \infty$ such that

$$\forall f \in \mathrm{H}, \quad \|f - F(f)\| \leq \mu \|f - \mathcal{P}_{\mathrm{T}} f\|,$$

where $\mathcal{P}_{\mathrm{T}}$ denotes the orthogonal projection on T.

(ii) F is *numerically stable*: there exists a constant $\kappa = \kappa(F) \ll \infty$ such that

$$\forall g \in \mathrm{H}, \quad \|F(g)\| \leq \kappa \|g\|.$$

As discussed in [1], the main feature of this approach is that it allows for arbitrary reconstruction subspaces T. In particular, the quality of the reconstruction is not determined by the approximation properties of the sampling frame, but rather those of T, which can be chosen according to the function f to be recovered. Often in practice, T may consist of wavelets since it is well-known that multidimensional images in applications such as MRI and CT are well represented using wavelets [49]. Quasi-optimality guarantees that the good intrinsic approximation properties of T are inherited by the reconstruction $F(f)$. Stability, on the other hand, is vital when dealing with noisy measurements.

3.1 NUGS framework

For convenience, here we recall the NUGS framework of [1] in more detail, and in particular, the construction of F. We commence with the definition of an admissible sampling operator:

Definition 3.1 (Admissible sampling operator)**.** *Let Ω be a sampling scheme, $\mathcal{S} : \mathrm{H} \to \mathrm{H}$ a linear operator and let* T *be a finite-dimensional subspace of* H*. Suppose that $\mathcal{S}$ satisfies:*

(i) *for each $f \in \mathrm{H}$, $\mathcal{S}f$ depends only on the sampling data $\{\hat{f}(\omega)\}_{\omega\in\Omega}$,*

(ii) *$\mathcal{S}$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$ and*

$$\forall f, g \in \mathrm{H}, \quad |\langle \mathcal{S}f, g\rangle|^2 \leq \langle \mathcal{S}f, f\rangle \langle \mathcal{S}g, g\rangle,$$

(iii) *there exists a positive constant $C_1 = C_1(\Omega, \mathrm{T})$ such that*

$$\forall f \in \mathrm{T}, \quad \langle \mathcal{S}f, f\rangle \geq C_1 \|f\|^2. \tag{3.1}$$

Then $\mathcal{S}$ is said to be an admissible sampling operator for (Ω, T).

For convenience, we shall assume that C_1 is the largest constant for which (3.1) holds. Given such an operator $\mathcal{S}$, we can also define the constant $C_2 = C_2(\Omega)$ by

$$\forall f \in \mathrm{H}, \quad \langle \mathcal{S}f, f \rangle \leq C_2 \|f\|^2. \tag{3.2}$$

Likewise, we assume this constant is the smallest possible. Note that C_2 exists since $\mathcal{S}$ is linear, and therefore bounded.

Following [1], given a sampling scheme Ω, a finite-dimensional subspace T and an admissible sampling operator $\mathcal{S}$ for the pair (Ω, T), we define the *NUGS reconstruction* $\tilde{f} \in \mathrm{T}$ by

$$\forall g \in \mathrm{T}, \quad \langle \mathcal{S}\tilde{f}, g \rangle = \langle \mathcal{S}f, g \rangle, \tag{3.3}$$

and write $F = F_{\Omega,\mathrm{T}}$ for the mapping $f \mapsto \tilde{f}$. If $\mathcal{S}$ is an admissible sampling operator with constants C_1 and C_2 given by (3.1) and (3.2) respectively, then the ratio

$$C(\Omega, \mathrm{T}) = \sqrt{\frac{C_2}{C_1}} \tag{3.4}$$

is referred to as the *NUGS reconstruction constant.* As we shall see from the following result, the constants C_1 and C_2 arising from an admissible sampling operator $\mathcal{S}$ determine the stability and quasi-optimality of the resulting NUGS reconstruction via the reconstruction constant $C(\Omega, \mathrm{T})$.

Theorem 3.2 ([1], Thm. 3.4)**.** *Let Ω be a sampling scheme and T a finite-dimensional subspace, and suppose that $\mathcal{S}$ is an admissible sampling operator for the pair (Ω, T). Then the NUGS reconstruction $F(f) = \tilde{f}$ defined by (3.3) exists uniquely for any $f \in \mathrm{H}$ and we have the bound*

$$\forall f, h \in \mathrm{H}, \quad \|f - F(f+h)\| \leq C(\Omega, \mathrm{T}) \left(\|f - \mathcal{P}_{\mathrm{T}} f\| + \|h\| \right), \tag{3.5}$$

where $C(\Omega, \mathrm{T})$ is the corresponding NUGS reconstruction constant.

Note that (3.5) implies numerical stability and quasi-optimality of the mapping F with $\kappa = \mu \leq C(\Omega, \mathrm{T})$. Therefore, this theorem provides existence and uniqueness of the stable and quasi-optimal reconstruction defined by (3.3).

Whilst in previous sections Ω has been countably infinite, in practice we are almost always faced with a finite sampling set $\Omega_N = \{\omega_n\}_{n=1}^N$, for some $N \in \mathbb{N}$. For convenience, let us now assume that Ω_N arises as a finite subset of points Ω forming a weighted Fourier frame, where $\Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega$. For such a sampling scheme Ω_N, we choose $\mathcal{S}$ to be a truncated version of the weighted Fourier frame operator (2.10) given by

$$\mathcal{S}_N : \mathrm{H} \to \mathrm{H}, \quad f \mapsto \mathcal{S}_N f = \sum_{n=1}^{N} \mu_n \hat{f}(\omega_n) e_{\omega_n}, \tag{3.6}$$

which converges to $\mathcal{S}$ strongly on H when $N \to \infty$. If this is the case, (3.3) becomes equivalent to the weighted least-squares data fit:

$$\tilde{f} = \underset{g \in \mathrm{T}}{\operatorname{argmin}} \sum_{n=1}^{N} \mu_n \left| \hat{f}(\omega_n) - \hat{g}(\omega_n) \right|^2. \tag{3.7}$$

For the operator $\mathcal{S}_N$ defined by (3.6), the properties (i) and (ii) from the Definition 3.1 are immediately satisfied. In what follows, by conveniently using the results on the weighted frames given in the previous section, we prove that $\mathcal{S}_N$ also satisfies property (iii), and thus ensures a stable and quasi-optimal reconstruction in (3.7), provided that the sampling scheme is sufficiently dense and wide in the frequency domain. By this, we shall extend the NUGS framework from [1] to the multidimensional setting.

We remark in passing that, although a finite set Ω_N always gives rise to a Fourier frame for its span, without weights the corresponding frame constants can deteriorate as $N \to \infty$. As an illustration, consider the set of points $\Omega = \Lambda_1 \cup \Lambda_2$ from the Example 2.8. If we take a sampling scheme $\Omega_N \subseteq \Omega$ defined as

$$\Omega_N = \{\omega \in \Omega : |\omega|_\infty \leq N\},$$

the upper frame bound B blows up as $N \to \infty$. However, as we see below, this issue is prevented by an appropriate choice of weights $\{\mu_\omega\}_{\omega \in \Omega_N}$.

Remark 3.3 Our purpose in this section is to provide analysis of the reconstruction of a function f from finitely-many samples in an arbitrary subspace T. Consequently, we shall not address the specific algorithmic details, besides from noting that $\tilde{f}$ can be computed by solving an algebraic least squares problem. Computational complexity depends on the choice of T; for example, if a wavelet basis is used then the number of operations is $\mathcal{O}(N \log N)$. We refer to [1] for more information.

3.2 Relation to previous work

The function recovery method NUGS used in this paper is based on the work of the authors [1]. This is a special instance of a more general approach of sampling and reconstruction in abstract Hilbert spaces, known as generalized sampling (GS). Although introduced by two of the authors in [3] it has its origins in earlier work of Unser & Aldroubi [48], Eldar [18], Eldar & Werther [19], Gröchenig [26, 27], Hrycak & Gröchenig [30], Shizgal & Jung [33], Aldroubi [7] and others.

In [26] (see also [27, 28, 21, 22]), the problem of recovering a bandlimited function from its own nonuniform samples was considered, where the arbitrary clustering is addressed by using weighted Fourier frames, exactly the same as we do in this paper. Specifically, Gröchenig et al. developed an efficient algorithm for the nonuniform sampling problem, known as the ACT algorithm (Adaptive weights, Conjugate gradients, Toeplitz) where they consider the reconstruction in a particular finite-dimensional space consisting of trigonometric polynomials. This corresponds to a specific instance of NUGS with a Dirac basis for T. Convergence and stability of the ACT algorithm [27, Thm. 7.1] are guaranteed by the sufficient sampling density and the explicit weighted frame bounds given in [27, Prop. 7.3] (Theorem 2.5 here). The result we prove below, Theorem 3.5, extends this in two ways. First, we have a less stringent density requirement based on the bounds derived in Theorem 1.2, which also directly improves the guarantees for ACT algorithm. Second, we allow for arbitrary choices of T which can be tailored to the particular function f to be recovered.

In MRI and several other applications, a popular algorithm for reconstruction from nonuniform Fourier samples is known as the iterative reconstruction technique [46]. This can also be viewed as an instance of NUGS, where T is a space of piecewise constant functions on a $M \times M$ grid (the term 'iterative' refers to the use of conjugate gradients to compute the reconstruction). Equivalently, when M is a power of 2, then T can be expressed as the space spanned by Haar wavelets up to some finite scale. As a result, Theorem 3.5 also provides guarantees for the iterative reconstruction technique. These improve existing estimates (see [34, 40]), which are based on Gröchenig's original bounds. Importantly, we shall also show how NUGS allows one to obtain better reconstructions, by replacing the Haar wavelet choice for the subspace T with higher-order wavelets.

Let us also note here that there exists a vast wealth of other methods for solving the same (or equivalent) recovery problem from nonuniform Fourier samples, which are fundamentally different than ours. Unlike some common approaches in MRI, such as gridding [31], resampling [42] or earlier mentioned iterative algorithms [46], we do not model f as a finite-length Fourier series, or as a finite array of pixels, but rather as a function in L^2-space. Hence, by using an appropriate approximation basis, we successfully avoid the unpleasant artefacts (e.g. Gibbs ringing) associated with gridding and resampling algorithms and also we gain more accuracy than with the iterative algorithms (see §5 Numerical results). On the other hand, there are approaches commonly found in nonuniform sampling theory which do use analog model but whose reconstruction is based on an iterative inversion of the frame operator [10, 11, 21, 8]. These approaches would be fine if one would be given infinitely-many samples and infinite processing time, but since one has only finite data in practice, they typically lead to large truncation errors (similar to Gibbs phenomena).

3.3 Sufficiently dense points for admissibility

We now wish to provide sufficient conditions for a finite sampling set to give an admissible sampling operator of the form (3.6). This extends the work of [1, §4] to the multidimensional setting.

Since we deal with finite sampling sets, which cannot be dense in the whole of $\hat{\mathbb{R}}^d$, in what follows we consider subsets of $\hat{\mathbb{R}}^d$. Therefore, for a given sampling bandwidth $K > 0$, we use the concept of (K, δ_*)-*density*:

Definition 3.4 ((K, δ_*)-density with respect to Y)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a set of sampling points, $K > 0$ and let $|\cdot|_*$ be an arbitrary norm on $\mathbb{R}^d$. If there exist a closed, simply connected set $Y \subseteq \hat{\mathbb{R}}^d$ with 0 in its interior such that:*

(i) $\max_{\hat{y}\in Y} |\hat{y}|_\infty = 1$,

(ii) $\Omega \subseteq Y_K$, *where* $Y_K = KY$, *and*

(iii) Ω *is* δ_**-dense in the domain* Y_K,

then we say that the set Ω is (K, δ_*)-dense with respect to Y.

Let now $\Omega_N = \{\omega_n\}_{n=1}^N$ be (K, δ_*)-dense with respect to Y, such that the corresponding set $\Omega = \{\omega_n\}_{n\in\mathbb{N}}$, $\Omega_N \subseteq \Omega$, yields a weighted Fourier frame with the weights $\{\mu_{\omega_n}\}_{n\in\mathbb{N}}$ and the frame bounds $A, B > 0$. For a finite-dimensional subspace $\mathrm{T} \subseteq \mathrm{H}$ we introduce the K*-residuals of* T:

$$
\begin{aligned}
R_K(\mathrm{T}) &= \sup\left\{ \|\hat{f}\|_{\hat{\mathbb{R}}^d \setminus Y_K} \ : \ f \in \mathrm{T},\ \|f\| = 1 \right\}, \quad \text{and} \\
\tilde{R}_K(\Omega_N, \mathrm{T}) &= \sup\left\{ \sqrt{\sum_{\omega\in\Omega\setminus\Omega_N} \mu_\omega |\hat{f}(\omega)|^2} \ : \ f \in \mathrm{T},\ \|f\| = 1 \right\}.
\end{aligned} \tag{3.8}
$$

Note that both of these residuals converge to zero when $K \to \infty$, since T is finite dimensional. Also, for all $K > 0$, we have that $R_K(\mathrm{T}) \leq 1$ and $\tilde{R}_K(\Omega_N, \mathrm{T}) \leq \sqrt{B}$.

We are ready to give our main result on NUGS.

Theorem 3.5. *Let $\mathrm{T} \subseteq \mathrm{H} = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \mathrm{supp}(f) \subseteq E\}$ be finite-dimensional, $E \subseteq \mathbb{R}^d$ compact, and let $\Omega_N = \{\omega_n\}_{n=1}^N$ be a sampling scheme.*

1. *Let E be also convex and symmetric, and Ω_N be (K, δ_{E°)-dense with respect to Y, with*

$$\delta_{E^\circ} < \frac{1}{4}.$$

Denote by A and B the frame bounds corresponding to the weighed Fourier frame arising from $\Omega = \{\omega_n\}_{n\in\mathbb{N}}$, $\Omega_N \subseteq \Omega$, and let $\epsilon > 0$ be such that

$$\epsilon < \sqrt{A}.$$

If $K > 0$ is large enough so that

$$\tilde{R}_K(\Omega_N, \mathrm{T}) \leq \epsilon,$$

then the operator $\mathcal{S}_N$, given by (3.6), is admissible sampling operator and the NUGS reconstruction constant satisfies

$$C(\Omega_N, \mathrm{T}) \leq \sqrt{\frac{B}{A - \epsilon^2}}. \tag{3.9}$$

2. *Let Ω_N be (K, δ_*)-dense with respect to Y, with*

$$\delta_* < \frac{\log(2)}{2\pi m_E c^*},$$

where $|\cdot|_*$ is an arbitrary norm on $\mathbb{R}^d$ and $c^* > 0$ is the smallest constant such that $|\cdot| \leq c^* |\cdot|_*$. Let also $\epsilon > 0$ be such that

$$\epsilon < \sqrt{\exp\left(2\pi m_E \delta_* c^*\right)\left(2 - \exp\left(2\pi m_E \delta_* c^*\right)\right)}.$$

If $K > 0$ is large enough so that

$$R_K(T) \leq \epsilon,$$

Then the operator $\mathcal{S}_N$ given by (3.6) is admissible and

$$C(\Omega_N, \mathrm{T}) \leq \frac{\exp\left(2\pi m_E \delta_* c^*\right)}{\sqrt{1 - \epsilon^2} + 1 - \exp\left(2\pi m_E \delta_* c^*\right)}. \tag{3.10}$$

Proof. For the first part, note that

$$\forall f \in \mathrm{H}, \quad \left(A - \sum_{\omega \in \Omega \setminus \Omega_N} \mu_\omega |\hat{f}(\omega)|^2 / \|f\|^2 \right) \|f\|^2 \leq \sum_{\omega \in \Omega_N} \mu_\omega |\hat{f}(\omega)|^2 \leq B \|f\|^2,$$

where the existence of $A, B > 0$ is provided by Theorem 1.1. Hence $C_2(\Omega) \leq B$ and for $C_1(\Omega, \mathrm{T})$ we have

$$C_1(\Omega, \mathrm{T}) \geq A - \tilde{R}_K(\Omega_N, \mathrm{T})^2 \geq A - \epsilon^2 > 0.$$

Since $C(\Omega, \mathrm{T}) = \sqrt{C_2/C_1}$ the first result now follows immediately.

For the second part, we follow the proof of Theorem 1.2 and define

$$\chi(\hat{y}) = \sum_{\omega \in \Omega_N} \hat{f}(\omega) \mathbf{1}_{V_\omega^*}(\hat{y}), \quad \hat{y} \in Y_K.$$

Hence

$$\|\chi\|_{Y_K}^2 = \sum_{\omega \in \Omega_N} \mu_\omega |\hat{f}(\omega)|^2.$$

Note that we have

$$\|f\|_{Y_K} - \|\hat{f} - \chi\|_{Y_K} \leq \|\chi\|_{Y_K} \leq \|\hat{f} - \chi\|_{Y_K} + \|f\|,$$

and also, by the same reasoning as in the proof of Theorem 1.2, we get

$$\|\hat{f} - \chi\|_{Y_K} \leq \left(\exp(2\pi m_E \delta_* c^*) - 1\right) \|f\|.$$

Therefore for all $f \in \mathrm{H}\backslash\{0\}$

$$\left(\sqrt{1-\frac{\|\hat{f}\|^2_{\mathbb{R}^d\backslash Y_K}}{\|f\|^2}}+1-\exp\left(2\pi m_E\delta_* c^*\right)\right)^2\|f\|^2 \leq \sum_{\omega\in\Omega_N}\mu_\omega|\hat{f}(\omega)|^2 \leq \exp(4\pi m_E\delta_* c^*)\|f\|^2.$$

Hence, we have $\sqrt{C_2(\Omega)} \leq \exp(2\pi m_E\delta_* c^*)$ and

$$\sqrt{C_1(\Omega,\mathrm{T})} \geq \sqrt{1-\epsilon^2}+1-\exp\left(2\pi m_E\delta_* c^*\right) > 0,$$

due to the definition of $R_K(\mathrm{T})$ and the assumption that

$$R_K(\mathrm{T}) \leq \epsilon < \sqrt{\exp\left(2\pi m_E\delta_* c^*\right)\left(2-\exp\left(2\pi m_E\delta_* c^*\right)\right)}.$$

Now the statement follows by using the definition of the NUGS reconstruction constant (3.4). □

By this theorem, for a fixed reconstruction space T, we are guaranteed a stable and quasi-optimal reconstruction via NUGS, for large enough sampling bandwidth K provided the density condition holds, even with a highly nonuniform sampling scheme. Namely, for a given nonuniform sampling scheme Ω_N which is (K, δ_{E°)-dense with respect to Y and $\delta_{E^\circ} < 1/4$, the first part of this theorem guarantees a stable and quasi-optimal reconstruction in any given reconstruction space T such that its K-residual $\tilde{R}_K(\Omega_N, \mathrm{T})$ is small enough. The downside, however, is that neither A nor B are known, and also the term $\tilde{R}_K(\Omega_N, \mathrm{T})$ depends on Ω_N. Conversely, by the second bound (3.10) – which is explicit – we are able to largely separate the geometric properties of the sampling scheme, i.e. the density, from intrinsic properties of the reconstruction space T, i.e. the K-residual $R_K(\mathrm{T})$. The latter is determined solely by the decay of functions $\hat{f}$, $f \in \mathrm{T}$, outside the domain Y_K. In other words, in the second case, once $R_K(\mathrm{T})$ is estimated for any given subspace T (see §6 for a discussion on this point), we can ensure a stable and quasi-optimal reconstruction for *any* nonuniform sampling scheme which is (K, δ_*)-dense with respect to Y with small enough δ_*.

4 Examples of sufficiently dense sampling schemes

In the next section, we illustrate NUGS on several numerical examples, where we use a number of sampling schemes commonly found in practice. Therein, we consider functions supported on $E = [-1, 1]^2$. According to Theorem 1.1 , a sampling scheme Ω must satisfy the condition

$$\delta_{E^\circ}(\Omega) < \frac{1}{4}, \tag{4.1}$$

where E° is the unit ball in ℓ^1-norm, or, according to Theorem 1.2, a more strict density condition

$$\delta_{\mathcal{B}_1}(\Omega) < \frac{\log(2)}{2\pi m_E} \tag{4.2}$$

(we have chosen $|\cdot|_* = |\cdot|$ for simplicity). Recall that $m_E = \sqrt{2}$ if $E = [-1, 1]^2$. In this section, we construct some sampling schemes which satisfy these conditions. Note that for $E = [-1, 1]^2$ we have

$$\delta_{E^\circ}(\Omega) \leq \sqrt{2}\delta_{\mathcal{B}_1}(\Omega).$$

Hence, to have (4.1) it is enough to enforce $\delta_{\mathcal{B}_1}(\Omega) < 1/(4\sqrt{2})$. The condition

$$\delta_{\mathcal{B}_1}(\Omega) < D, \tag{4.3}$$

where $D > 0$ is a given constant, can be easily checked on a computer for an arbitrary nonuniform sampling scheme Ω. Moreover, as we shall show below, for special sampling schemes, e.g. polar and spiral, it is always possible to construct them so that they satisfy the condition (4.3). The advantage of considering density condition in the Euclidean norm lies in its symmetry.

We mention that in [11], one can find a construction of a spiral sampling scheme satisfying condition (4.3). Here, we use a slightly different spiral scheme, one which has an accumulation point at the origin and cannot be treated without weights. More precisely, we use the constant angular velocity spiral, whereas Benedetto & Wu [11] use the constant linear velocity spiral (see [16, Fig 2]). Also, beside giving a sufficient condition for a spiral sampling scheme in order to satisfy (4.3), we provide both sufficient and necessary condition such that polar and jittered sampling schemes are appropriately dense.

4.1 Jittered sampling scheme

This sampling scheme is a standard model for jitter error, which appears when the measurement device is not scanning exactly on a uniform grid; see Figure 1. Due to its simplicity, we do not necessarily need to use Euclidean norm in this case, therefore we consider directly the condition (4.1), and then, for completeness, we also consider (4.2). For a given sampling bandwidth $K > 0$ and parameters $\epsilon > 0$ and $\eta \geq 0$, we define the jittered sampling scheme as

$$\Omega_K = \{(n,m)\epsilon + \eta_{n,m} : n, m = -\lfloor K/\epsilon \rfloor, \ldots, \lfloor K/\epsilon \rfloor\}, \tag{4.4}$$

where $\eta_{n,m} = (\eta^x_{n,m}, \eta^y_{n,m})$ with $\eta^x_{n,m}$ and $\eta^y_{n,m}$ such that $|\eta^x_{n,m}|, |\eta^y_{n,m}| \leq \eta$. Note that $\Omega_K \subseteq Y_{K'} = K'[-1,1]^2$, where $K' = \epsilon\lfloor K/\epsilon \rfloor + \eta$. Now, the following can easily be seen:

Proposition 4.1. *Let $E = [-1,1]^2$. Let also $K > 0$, $\epsilon > 0$ and $\eta \geq 0$ be given. The sampling scheme Ω_K defined by* (4.4) *is*

1. *$(\delta_{E^\circ}, \epsilon\lfloor K/\epsilon \rfloor + \eta)$-dense with respect to $Y = [-1,1]^2$ and with $\delta_{E^\circ}(\Omega_K) < 1/4$ if and only if $\epsilon + 2\eta < 1/4$.*
2. *$(\delta_{\mathcal{B}_1}, \epsilon\lfloor K/\epsilon \rfloor + \eta)$-dense with respect to $Y = [-1,1]^2$ and with $\delta_{\mathcal{B}_1}(\Omega_K) < \log(2)/(2\pi\sqrt{2})$ if and only if $\epsilon + 2\eta < \log(2)/(2\pi)$.*

4.2 Polar sampling scheme

Here, we discuss an important type of sampling scheme used in MRI and also whenever the Radon transform is involved in sampling process, see Figure 1. For a given sampling bandwidth $K > 0$ and separation between consecutive concentric circles $r > 0$ we define a polar sampling scheme as

$$\Omega_K = \left\{ m r e^{\mathrm{i} n \Delta\theta} \ : \ m = -\lfloor K/r \rfloor, \ldots, \lfloor K/r \rfloor, \ n = 0, \ldots, N-1 \right\}, \tag{4.5}$$

where $\Delta\theta = \pi/N \in (0,\pi)$ is the angle between neighbouring radial lines and $N \in \mathbb{N}$ is the number of radial lines in the upper half-plane. Note that $\Omega_K \subseteq \mathcal{B}_{r\lfloor K/r \rfloor} \subseteq \hat{\mathbb{R}}^d$. In what follows we shall assume that $K/r \in \mathbb{N}$ for simplicity.

Proposition 4.2. *Let $D > 0$, $K > D$, and $r \in (0, 2D)$ be given such that $K/r \in \mathbb{N}$. The sampling scheme Ω_K given by* (4.5) *is $(K, \delta_{\mathcal{B}_1})$-dense with respect to $Y = \mathcal{B}_1$ and with*

$$\delta_{\mathcal{B}_1}(\Omega_K) < D$$

if and only if

$$\Delta\theta < 2 \arctan \frac{\sqrt{D^2 - (r/2)^2}}{K - r/2}. \tag{4.6}$$

Proof. To prove this claim, we need to calculate

$$\sup_{\hat{y}\in\mathcal{B}_K}\inf_{\omega\in\Omega_K}|\hat{y}-\omega|_{\mathcal{B}_1}. \tag{4.7}$$

Since $\mathcal{B}_1$ is symmetric with respect to any direction, and since the symmetry of polar sampling scheme, in (4.7) without loss of generality we may assume:

1. $\hat{y}\in\left\{s\mathrm{e}^{\mathrm{i}\theta}: s\in[0,K],\ \theta\in[0,\Delta\theta/2]\right\}$,
2. $\omega\in\{mr: m=0,\dots,K/r\}$.

Now it is easily seen that the supremum (4.7) is attained for $\hat{y}_0=(m+d_0)r\mathrm{e}^{\mathrm{i}\theta_0}$, where $(m+1)r=K$, $\theta_0=\Delta\theta/2$ and $d_0\in(0,1)$ is such that

$$|(m+d_0)r\mathrm{e}^{\mathrm{i}\theta_0}-mr|=|(m+d_0)r\mathrm{e}^{\mathrm{i}\theta_0}-(m+1)r|.$$

This equation gives

$$m+d_0=\frac{2m+1}{2\cos\theta_0},$$

and therefore

$$\sup_{\hat{y}\in\mathcal{B}_K}\inf_{\omega\in\Omega_K}|\hat{y}-\omega|=r\left|\frac{2m+1}{2\cos\theta_0}\mathrm{e}^{\mathrm{i}\theta_0}-m\right|=\sqrt{(r/2)^2+\frac{r^2(2m+1)^2}{4}(\tan\theta_0)^2}.$$

Hence, having $\delta_{\mathcal{B}_1}(\Omega_K)<D$ in the domain $\mathcal{B}_K$ is equivalent to

$$\sqrt{(r/2)^2+((K-r/2)\tan(\Delta\theta/2))^2}<D.$$

i.e.

$$\Delta\theta<2\arctan\frac{\sqrt{D^2-(r/2)^2}}{K-r/2},$$

which proves our claim. □

Also, note that in order Ω_K to be $(K,\delta_{\mathcal{B}_1})$-dense with respect to $Y=\mathcal{B}_1$ and $\delta_{\mathcal{B}_1}(\Omega_K)<D$, it is enough to take the number of radial lines corresponding to the following formula

$$N\geq\left\lceil\frac{\pi}{2\arctan\frac{\sqrt{D^2-(r/2)^2}}{K-r/2}}\right\rceil+1.$$

This proposition asserts that δ-density of polar sampling scheme is satisfied if and only if the angle $\Delta\theta$ is sufficiently small and taken according to the formula (4.6). From (4.6), it is evident that the angle $\Delta\theta$ goes to zero linearly in $1/K$ when $K\to\infty$. Therefore, the condition $\delta_{\mathcal{B}_1}(\Omega_K)<D$ implies that the points Ω_K accumulate at the inner concentric circles as we increase K. Thus, the unweighted frame bounds for the frame sequence corresponding to Ω_K clearly blow up as $K\to\infty$, which can be prevented by using the weights.

4.3 Spiral sampling scheme

For a given $r>0$,

$$S_r(\theta)=r\tfrac{\theta}{2\pi}\mathrm{e}^{\mathrm{i}\theta},\quad \theta\geq 0, \tag{4.8}$$

is a spiral trajectory in $\hat{\mathbb{R}}^2$ with the constant separation r between the spiral turns. If $\theta\in[0,2\pi k]$ for $k\in\mathbb{N}$, then the number of turns in the spiral is exactly k. For given $r>0$ and $k\in\mathbb{N}$, let $Y_{rk}\subseteq\hat{\mathbb{R}}^2$ be defined as

$$Y_{rk}=\left\{S_\rho(\theta):\rho\in[0,r],\ \theta\in[0,2\pi k]\right\}. \tag{4.9}$$

Then $S_r(\theta) \subseteq Y_{rk} \subseteq \mathcal{B}_{rk}$, for $\theta \in [0, 2\pi k]$.

Now, let $K > 0$ and $r > 0$ be given, and for simplicity assume that they are such that $K/r = k \in \mathbb{N}$. We define a spiral sampling scheme as

$$\Omega_K = \left\{ r \tfrac{n\Delta\theta}{2\pi} \mathrm{e}^{\mathrm{i}n\Delta\theta} : n = 0, \dots, Nk \right\}. \tag{4.10}$$

where $\Delta\theta = 2\pi/N \in (0, \pi)$, $N \in \mathbb{N}$, is a discretization angle. Note that this Ω_K represents a discretization of the spiral trajectory (4.8), which consists of k turns with the constant separation r between them and with a constant angular distance $\Delta\theta$. Also, note that $\Omega_K \subseteq Y_K = KY \subseteq \mathcal{B}_K \subseteq \hat{\mathbb{R}}^2$, where Y is

$$Y = \left\{ \rho \tfrac{\theta}{2\pi} \mathrm{e}^{\mathrm{i}\theta} : \rho \in [0,1],\ \theta \in [0, 2\pi] \right\} \tag{4.11}$$

i.e. Y is given by (4.9) for $r = k = 1$.

Proposition 4.3. *Let $D > 0$, $K > 4/5D$ and let $r \in (0, 2D)$ be given such that $K/r = k \in \mathbb{N}$. The sampling scheme Ω_K defined as* (4.10) *is $(\delta_{\mathcal{B}_1}, K)$-dense with respect to Y given by* (4.11) *and with*

$$\delta_{\mathcal{B}_1}(\Omega_K) < D$$

if the angle $\Delta\theta$ is chosen small enough depending on K.

Proof. To prove this claim, we want to estimate $\delta_{\mathcal{B}_1}(\Omega_K)$. Similarly as in [11], note that, by the triangle inequality

$$\delta_{\mathcal{B}_1}(\Omega_K) \le \frac{r}{2} + |S_r(2\pi k) - S_r(2\pi k - \Delta\theta/2)|$$

where $S_r(\cdot)$ is given by (4.8). Therefore, the density condition is satisfied if $\Delta\theta$ is such that

$$d_{r,k}(\Delta\theta) = |S_r(2\pi k) - S_r(2\pi k - \Delta\theta/2)| < D - \frac{r}{2}.$$

Hence, it is enough to choose $\Delta\theta$ as

$$\Delta\theta < \tilde{\theta},$$

where $\tilde{\theta}$ is such that $d_{r,k}(\tilde{\theta}) = D - r/2$. This $\tilde{\theta}$ exists and it is unique on the interval $(0, \pi)$, since the function $d_{r,k}(\cdot)$ is continuous and strictly increasing on $(0, \pi)$ and also

$$\lim_{\Delta\theta \to 0} d_{r,k}(\Delta\theta) = 0 < D - \frac{r}{2}, \quad \lim_{\Delta\theta \to \pi} d_{r,k}(\Delta\theta) = r\sqrt{k^2 + \left(k - \frac{1}{4}\right)^2} \ge \frac{5}{4}K > D - \frac{r}{2}.$$

□

Let us mention here that in a similar manner an interleaving spiral sampling scheme can be annalyzed. An interleaving spiral consists of multiple single spirals. Both of these spiral sampling schemes are shown in Figure 1.

5 Numerical results

Finally, in this section, we present some numerical experiments illustrating the developed theory.

Some of the advantages of using weights have been already reported earlier in the literature, see for example [21, 22, 28] and also [31, 46]. In a different setting, in Figure 2, we provide further insight on the necessity of using weights. To this end, we test a spiral sampling scheme, which gains an accumulation point as we increase the sampling bandwidth and which is constructed as in §4.3 so that it satisfies the density condition provided by Theorem 1.1. For this experiment, we perform function recovery using NUGS with boundary corrected Daubechies wavelets of order 3 (DB3).

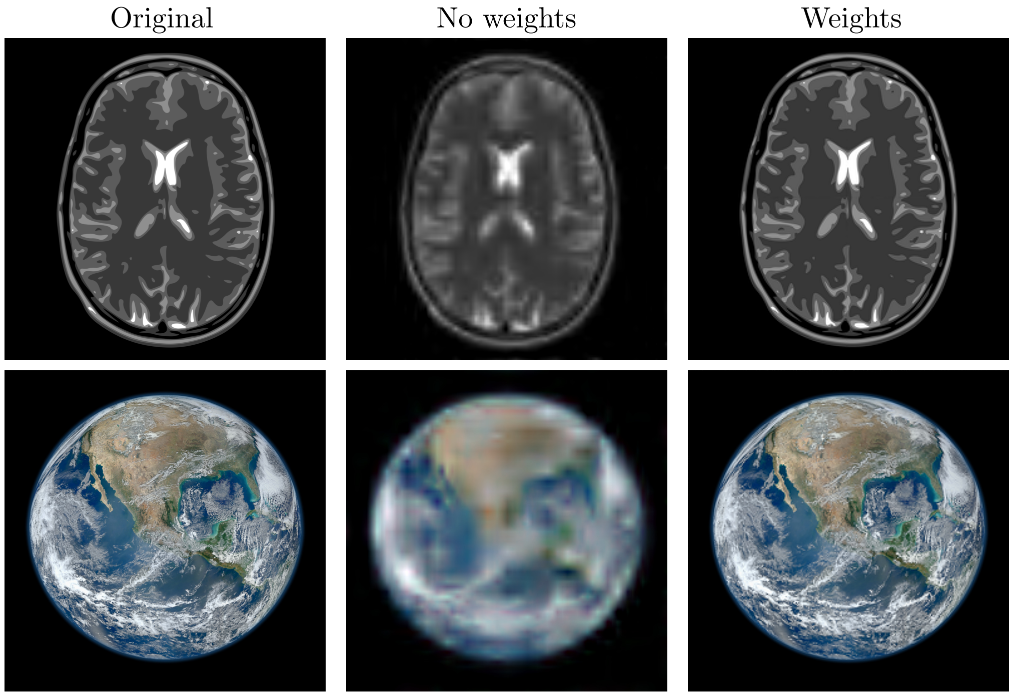

Figure 2: Analytic phantom and a high resolution image of earth are reconstructed by NUGS with 256×256 DB3 wavelets. Left: original rasterized images; middle: NUGS reconstruction without weights and 40 iterations; right: NUGS with weights and 20 iterations. In all cases, samples are taken according to spiral sampling scheme with $K = 128$ and $\delta_1 < 0.25$.

As discussed in [1], the condition number of the least-squares system (3.7) of the NUGS approximation determines the number of iterations required in an iterative solver such as conjugate gradients. As shown in Figure 2, even with the doubled number of iterations, the reconstruction obtained without weights is distinctly inferior. The fact that the reconstruction is bad even with a large number of iterations is due to two effects. First, many iterations are required for convergence of the iterative solver. Second, even if iterated to convergence, the reconstruction constant $C(\Omega, \mathrm{T})$, which determines the reconstruction error and depends on the ratio B/A, is also large in the unweighted case. On the other hand, if weights are used, we obtain a good approximation which is guaranteed by controllable weighted Fourier frame bounds.

In Figure 3, we illustrate NUGS method when applied to a two-dimensional continuous function. We also compare it with the earlier mentioned common heuristic approach known as gridding [31]. The NUGS is used with Haar wavelets and with boundary corrected DB2 wavelets. As noted earlier, the NUGS with Haar wavelets is equivalent to the iterative algorithms such as the one found in [46]. In all cases, the same set of samples acquired on a polar sampling scheme is used. Moreover, we present the results when noise is added to the samples (SNR = 40dB). As demonstrated in Figure 3 for the two-dimensional setting (see [1] for univariate examples), the major advantage of NUGS is the possibility to change the approximation space T and achieve better reconstructions.

6 Conclusions

In the paper, we provide new theoretical insight of when a given countable set of sampling points yields a weighted Fourier frame, and therefore permits a multidimensional function recovery. To have a weighted Fourier frame for the space of L^2 functions supported on a compact convex and symmetric set E, it is enough to take pointwise measurements of its Fourier transform at points with density $\delta_{E^\circ} < 1/4$. Separation of sampling points is not required. Moreover, the

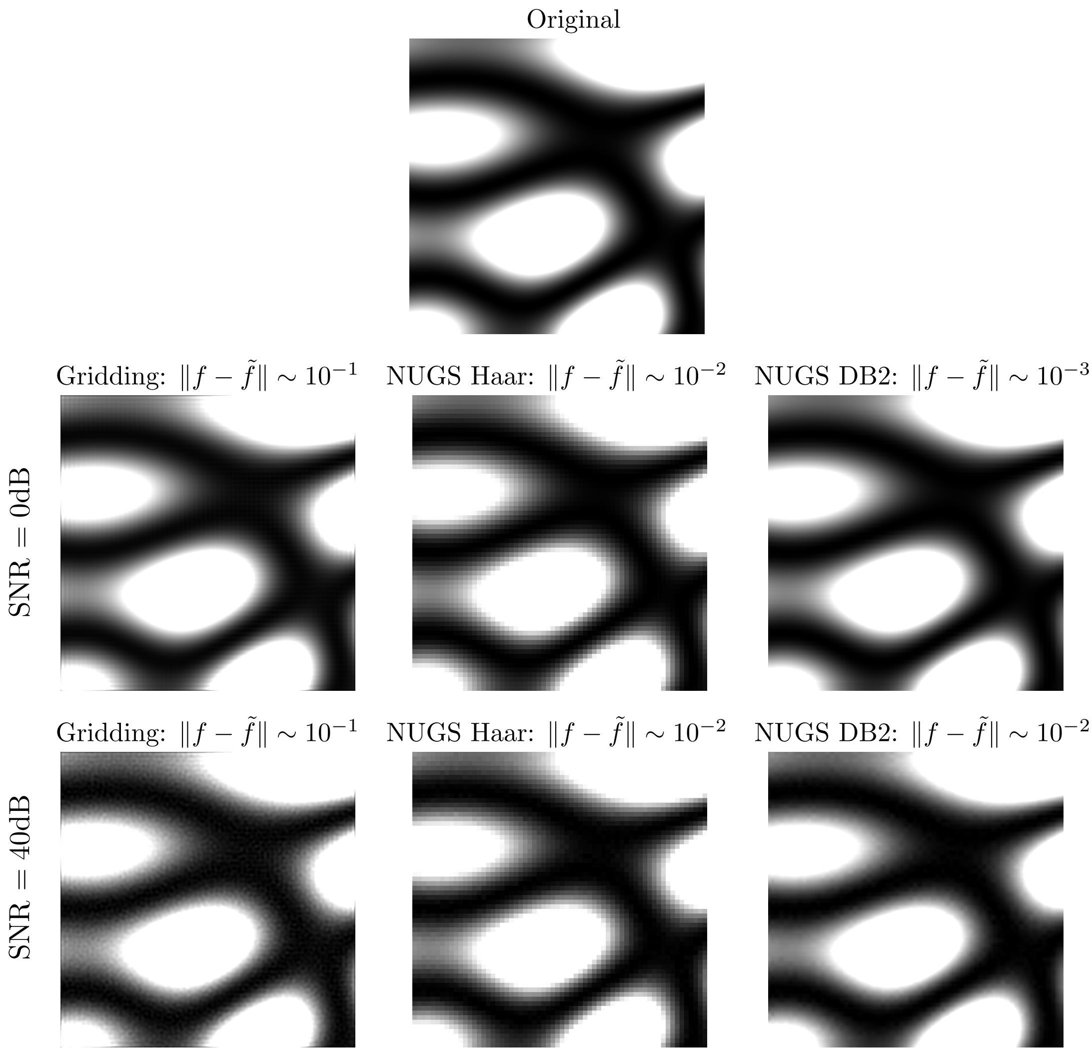

Figure 3: The function $f(x, y) = (\sin(3\pi(y + 1/2)) + (x + 1/4)\cos(2\pi x(y + 1/2))))^2$, supported on $E = [-1, 1]^2$, is reconstructed by gridding, NUGS with 64×64 Haar wavelets and NUGS with 64×64 DB2 wavelets. On the lower set of pictures, white Gaussian noise is added to the samples with the signal-to-noise ratio (SNR) of 40dB. Next to each reconstruction $\tilde{f}$ the order of the L^2-error $\|f - \tilde{f}\|$ is written. The polar sampling scheme is used with $K = 32$ and $\delta_1 < 0.25$.

weighted Fourier frame bounds are explicitly estimated in the case of smaller densities than previously known, and in particular, their dimension dependence is removed for the space of functions supported on spheres. However, it remains an open problem to explicitly estimate frame bounds for even smaller densities (larger δ), closer to the condition $\delta_{E^\circ} < 1/4$.

By exploiting these novel results on weighted Fourier frames, the method for recovering a function in any given finite dimensional space, known as NUGS, is analysed in multivariate setting. Its stability and accuracy are guaranteed provided that finitely many samples are taken with both density and bandwidth large enough. The density required is the same as the one that guarantees weighted Fourier frames.

It remains an open question how to choose the sampling bandwidth K depending on the specific reconstruction space. In [1], the authors considered important case of reconstruction spaces T consisting of compactly supporting wavelets in the one-dimensional setting. For any $\epsilon > 0$, it was shown that $R_K(\mathrm{T}) < \epsilon$, provided $K \geq c(\epsilon)M$, where $M = \dim(\mathrm{T})$ and $c(\epsilon) > 0$ is a constant depending on ϵ only (see [1, Thm. 5.3 and Thm. 5.4]). This means that a linear scaling of the sampling bandwidth K with the wavelet dimension M is sufficient for stable recovery (necessity was also shown – see [1, Thm. 6.1]). For this reason, wavelets subspaces are up to constant factors optimal spaces for reconstruction. We expect these results to extend to the multivariate case, but this is left for further investigations.

Having developed the NUGS framework in multivariate setting, it is possible to consider recoveries from nonuniform samples in any finite dimensional space one desires. Besides wavelets, one can consider spaces consisting of algebraic or trigonometric polynomials as well as important generalizations of wavelets, such as curvelets and shearlets. This is also left for future work.

Acknowledgements

The authors would like to thank Karlheinz Gröchenig and Gil Ramos for useful discussions and Clarice Poon for Matlab code used in an initial stage of our implementation. BA acknowledges support from the NSF DMS grant 1318894. MG acknowledges support from the UK EPSRC grant EP/H023348/1 for the University of Cambridge Centre for Doctoral Training, the Cambridge Centre for Analysis. AH acknowledges support from a Royal Society University Research Fellowship as well as the EPSRC grant EP/L003457/1.

References

[1] B. Adcock, M. Gataric, and A. C. Hansen. On stable reconstructions from nonuniform Fourier measurements. *arXiv:1310.7820*, 2013.

[2] B. Adcock and A. C. Hansen. Reduced consistency sampling in Hilbert spaces. In *Proceedings of the 9th International Conference on Sampling Theory and Applications*, 2011.

[3] B. Adcock and A. C. Hansen. A generalized sampling theorem for stable reconstructions in arbitrary bases. *J. Fourier Anal. Appl.*, 18(4):685–716, 2012.

[4] B. Adcock and A. C. Hansen. Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon. *Appl. Comput. Harmon. Anal.*, 32(3):357–388, 2012.

[5] B. Adcock, A. C. Hansen, and C. Poon. Beyond consistent reconstructions: optimality and sharp bounds for generalized sampling, and application to the uniform resampling problem. *SIAM J. Math. Anal.*, 45(5):3114–3131, 2013.

[6] B. Adcock, A. C. Hansen, and C. Poon. On optimal wavelet reconstructions from Fourier samples: linearity and universality of the stable sampling rate. *Appl. Comput. Harmon. Anal.*, 36(3):387–415, 2014.

[7] A. Aldroubi. Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces. *Appl. Comput. Harmon. Anal.*, 13:151–161, 2002.

[8] A. Aldroubi and K. Gröchenig. Nonuniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.*, 43:585–620, 2001.

[9] R. F. Bass and K. Gröchenig. Random sampling of multivariate trigonometric polynomials. *SIAM J. Math. Anal.*, 36(3):773–795 (electronic), 2004/05.

[10] J. J. Benedetto. Irregular sampling and frames. In *Wavelets*, volume 2 of *Wavelet Anal. Appl.*, pages 445–507. Academic Press, Boston, MA, 1992.

[11] J. J. Benedetto and H. C. Wu. Non-uniform sampling and spiral MRI reconstruction. *Proc. SPIE*, 4119:130–141, 2000.

[12] J.J. Benedetto and P.J.S.G. Ferreira. *Modern Sampling Theory: Mathematics and Applications*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, 2001.

[13] A. Beurling. Local harmonic analysis with some applications to differential operators. In *Some Recent Advances in the Basic Sciences, Vol. 1 (Proc. Annual Sci. Conf., Belfer Grad. School Sci., Yeshiva Univ., New York, 1962–1964)*, pages 109–125. Belfer Graduate School of Science, Yeshiva Univ., New York, 1966.

[14] A. Beurling. *The collected works of Arne Beurling. Vol. 2.* Contemporary Mathematicians. Birkhäuser Boston Inc., Boston, MA, 1989. Harmonic analysis, Edited by L. Carleson, P. Malliavin, J. Neuberger and J. Wermer.

[15] O. Christensen. Frames, Riesz bases, and discrete Gabor/wavelet expansions. *Bull. Amer. Math. Soc*, 38(3):273–291, 2001.

[16] B. M. A. Delattre, R. M. Heidemann, L. A. Crowe, J.-P. Vallée, and J.-N. Hyacinthe. Spiral demystified. *Magn. Reson. Imaging*, 28(6):862–881, 2010.

[17] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[18] Y. C. Eldar. Sampling without input constraints: Consistent reconstruction in arbitrary spaces. In A. I. Zayed and J. J. Benedetto, editors, *Sampling, Wavelets and Tomography*, pages 33–60. Boston, MA: Birkhäuser, 2004.

[19] Y. C. Eldar and T. Werther. General framework for consistent sampling in Hilbert spaces. *Int. J. Wavelets Multiresolut. Inf. Process.*, 3(3):347, 2005.

[20] C. L. Epstein. *Introduction to the mathematics of medical imaging.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2008.

[21] H. G. Feichtinger and K. Gröchenig. Theory and practice of irregular sampling. In J. J. Benedetto and M. Frazier, editors, *Wavelets: Mathematics and Applications*, pages 305–363. Boca Raton, FL: CRC, 1994.

[22] H. G. Feichtinger, K. Gröchenig, and T. Strohmer. Efficient numerical methods in nonuniform sampling theory. *Numer. Math.*, 69:423–440, 1995.

[23] J. A. Fessler and B. P. Sutton. Nonuniform fast Fourier transforms using min-max interpolation. *IEEE Trans. Signal Process.*, 51(2):560–574, 2003.

[24] J.-P. Gabardo. Weighted tight frames of exponentials on a finite interval. *Monatsh. Math.*, 116(3-4):197–229, 1993.

[25] K. Gröchenig. Reconstruction algorithms in irregular sampling. *Math. Comp.*, 59:181–194, 1992.

[26] K. Gröchenig. Irregular sampling, Toeplitz matrices, and the approximation of entire functions of exponential type. *Math. Comp.*, 68(226):749–765, 1999.

[27] K. Gröchenig. Non-uniform sampling in higher dimensions: from trigonometric polynomials to bandlimited functions. In J. J. Benedetto, editor, *Modern Sampling Theory*, chapter 7, pages 155–171. Birkhöuser Boston, 2001.

[28] K. Gröchenig and T. Strohmer. Numerical and theoretical aspects of non-uniform sampling of band-limited images. In F. Marvasti, editor, *Nonuniform Sampling: Theory and Applications*, chapter 6, pages 283–324. Kluwer Academic, Dordrecht, The Netherlands, 2001.

[29] M. Guerquin-Kern, M. Häberlin, K. P. Pruessmann, and M. Unser. A fast wavelet-based reconstruction method for Magnetic Resonance Imaging. *IEEE Trans. Med. Imaging*, 30(9):1649–1660, 2011.

[30] T. Hrycak and K. Gröchenig. Pseudospectral Fourier reconstruction with the modified inverse polynomial reconstruction method. *J. Comput. Phys.*, 229(3):933–946, 2010.

[31] J. I. Jackson, C. H. Meyer, D. G. Nishimura, and A. Macovski. Selection of a convolution function for Fourier inversion using gridding. *IEEE Trans. Med. Imaging*, 10:473–478, 1991.

[32] S. Jaffard. A density criterion for frames of complex exponentials. *Mich. Math. J.*, 38(3):339–348, 1991.

[33] Jae-Hun Jung and Bernie D. Shizgal. Generalization of the inverse polynomial reconstruction method in the resolution of the Gibbs phenomenon. *J. Comput. Appl. Math.*, 172(1):131–151, 2004.

[34] S. Kunis and D. Potts. Stability results for scattered data interpolation by trigonometric polynomials. *SIAM J. Sci. Comput.*, 29(4):1403–1419 (electronic), 2007.

[35] H. J. Landau. Necessary density conditions for sampling and interpolation of certain entire functions. *Acta Math.*, 117:37–52, 1967.

[36] N. Levinson. *Gap and Density Theorems*. American Mathematical Society Colloquium Publications, v. 26. American Mathematical Society, New York, 1940.

[37] F. Marvasti. *Nonuniform Sampling: Theory and Practice*. Number v. 1 in Information Technology Series. Springer US, 2001.

[38] A. Olevskii and A. Ulanovskii. On multi-dimensional sampling and interpolation. *Anal. Math. Phys.*, 2(2):149–170, 2012.

[39] R. E. A. C. Paley and N. Wiener. *Fourier transforms in the complex domain*, volume 19 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1987. Reprint of the 1934 original.

[40] D. Potts and M. Tasche. Numerical stability of nonequispaced fast Fourier transforms. *J. Comput. Appl. Math.*, 222(2):655–674, 2008.

[41] V. Rasche, R. Proksa, R. Sinkus, P. Bornert, and H. Eggers. Resampling of data between arbitrary grids using convolution interpolation. *IEEE Trans. Med. Imaging*, 18(5):385–392, 1999.

[42] D. Rosenfeld. New approach to gridding using regularlization and estimation theory. *Magn. Reson. Med.*, 48(1):193–202, 2002.

[43] K. Seip. On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$. *J. Funct. Anal.*, 130(1):131–160, 1995.

[44] K. Seip. *Interpolation and sampling in spaces of analytic functions*, volume 33 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2004.

[45] T. Strohmer. Numerical analysis of the non-uniform sampling problem. *J. Comput. Appl. Math.*, 122(1–2):297–316, 2000.

[46] B. P. Sutton, D. C. Noll, and J. A. Fessler. Fast, iterative image reconstruction for MRI in the presence of field inhomogeneities. *IEEE Trans. Med. Imaging*, 22(2):178–188, 2003.

[47] M. Unser. Sampling—50 Years after Shannon. *Proceedings of the IEEE*, 88(4):569–587, 2000.

[48] M. Unser and A. Aldroubi. A general sampling theory for nonideal acquisition devices. *IEEE Trans. Signal Process.*, 42(11):2915–2925, 1994.

[49] M. Unser and A. Aldroubi. A review of wavelets in biomedical applications. *Proc. IEEE*, 84(4):626–638, 1996.

[50] R. M. Young. *An Introduction to Nonharmonic Fourier Series*. Academic Press Inc., first edition, 2001.