Convergence theory for spectral deferred correction

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Abstract

In a recent article [2] Dutt, Greengard and Rohklin define two new methods of deferred correction. Convergence for the the first method, using one step methods, has been proven in Hansen [8]. In this paper we augment the theory presented in [8] and use this to prove convergence for the second deferred correction method in [2] using linear k-step methods. This method has been known as Spectral Deferred Correction.

1 Introduction

Deferred correction methods for solving the initial value problem (IVP)

$$y'(t) = f(t, y), \quad y(t_0) = y_0 \in \mathbb{R}^N \tag{1.1}$$

for a system of ordinary differential equations are presented. In particular, a modification of previous deferred correction methods is considered and convergence of the modified method is proved.

Deferred correction methods were investigated quite intensively in the 1970's and the 1980's [19], [4], [5], [7], [9], [17]. Lately new deferred correction methods have been derived [2] with very promising numerical results [2], [12], [13], [14]. The methods presented in [2] are inspired by Zadunaisky's Iterated Defect Correction (IDeC) [19] however, the convergence proofs of IDeC in Frank and Ueberhuber [4], [6] do not justify convergence for the methods invented by Dutt, Greengard and Rohklin. For an introduction to IDeC we refer to Bohmer and Stetter [1]. The convergence proofs in this paper are quite different from the proofs presented by Frank and Ueberhuber even though both approaches make use of Stetter's framework for numerical ODEs [18]

Our work is inspired by the theory developed in Skeel [17] and the theory presented in this article augment Skeel's ideas and seems to be more general. We follow Skeel closely in his emphasis on the smoothness of the global error rather than the existence of asymptotic expansion of the global error. The smoothness of the global error can be measured by means of the "discrete sobolev" defined in terms of divided differences (Lindberg, Skeel and Van Rosendale [16]).

The definitions in [16] are generalizations of the one-step Spijker norm [18]. One problem that occurs using this norm is that the one-step Spijker norm is not suitable for linear k-step methods. A generalization of the k-step Spijker norm is therefore needed. By using our generalized k-step Spijker norm together with the framework developed in [8] we can show convergence for the two deferred correction methods of Dutt, Greengard and Rohklin, using linear k-step methods.

Even though Skeel suggest the use of linear k-step methods in [17], ex 4.1, it is straight forward to show that the linear k-step methods are not stable in the generalized one-step Spijker

norm. We emphasize that we use Stetter's definition of Stability, and this will become clear in Section 3. However, Skeel's suggestion of using linear k-step methods make perfectly sense, we only have to use a generalized k-step Spijker norm to prove convergence. Stability for linear k-step methods, using this norm, can be shown.

2 Spectral Deferred Correction

The idea of the Spectral Deferred Correction (SDC) method is to avoid the interpolation and differentiation used in Dutt-Greengard-Rohklin (DGR) correction [2]. In [2] the SDC algorithm is derived via an integral equation. We derive the method in a slightly different way, however the final algorithm is the same. The approach presented here makes the convergence proof for this method fit into the same framework as used to prove convergence for DGR-Correction [8].

2.1 Description of the method

We consider the same error equation from [2] and [8], namely

$$\begin{aligned} \tilde{\delta}'(t) &= f(t, \tilde{\delta}(t) + g(t)) - g'(t) \\ \tilde{\delta}(0) &= 0 \end{aligned} \tag{2.1}$$

where $g(t)$ is a previously calculated numerical solution to (1.1). This can be written as

$$\tilde{\delta}'(t) - \varepsilon'(t) = f(t, \tilde{\delta}(t) + g(t)) - f(t, g(t)),$$

where $\varepsilon'(t) = f(t, g(t)) - g'(t)$. Now

$$\varepsilon(t) = y_0 + \int_0^t f(s, g(s))\, ds - g(t).$$

Letting $\tilde{\xi}(t) = \tilde{\delta}(t) - \varepsilon(t)$ we get

$$\begin{aligned} \tilde{\xi}'(t) &= f(t, \tilde{\xi}(t) + \varepsilon(t) + g(t)) - f(t, g(t)) \\ \tilde{\xi}(0) &= \tilde{\xi}_0 = 0. \end{aligned} \tag{2.2}$$

Solving this problem numerically with solution $\xi \in \mathbb{R}^{n+1}$ we now define $\delta \in \mathbb{R}^{n+1}$ such that $\delta_\nu = \xi_\nu + \varepsilon(t_\nu)$ assuming ε is known. Hence

$$\begin{aligned} \delta_\nu &\approx \tilde{\xi}(t_\nu) + \varepsilon(t_\nu) \\ &= \tilde{\xi}_0 + \int_0^{t_\nu} f(s, \tilde{\xi}(s) + \varepsilon(s) + g(s)) - f(s, g(s))\, ds + \varepsilon(t_\nu) \\ &= \tilde{\delta}(t_\nu) \end{aligned}$$

where $\tilde{\delta}$ is the solution of (2.1).

As discussed in [2] one wants to avoid interpolation and differentiation. This approach clearly eliminates the differentiation, but if a nonlinear numerical method is used, such as Runge-Kutta methods, we may need values of g outside the mesh. This suggests that SDC with

nonlinear methods might not solve the interpolation problem. However, it is very tempting to use linear methods such as linear k-step methods. Using Euler's method we get

$$\frac{1}{h}(\xi_\nu - \xi_{\nu-1}) = f(t_{\nu-1}, \xi_{\nu-1} + \varepsilon(t_{\nu-1}) + g(t_{\nu-1})) - f(t_{\nu-1}, g(t_{\nu-1}))$$

which by the definition of δ is equivalent to

$$\begin{aligned}\frac{1}{h}(\delta_\nu - \varepsilon(t_\nu) - (\delta_{\nu-1} - \varepsilon(t_{\nu-1}))) \\ = f(t_{\nu-1}, \delta_{\nu-1} + g(t_{\nu-1})) - f(t_{\nu-1}, g(t_{\nu-1})).\end{aligned} \tag{2.3}$$

This clearly solves the interpolation and differentiation problem but still we have assumed that ε is known. A more natural assumption is that an approximation $\sigma \in \mathbb{R}^{n+1}$ to ε is known. In other words $\sigma_\nu \approx \varepsilon(t_\nu)$. Using this in (2.3) and letting $g(t_\nu) = \eta_\nu$ we get

$$\frac{1}{h}(\delta_\nu - \sigma_\nu - (\delta_{\nu-1} - \sigma_\nu)) = f(t_{\nu-1}, \delta_{\nu-1} + \eta_{\nu-1}) - f(t_{\nu-1}, \eta_{\nu-1}). \tag{2.4}$$

For a detailed discussion on how σ is calculated see Dutt, Greengard and Rokhlin [2]. We may now proceed with the algorithm for SDC using Euler's method.

Algorithm 2.1 `do j = 1,...,J`

- *Compute* σ.
- *Solve (2.4).*
- *Update* $\eta^{[j]} = \eta^{[j-1]} + \delta$.

`end do`

3 Theoretical framework

We will throughout the paper follow Stetter [18] in his formalism for analyzing numerical solutions of IVP. The following ideas are inspired by Skeel [17] and Lindberg [15] and their deferred correction approach. Some of our definitions are generalizations of Skeel's ideas. Stetter's idea is to write the IVP (1.1) as an operator equation

$$Fy = 0 \tag{3.1}$$

where $F : Y \to X$ is an operator from a normed linear space Y into a normed linear space X. As an approximation to this problem we consider

$$\phi_n(F)\eta = 0$$

where $\phi_n(F) : Y_n \to X_n$ and Y_n, X_n are discrete finite dimensional normed linear spaces and the subscript n is related to the dimension of the spaces.

Remark 3.1 *Stetter requires the spaces Y, X, Y_n and X_n to be Banach spaces. Completeness is not needed in general for analyzing the discretization error of the mapping $\phi_n(F)$, and one may therefore relax the completeness assumption.*

We will now give the definitions of the required operator and spaces. The operator F and the spaces Y and X will be defined such that the IVP (1.1) can be written as (3.1). Here we follow Skeel's definitions with minor alterations. As usual

$$\|z\|_\infty = \max_{t\in[a,b]} |z(t)|, \quad z \in C[a,b].$$

Define $Y^m := C^m[a,b]$ and let for $z \in Y^m$

$$\|z\|_{Y^m} = \max\{\|z\|_\infty, \frac{1}{2!}\|z'\|_\infty, \ldots, \frac{1}{m!}\|z^{[m]}\|_\infty\}. \tag{3.2}$$

The operator F should now be designed such that $Fy = 0$ gives us the IVP. To assure the uniqueness of y we define

$$Fz := \begin{cases} -z(0) + y_0 \\ -z'(t) + f(t, z(t)) \end{cases} \tag{3.3}$$

where $f : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$. We will throughout the paper assume that f is sufficiently smooth on $[a,b] \times \mathbb{R}^N$ with bounded derivatives. The inclusion of the initial value y_0 in F and the rather strong smoothness assumption on f make y the unique solution.

The previous definitions of F and Y^m make the following a natural definition: $X^m := \mathbb{R} \times C^{m-1}[a,b]$ and for $g = g_0 \times g(t) \in X^m$

$$\|g\|_{X^m} = \|Lg\|_{Y^m}, \tag{3.4}$$

where

$$(Lg)(t) = g_0 + \int_0^t g(s)\, ds.$$

As we mentioned in the beginning of this section, we will approximate the continuous operator equation $Fy = 0$ with a discrete problem $\phi_n(F)\eta = 0$. But before we can proceed with the definitions we need the definition of a grid.

Definition 3.1 *A grid $\mathbb{G}_n$ on $[a,b]$ is any finite set of grid-points $t_\nu \in [a,b]$, $\nu = 0, \ldots, n$, with*

$$t_0 = a, \quad t_{\nu-1} < t_\nu, \quad t_n = b, \quad \nu = 1, \ldots, n. \tag{3.5}$$

The quantities $h_\nu := t_\nu - t_{\nu-1} > 0, \quad \nu = 1, \ldots, n$, are the steps of the grid $\mathbb{G}_n = \{t_\nu : \nu = 0, \ldots, n\}$,

$$h := \max_{\nu=1,\ldots,n} h_\nu \tag{3.6}$$

is the maximal step of $\mathbb{G}_n$.

For the discrete space $Y_{n,m}$ and $X_{n,m}$ we let

$$Y_{n,m} = X_{n,m} := \mathbb{R}^{N(n+1)}$$

and define $\Delta_n : Y^m \to Y_{n,m}$ and $\Lambda_n : X^m \to X_{n,m}$ by $(\Delta_n z)_\nu = z(t_\nu)$ and

$$(\Lambda_n g)_\nu = \begin{cases} g_0 & \nu = 0 \\ g(t_{\nu-1}) & \nu = 1, \ldots, n \end{cases} \text{for} \quad g = \begin{pmatrix} g_0 \\ g(t) \end{pmatrix} \in X,$$

where $t_\nu \in \mathbb{G}_n$.

The definitions of the norms on Y^m and X^m give a natural choice of norms on $Y_{n,m}$ and $X_{n,m}$ using backward divided differences. For one step methods the obvious choice is discussed in [8], but when using multi-step methods one has to be a little careful. It is the crucial stability inequality one has to have in mind when choosing the norms. Now, it is not difficult to show that the norms used in [8] are insufficient when using multi-step methods. Even the simplest Adams methods will not satisfy the stability inequality with those norm. Thus, a little alteration is needed. As usual

$$\|\eta\|_\infty = \max_{0\le\nu\le n} |\eta_\nu|.$$

Now for $\eta \in Y_{n,m}$ (when considering a k-step method)

$$\|\eta\|_{Y_{n,m}} := \max \|\eta\|_\infty, \|\tilde{D}\eta\|_\infty, \ldots, \|\tilde{D}^m\eta\|_\infty, \tag{3.7}$$

where

$$\tilde{D}^m\eta = (\eta_0, \eta_1, \ldots, \eta_{k-1}, D\eta_k, D^2\eta_{k+1} \ldots, D^m\eta_m)^T$$

where D^m denotes the usual backward divided differences defined recursively by

$$D^0\eta_\nu = \eta_\nu$$

$$D^m\eta_\nu = \frac{D^{m-1}\eta_\nu - D^{m-1}\eta_{\nu-1}}{t_\nu - t_{\nu-m}}, \quad m = 1, 2, \ldots.$$

The norm on $X_{n,m}$ needs to be picked carefully. Clearly the stability inequality (3.3) depends on the norms on $Y_{n,m}$ and $X_{n,m}$ and on the mapping $\phi_n(F) : Y_{n,m} \to X_{n,m}$. It is therefore natural to let the norms depend on ϕ. In particular we will use a generalization of the norm defined in Skeel [17] which is a generalization of the one-step Spijker norm. Our norm is a generalization of the k-step Spijker norm. In particular for $\delta \in X_{n,m}$

$$\|\delta\|_{X_{n,m}} := \|A \otimes I\delta\|_{Y_{n,m}}, \tag{3.8}$$

where

$$A = \begin{bmatrix} I_{k-1} & 0 \\ 0 & \tilde{A} \end{bmatrix}, \quad k = 1, 2, 3, \ldots,$$

and

$$\tilde{A} = \begin{bmatrix} 1 & & & & \\ 1 & h & & & \\ 1 & h & h & & \\ \vdots & \vdots & \vdots & & \\ 1 & h & h & \ldots & h \end{bmatrix}.$$

I_{k-1} is a $(k-1) \times (k-1)$ identity matrix. In particular

$$\|\delta\|_{X_{n,0}} = \max\{|\delta_0|, |\delta_1|, \ldots, |\delta_{k-2}|, \max_{0\le j\le n'} |\delta_{k-1} + \sum_{i=1}^{j} h\delta_{k-1+i}|\}$$

where $n' = n - k + 1$ and

$$\|\delta\|_{X_{n,1}} = \max\{|\delta_0|, |\delta_1|, \ldots, |\delta_{k-1}|, |\delta_k|, |\delta_{k+1}|, \ldots, |\delta_n|\}\}.$$

The relations between the spaces and mappings defined above can be visualized by the following diagram and example:

$$\begin{array}{ccc} Y & \xrightarrow{F} & X \\ {\scriptstyle \Delta_n}\downarrow & & \downarrow{\scriptstyle \Lambda_n} \\ Y_n & \xrightarrow{\phi_n(F)} & X_n \end{array}$$

Remark 3.2 *As mentioned in Stetter [18] this is not a commutative diagram, but an asymptotic (as $n \to \infty$) commutative diagram.*

Example 3.1 *Let ϕ_n in the discretization method ϕ be defined by Euler's method. We may define the following:*

For $n \in \mathbb{N}$, let

$$\mathbb{G}_n := \{t_\nu, \nu = 0, \ldots, n\}, \quad Y_n = X_n = \mathbb{R}^{n+1}$$

$$(\Delta_n y)_\nu = y(t_\nu) \quad \textit{for} \quad y \in Y,$$

$$(\Lambda_n d)_\nu = \begin{cases} d_0 & \nu = 0 \\ d(t_{\nu-1}) & \nu = 1, \ldots, n \end{cases} \textit{for} \quad d = \begin{pmatrix} d_0 \\ d(t) \end{pmatrix} \in X$$

$$\phi_n(F)(\eta)_\nu = \begin{cases} -\eta_0 + y_0 & \nu = 0 \\ -\frac{\eta(t_\nu) - \eta(t_{\nu-1})}{h} + f(t_{\nu-1}, \eta(t_{\nu-1})) & \nu = 1, \ldots, n. \end{cases}$$

In the analysis of discretization methods we are interested in how well the solution of $\phi_n(F)\eta = 0$ approximates the solution of $Fy = 0$. Two important definitions are therefore the definitions of the local discretization error and the global discretization error.

Definition 3.2 *The sequence $\{l_n\}_{n\in\mathbb{N}}$, $l_n \in X$ with*

$$l_n = \phi_n(F)\Delta_n y, \quad n \in \mathbb{N}$$

where $Fy = 0$ is called the local discretization error.

In the analysis of convergent discretization the concept of stability and consistency is very useful. As we know convergence is often shown indirectly by using stability and consistency. We emphasize that the following definitions may differ from other definitions in the literature. (For details see Stetter [18]).

Definition 3.3 *A discretization method ϕ is called stable at η if there exist constants S and $r > 0$ such that uniformly for all $n \in \mathbb{N}$,*

$$\|\eta_n^1 - \eta_n^2\|_{Y_n} \leq S\|\phi(F)_n\eta_n^1 - \phi(F)_n\eta_n^2\|_{X_n} \tag{3.9}$$

for all $\eta^{(i)}, \quad i = 1, 2$ such that

$$\|F_n\eta_n^i - F_n\eta_n\|_{X_n} < r \tag{3.10}$$

S and r are called stability bound and stability threshold.

3.1 The Local Error of Multistep methods

In Dutt, Greengard and Rohklin [2], the authors suggest use of multi-step methods as an improvement to the one step methods used in their experiment. We will in this section develop the theory needed to prove convergence for DGR-Correction using multi-step methods.

Definition 3.4 *A numerical integrator ϕ is called a linear k-step method when*

$$\begin{aligned}&\phi_n(F)(\eta)_\nu \\ &\quad= \begin{cases} -\eta_\nu + s_{n\nu}(F) & \nu = 0,\dots,k-1 \\ -\frac{1}{h}\sum_{j=0}^{k}\alpha_j\eta_{\nu-j} + \sum_{j=0}^{k}\beta_j f(t_{\nu-j},\eta_{\nu-j}) & \nu = k,\dots,n, \end{cases}\end{aligned} \tag{3.11}$$

where α_j and β_j are real coefficients and $s_n(F) \in \mathbb{R}^k$ are the starting values. Also, the integrator ϕ is said to be of order p if

$$-\frac{1}{h}\sum_{j=0}^{k}\alpha_j q(t_{\nu-j}) + \sum_{j=0}^{k}\beta_j q'(t_{\nu-j}) = 0$$

for all polynomials $q(t)$ of degree $\leq p$. The definition of the order allows a Taylor expansion argument, which gives the following expression for the local discretization error of a p-th order linear k-step method:

$$\begin{aligned}&\phi_n(F)(\Delta_n y)_\nu \\ &\qquad= \begin{cases} -s_{n\nu}(F) + (\Delta_n y)_\nu & \nu = 0,\dots,k-1 \\ \sum_{j=p}^{\mu-1}\frac{h^j}{j!}c_j y^{[j+1]}(t_{\nu-k}) + \mathcal{O}(h^{\mu+p}) & \nu = k,\dots,J \end{cases}\end{aligned} \tag{3.12}$$

where

$$c_j = \sum_{\mu=1}^{k}\frac{\mu^{j+1}\alpha_\mu}{j+1} - \mu^j\beta_\mu. \tag{3.13}$$

The question is, how to pick the starting values? We will see in the later theoretical framework that smoothness of the local discretization error will be crucial. In particular we will have to require that the starting values $s_n(F)$ is chosen such that the local error is both smooth and accurate. More precisely we need

$$\|\phi_n(F)(\Delta_n y)\|_{X_{n,m}} \leq Ch^p\|y\|_{Y^{m+p}} + \max_{0\leq\nu\leq k-1}|s_{n\nu}(F) - (\Delta_n y)_\nu| \tag{3.14}$$

for some $C > 0$, depending on α_j and β_j, bounding all the partial derivatives of f.

3.2 Stability of linear k-step methods

The stability inequality

$$\|\eta_n^1 - \eta_n^2\|_{Y_{n,m}} \leq S\|\phi_n(F)\eta_n^1 - \phi_n(F)\eta_n^2\|_{X_{n,m}} \tag{3.15}$$

is not trivial to show for linear k-step methods. Obviously the choice of norms on $Y_{n,m}$ and $X_{n,m}$ is important. The norms defined in (3.7) and (3.8) are inspired by the norms suggested in Skeel [17] but are more general. Skeel's norm on the space $X_{n,m}$ is a generalization of the one-step Spijker norm, but for linear k-step methods we cannot expect to be able to prove

the desired inequality using this norm. Our choice of norm on $X_{n,m}$ is a generalization of of the k-step Spijker norm and is therefore more suitable for k-step methods. As discussed in Stetter [18] strongly D-stability is necessary for the stability inequality to be satisfied using the Spijker norm. Hence we will have to restrict ourselves to k-step methods with very special stability properties.

Adams methods are strongly D-stable and as discussed in [2] they give excellent numerical results when using deferred correction. Following and extending example 4.2 in Skeel [17] we can show that for Adams methods the stability inequality holds for arbitrary m. In particular one can show that (3.15) is valid with $S = S(\theta)$ where θ is a bound on all the partial derivatives of f up to m and $S : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function. Throughout the article we will assume that the numerical methods used are stable with increasing function S.

4 Accuracy and convergence for SDC-method

4.1 SDC on operator form

The idea of SDC-methods is to avoid the interpolation and differentiation used in DGR-correction. We consider the same equation

$$\begin{aligned}\tilde{\delta}'(t) &= f(t, \tilde{\delta}(t) + g(t)) - g'(t)\\ \tilde{\delta}(0) &= \tilde{\delta}_0.\end{aligned}$$

which can be written as

$$\tilde{\delta}'(t) - \varepsilon'(t) = f(t, \tilde{\delta}(t) + g(t)) - f(t, g(t)),$$

where $\varepsilon'(t) = f(t, g(t)) - g'(t)$. Now

$$\varepsilon(t) = y_0 + \int_0^t f(s, g(s))\, ds - g(t).$$

Letting $\tilde{\xi}(t) = \tilde{\delta}(t) - \varepsilon(t)$ we get

$$\begin{aligned}\tilde{\xi}'(t) &= f(t, \tilde{\xi}(t) + \varepsilon(t) + g(t)) - f(t, g(t))\\ \tilde{\xi}(0) &= \tilde{\xi}_0 = 0.\end{aligned} \tag{4.1}$$

We now define $\hat{F}_g : Y^m \to X^m$ by

$$\hat{F}_g\tilde{\xi} := \begin{cases} -\tilde{\xi}(0) + \tilde{\xi}_0 \\ -\tilde{\xi}'(t) + f(t, \tilde{\xi}(t) + \varepsilon(t) + g(t)) - f(t, g(t)) \quad 0 \le t \le 1. \end{cases} \tag{4.2}$$

For notational purposes let

$$\phi_n(F) = L_n + G_n$$

such that

$$(L_n)(\eta)_\nu = \begin{cases} 0 & \nu = 0, \dots, k-1 \\ -\frac{1}{h}\sum_{j=0}^k \alpha_j \eta_{\nu-j} & \nu = k(1)n. \end{cases}$$

and

$$G_n(F)(\eta)_\nu = \begin{cases} -\eta_\nu + s_{n\nu}(F) & \nu = 0, \dots, k-1 \\ \sum_{j=0}^k \beta_j f(t_{\nu-j}, \eta_{\nu-j}) & \nu = k(1)n. \end{cases}$$

Solving $\phi_n(\hat{F})\xi_n = 0$ assuming ε is known we define $\delta_n = \xi_n + \Delta_n\varepsilon$. Thus

$$\begin{aligned} 0 &= \phi_n(\hat{F})(\xi)_\nu \\ &= (L_n)(\delta_n - \Delta_n\varepsilon)_\nu + (G_n)(\delta_n - \Delta_n\varepsilon)_\nu \end{aligned}$$

for $\nu = 0, \ldots, n$. If we want to advance the previously calculated solution from order r accuracy to order $p + r$ we will need starting values such that

$$\|\phi_n(F)(\Delta_n\tilde{\xi})\|_{X_{n,m}} \leq Ch^p\|\tilde{\xi}\|_{Y^{m+p}} + \tilde{C}h^{r+p}. \tag{4.3}$$

In practice ε is not exact, but calculated numerically, and we obtain $\sigma_n \in Y_{n,m}$ as an approximation to $\Delta_n\varepsilon$. For details see [2]. We will have to require

$$\|\sigma_n - \Delta_n\varepsilon\|_{Y_{n,m}} \leq Ch^{r+p} \tag{4.4}$$

for some $C > 0$. Then δ_n is the solution of the error equation

$$(L_n)(\sigma_n - \delta_n)_\nu = (G_n)(\delta_n - \Delta_n\varepsilon)_\nu \quad \nu = 0, \ldots, n. \tag{4.5}$$

Theorem 4.1 *Let y be the unique solution of $Fy = 0$ and suppose ϕ is stable and consistent of order p. Suppose also that $n \geq m + p + r$, $\|\eta_n - \Delta_n y\|_{Y_{n,m+p}} \leq Ch^r$ and δ_n satisfies (4.5) with (4.4) and (4.3) satisfied, then*

$$\|\delta_n + \eta_n - \Delta_n y\|_{Y_{n,m}} \leq Ch^{r+p}$$

for some $C > 0$.

PROOF. Let

$$g \in E_n = \{z \in Y^{m+p} : \|z - y\|_{Y^{m+p}} \leq Ch^r,\ \Delta_n z = \eta_n\}.$$

It is straightforward to show that there exists a $C > 0$ such that $E_n \neq \emptyset$, e.g. by using splines. Let

$$\varepsilon(t) = y(a) + \int_a^t f(s, g(s))\, ds - g(t).$$

By stability we have

$$\begin{aligned} &\|\delta_n + \eta_n - \Delta_n y\|_{Y_{n,m}} \\ &\quad = \|\delta_n - \Delta_n\varepsilon + \eta_n + \Delta_n\varepsilon - \Delta_n y\|_{Y_{n,m}} \\ &\quad \leq S(\hat{\theta})\|\phi_n(\hat{F})(\delta_n - \Delta_n\varepsilon) - \phi_n(\hat{F})(\Delta_n y - (\eta_n + \Delta_n\varepsilon))\|_{X_{n,m}}. \end{aligned}$$

We recognize the second term as the local discretization error (4.3) since $\tilde{\xi} = y - (g + \varepsilon)$ satisfies $\hat{F}_g\tilde{\xi} = 0$ and $\Delta_n g = \eta_n$. But before we can proceed, a little manipulation is needed. Splitting ϕ_n as in (4.1) and adding and subtracting the approximation of $\Delta_n\varepsilon$ gives

$$\begin{aligned} &\|\phi_n(\hat{F}_g)(\delta_n - \Delta_n\varepsilon) - \phi_n(\hat{F}_g)(\Delta_n y - (\eta_n + \Delta_n\varepsilon))\|_{X_{n,m}} \\ &\quad \leq \|L_n(\sigma_n - \Delta_n\varepsilon)\|_{X_{n,m}} + \|L_n(\delta_n - \sigma_n) + G_n(\delta - \Delta_n\varepsilon)\|_{X_{n,m}} \\ &\qquad + \|\phi_n(\hat{F}_g)(\Delta_n y - (\eta_n + \Delta_n\varepsilon))\|_{X_{n,m}} \\ &\quad = \|L_n(\sigma_n - \Delta_n\varepsilon)\|_{X_{n,m}} + \|\phi_n(\hat{F}_g)(\Delta_n y - (\eta_n + \Delta_n\varepsilon))\|_{X_{n,m}} \end{aligned}$$

and by the definition of δ_n we have $L_n(\delta_n - \sigma_n) + G_n(\delta - \Delta_n\varepsilon) = 0$. Hence

$$\begin{aligned}&\|\delta_n + \eta_n - \Delta_n y\|_{Y_{n,m}}\\ &\leq S(\hat{\theta}) \left[\|L_n(\sigma_n - \Delta_n\varepsilon)\|_{X_{n,m}} + \|\phi_n(\hat{F}_g)(\Delta_n y - (\eta_n + \Delta_n\varepsilon))\|_{X_{n,m}}\right].\end{aligned}$$

It is clear that

$$\|L_n\|_{X_{n,0}\to X_{n,m}} \leq Ch^{-m}$$

and thus by assumption $\|L_n(\sigma_n - \Delta_n\varepsilon))\|_{X_{n,m}} = Ch^{r+p}$. We may now proceed with the local discretization error (4.3). Now the exact solution of (4.1) is

$$\begin{aligned}\tilde{\xi}(t) &= y(t) - g(t) - \varepsilon(t)\\ &= \int_a^t f(s, \tilde{\xi}(s) - \varepsilon(s) + g(s)) - f(s, g(s))\, ds\\ &= \int_a^t f(s, y(s)) - f(s, g(s))\, ds\end{aligned}$$

and thus, by using a Taylor expansion argument, it is straightforward to show that

$$\|\tilde{\xi}\|_{Y^{m+p}} \leq C\|g - y\|_{Y^{m+p}}.$$

where C bounds the derivatives of f. Hence we get by assumption and (4.3) that

$$\begin{aligned}\|\phi_n(F)(\Delta_n\tilde{\xi})\|_{X_{n,m}} &\leq Ch^p\|\tilde{\xi}\|_{Y^{m+p}} + \tilde{C}h^{r+p}\\ &\leq Ch^p\|g - y\|_{Y^{m+p}} + \tilde{C}h^{r+p}.\end{aligned}$$

Note that $S(\hat{\theta}) \leq K$ since θ is bounded by the derivatives of f.

The theorem follows since $g \in E_n$.

References

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