Isn't the continuum in fact simpler than the discrete world?

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Sums versus integrals

How do you compute $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ for $n = 10^8$ $(=$ cube root of the Avogadro number $10^{23})$?

You *do not compute the sum for some large n* ,
say $n = 20, 50, 100$, having hopes you will guess the result in this way, because you know that the series diverges.

Sums versus integrals

How do you compute $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ for $n = 10^8$?

You will use Euler's formula $S_n = \log n + C + o(1)$ with $C = 0.5772\ldots$ to get

$$S \approx \log 10^8 + C = 18.9979$$

or you will simply replace the sum by an integral to obtain

$$\begin{aligned} S &\approx 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \int_6^{10^8} \frac{1}{x}\,dx \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \log 10^8 - \log 6 = 18.9123. \end{aligned}$$

A large Toeplitz matrix

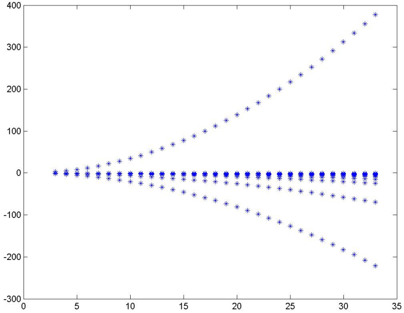
How do you compute the eigenvalues of the $n \times n$ Hermitian Toeplitz matrix

$$T_n = \begin{pmatrix} 0 & 1 & 2 & \dots & n-1 \\ 1 & 0 & 1 & \dots & n-2 \\ 2 & 1 & 0 & \dots & n-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-2 & n-3 & \dots & 0 \end{pmatrix}$$

for $n = 10^8$?

This time you *compute the eigenvalues for some large n* in order to get a conjecture, and then you look into one of the books by Böttcher et al. having hopes to find a theorem which confirms your conjecture. But you won't find such a theorem there, because T_n is not the truncation of an infinite Toeplitz matrix that is generated by an L^1 function.

Numerical results and the conjecture



The eigenvalues $\lambda_1(T_n) \le \lambda_2(T_n) \le \ldots \le \lambda_n(T_n)$ of T_n for $3 \le n \le 33$. They behave like constants times n^2,

$$\lambda_n(T_n) \sim \mu_1 n^2,$$
$$\lambda_k(T_n) \sim -\nu_k n^2 \quad \text{for} \quad k = 1, 2, \ldots.$$

Guessing the constants

To get the constants μ_1 and ν_1, ν_2, we plot eigenvalues / n^2.

Guess: $\mu_1 = 0.347$, $\nu_1 = -0.203$, $\nu_2 = -0.064$.

From matrices to integral operators

To prove what we have seen, we have recourse to an old but apparently forgotten trick due to

Harold Widom 1958

and independently to

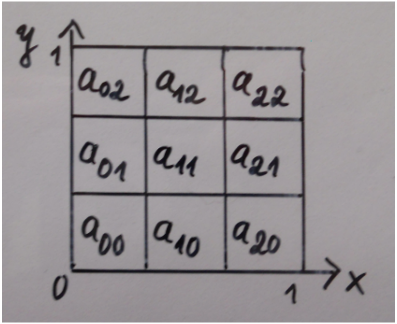
Lawrence Shampine 1965.

Incidentally, Shampine wrote all the codes associated with differential equations in Matlab and the default solvers of Maple.

From matrices to integral operators: the trick

Given a matrix $A_n = (a_{jk})_{j,k=0}^{n-1}$, consider the integral operator defined by

$$(K_n f)(x) = \int_0^1 a_{[nx],[ny]}\, f(y)\, dy \quad \text{on} \quad L^2(0,1).$$



From matrices to integral operators: the trick

The nonzero eigenvalues of

$$A_n = (a_{jk})_{j,k=0}^{n-1}$$

are just n times the eigenvalues of K_n given by

$$(K_n f)(x) = \int_0^1 a_{[nx],[ny]}\, f(y)\, dy \quad \text{on} \quad L^2(0,1).$$

From matrices to integral operators

In the case at hand,

$$T_n = \begin{pmatrix} 0 & 1 & 2 & \dots & n-1 \\ 1 & 0 & 1 & \dots & n-2 \\ 2 & 1 & 0 & \dots & n-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-2 & n-3 & \dots & 0 \end{pmatrix}$$

is the matrix

$$T_n = (|j-k|)_{j,k=0}^{n-1}.$$

From matrices to integral operators

Thus, we replace the matrix $T_n = (|j-k|)_{j,k=0}^{n-1}$ by the integral operator on $L^2(0,1)$ given by

$$(K_n f)(x) = \int_0^1 \Big| [nx] - [ny] \Big| f(y)\, dy, \quad x \in (0,1).$$

The nonzero eigenvalues of T_n are just n times the eigenvalues of K_n. It is easily seen that K_n/n converges uniformly to the integral operator

$$(Kf)(x) = \int_0^1 |x-y| f(y)\, dy, \quad x \in (0,1).$$

Consequently, the extreme eigenvalues of K_n are asymptotically n times the extreme eigenvalues of K, and hence those of T_n are n^2 times the extreme eigenvalues of K.

From matrices to integral operators

The extreme eigenvalues of T_n are asymptotically n^2 times the extreme eigenvalues of K.

This already proves part of the conjectured behaviour: the eigenvalues behave like constants times n^2,

$$\begin{aligned} \lambda_n(T_n) &\sim \mu_1 n^2, \\ \lambda_k(T_n) &\sim -\nu_k n^2 \quad \text{for} \quad k = 1, 2, \ldots. \end{aligned}$$

We are left with determining the constants $\mu_1, \nu_1, \nu_2, \ldots$, which are the eigenvalues of

$$(Kf)(x) = \int_0^1 |x - y| f(y)\, dy, \quad x \in (0, 1).$$

Eigenvalues of the integral operator

These can be determined in the standard way.

Twice differentiating the equation

$$\begin{aligned}\lambda f(x) &= \int_0^1 |x-y| f(y)\, dy \\ &= \int_0^x (x-y) f(y)\, dy + \int_x^1 (y-x) f(y)\, dy\end{aligned}$$

we get $\lambda f''(x) = 2f(x)$ with the general solution

$$f(x) = A\cos(\omega x) + B\sin(\omega x), \quad \lambda = -2/\omega^2.$$

Inserting this in the integral equation yields a homogeneous linear system for A and B with coefficients depending on ω. The system has a nontrivial solution if and only if $2 + 2\cos\omega + \omega\sin\omega = 0$.

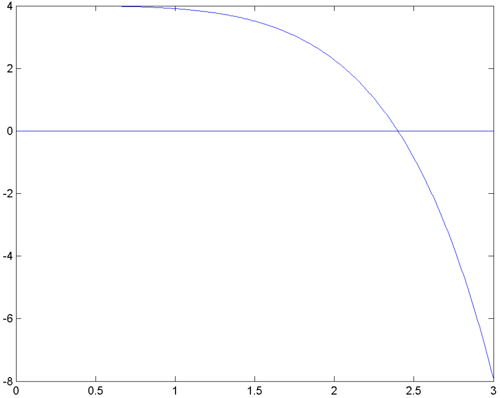
$$\lambda > 0 \iff \omega \in i\mathbf{R}, \quad \lambda < 0 \iff \omega \in \mathbf{R}$$

The positve eigenvalues of the integral operator

$\mu_1 = -2/\omega^2$ with $\omega = i\sigma \Longleftrightarrow \mu = 2/\sigma^2$ with

$$2 + 2\cos\omega + \omega\sin\omega = 2 + 2\cosh\sigma - \sigma\sinh\sigma = 0.$$

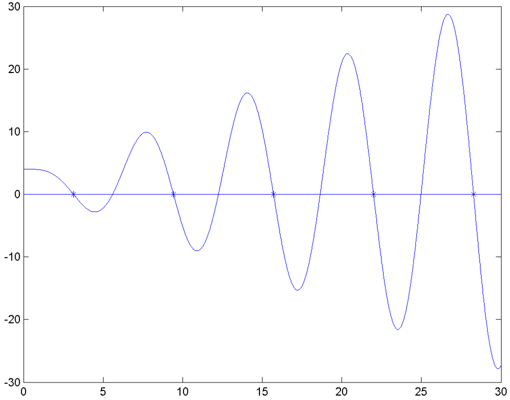
$\mu_1 = 0.3471$ (the guess was $\mu_1 = 0.347$)



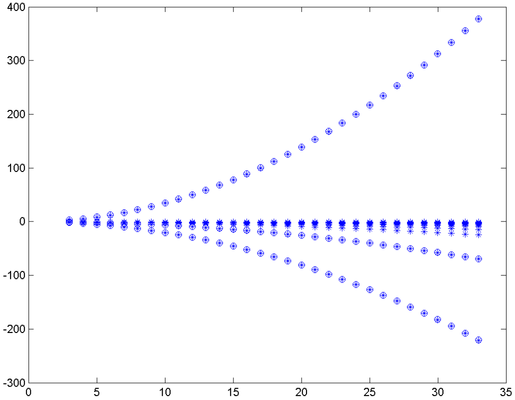
The negative eigenvalues of the integral operator

$\nu_k = -2/\omega_k^2$, where ω_k is the kth positive solution of $2 + 2\cos\omega + \omega\sin\omega = 0$. In particular, $\nu_k = -2/(k\pi)^2$ for odd k.

$\nu_1 = -2/\pi^2 = -0.2026$ (guess: -0.203),
$\nu_2 = -0.0638$ (guess: -0.064)



Numerical result and theory



The eigenvalues of T_n for $2 \leq n \leq 33$ (asterisks) and the values of $\mu_1 n^2$, $-\nu_1 n^2$, $-\nu_2 n^2$ (circles).

Andrei Markov 1890

The story started in Saint Petersburg with Dmitri Ivanovich Mendeleev, best known for the creation of the periodic table of elements. He posed a mathematical problem, which was solved by Andrei Andreevich Markov, then also living in Saint Petersburg.

Paper “On a question by D. I. Mendeleev” by Andrei Markov:

$\mathcal{P}_n = \mathbf{C}_n[x]$ analytic polynomials of degree $\leq n$

$\|f\|_\infty = \max_{x \in [-1,1]} |f(x)|$

$$\|f_n'\|_\infty \leq n^2 \|f_n\|_\infty \text{ for all } f_n \in \mathcal{P}_n$$

The constant n^2 is best possible.

Vladimir Markov 1916

The inequality $\|f_n'\|_\infty \le n^2 \|f_n\|_\infty$ implies

$$\|f_n''\|_\infty \le (n-1)^2 \|f_n'\|_\infty \le n^2(n-1)^2 \|f_n\|_\infty$$

but the constant $n^2(n-1)^2$ is not best possible. Vladimir Markov, the younger brother of Andrei Markov, proved:

$$\|f_n''\|_\infty \le \frac{n^2(n^2-1)}{3} \|f_n\|_\infty \text{ for all } f_n \in \mathcal{P}_n$$

$$\|f_n^{(\nu)}\|_\infty \le \frac{n^2(n^2-1)(n^2-2^2)\cdots(n^2-(\nu-1)^2)}{(2\nu-1)!!} \|f_n\|_\infty \text{ for all } f_n \in \mathcal{P}_n$$

The constant is best possible. It equals $T_n^{(\nu)}(1)$ with $T_n(x) = \cos(n \arccos x)$. Denote the constant by $\mu_n^{(\nu)}$. Then

$$\mu_n^{(\nu)} \sim \frac{1}{(2\nu-1)!!} n^{2\nu} \quad \text{as} \quad n \to \infty.$$

Erhard Schmidt

Die asymptotische Bestimmung des Maximums des Integrals über das Quadrat der Ableitung eines normierten Polynoms, dessen Grad ins Unendliche wächst. *1932*

Über die nebst ihren Ableitungen orthogonalen Polynomensysteme und das zugehörige Extremum. *1944*

Studied $\|f_n'\| \leq$ **best constant** $\|f_n\|$ for

Hermite $\int_{-\infty}^{\infty} \|f(x)\|^2 e^{-x^2}\, dx$	Laguerre $\int_0^{\infty} \|f(x)\|^2 e^{-x}\, dx$	Legendre $\int_{-1}^{1} \|f(x)\|^2\, dx$
$\eta_n^{(1)}$	$\lambda_n^{(1)}$	$\gamma_n^{(1)}$

and proved

$$\eta_n^{(1)} = \sqrt{2}\, n^{1/2}, \qquad \lambda_n^{(1)} \sim \frac{2}{\pi}\, n, \qquad \gamma_n^{(1)} \sim \frac{1}{\pi}\, n^2.$$

Turán and Shampine

Laguerre case: $\|f_n^{(\nu)}\| \le \lambda_n^{(\nu)} \|f_n\|$.

Erhard Schmidt 1944: $\lambda_n^{(1)} \sim \frac{2}{\pi}\, n$.

Pál Turán 1960:

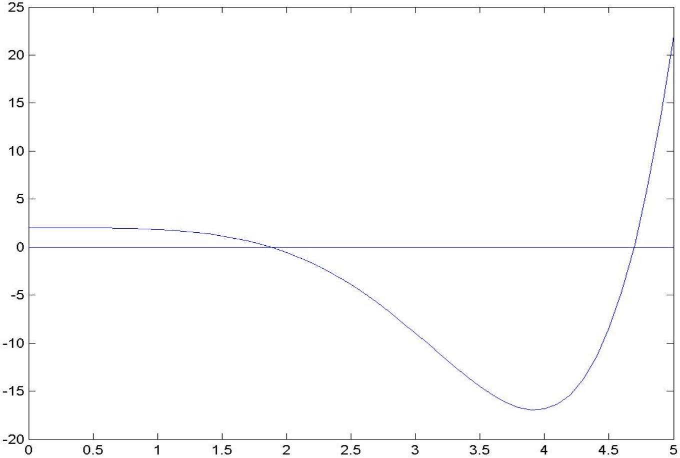
$$\lambda_n^{(1)} = \frac{1}{2\,\sin\frac{\pi}{4n+2}} \sim \frac{1}{2\,\frac{\pi}{4n}} = \frac{2}{\pi}\, n.$$

Lawrence Shampine 1965:

$$\lambda_n^{(2)} \sim \frac{1}{\omega_0^2}\, n^2$$

where ω_0 is the smallest positive zero of $1 + \cos\omega\, \cosh\omega = 0$.

$1 + \cos\omega \, \cosh\omega = 0$



$$\omega_0 \approx 1.8737, \quad 1/\omega_0^2 \approx 0.2848$$

Peter Dörfler (Austria) since 1987

Laguerre case: $\|f_n^{(\nu)}\| \le \lambda_n^{(\nu)} \|f_n\|$ with $\|f\|^2 = \int_0^\infty |f(x)|^2 e^{-x}\, dx$.

$$\frac{1}{2\nu!}\sqrt{\frac{4}{2\nu+1}} \le \liminf_{n\to\infty} \frac{\lambda_n^{(\nu)}}{n^\nu} \le \limsup_{n\to\infty} \frac{\lambda_n^{(\nu)}}{n^\nu} \le \frac{1}{2\nu!}\sqrt{\frac{2\nu}{2\nu-1}}$$

Recall:

Schmidt

$$\lim_{n\to\infty} \frac{\lambda_n^{(1)}}{n} = \frac{2}{\pi}$$

Shampine

$$\lim_{n\to\infty} \frac{\lambda_n^{(2)}}{n^2} = \frac{1}{\omega_0^2}$$

Peter Dörfler (Austria) since 1987

Laguerre case: $\|f_n^{(\nu)}\| \leq \lambda_n^{(\nu)} \|f_n\|$ with $\|f\|^2 = \int_0^\infty |f(x)|^2 e^{-x}\,dx$.

Let $\{L_0, L_1, \ldots, L_n\}$ be the orthonormal basis of Laguerre polynomials in $\mathcal{P}_n$:

$$L_k(x) = 1 - \binom{k}{1}\frac{x}{1!} + \binom{k}{2}\frac{x^2}{2!} + \cdots + (-1)^k \binom{k}{k}\frac{x^k}{k!}.$$

Then

$$L_k' = -L_0 - L_1 - \cdots - L_{k-1}$$

and hence the matrix representation of $D : \mathcal{P}_n \to \mathcal{P}_n$ in this basis is the Toeplitz matrix

$$\begin{pmatrix} 0 & -1 & -1 & \ldots & -1 \\ & 0 & -1 & \ldots & -1 \\ & & & & \ldots \\ & & & & -1 \\ & & & & 0 \end{pmatrix}.$$

Peter Dörfler (Austria) since 1987

Laguerre case: $\|f_n^{(\nu)}\| \le \lambda_n^{(\nu)} \|f_n\|$ with $\|f\|^2 = \int_0^\infty |f(x)|^2 e^{-x}\,dx$.

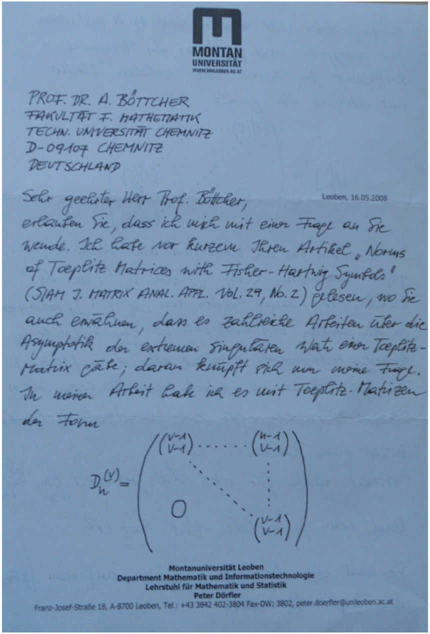
Let $\{L_0, L_1, \ldots, L_n\}$ be the orthonormal basis of Laguerre polynomials in $\mathcal{P}_n$:

Matrix representation of $D^\nu : \mathcal{P}_n \to \mathcal{P}_n$ in this basis is the Toeplitz matrix

$$T_n = (-1)^\nu \begin{pmatrix} 0 & \binom{0}{\nu-1} & \binom{1}{\nu-1} & \cdots & \binom{n-1}{\nu-1} \\ & 0 & \binom{0}{\nu-1} & \cdots & \binom{n-2}{\nu-1} \\ & & & \ddots & \vdots \\ & & & & \binom{0}{\nu-1} \\ & & & & 0 \end{pmatrix}.$$

We have $\lambda_n^{(\nu)} = \|T_n\|_\infty = s_{\max}(T_n)$.

How I became involved: letter by Peter Dörfler 2008



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Leoben, 16.05.2008

Sehr geehrter Herr Prof. Böttcher,
erlauben Sie, dass ich mich mit einer Frage an Sie wende. Ich habe vor kurzem Ihren Artikel „Norms of Toeplitz Matrices with Fisher-Hartwig Symbols" (SIAM J. MATRIX ANAL. APPL. Vol. 29, No. 2) gelesen, wo Sie auch erwähnen, dass es zahlreiche Arbeiten über die Asymptotik der extremen singulären Werte einer Toeplitz-Matrix gäbe; daran knüpft sich nun meine Frage. In meiner Arbeit habe ich es mit Toeplitz-Matrizen der Form

$$D_n^{(\nu)} = \begin{pmatrix} \binom{\nu-1}{\nu-1} & \cdots\cdots & \binom{n-1}{\nu-1} \\ & \ddots & \vdots \\ 0 & & \binom{\nu-1}{\nu-1} \end{pmatrix}$$

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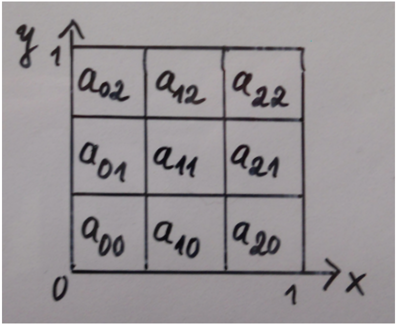
From matrices to integral operators

Harold Widom 1950s and Lawrence Shampine 1965:

Given a matrix $A_n = (a_{jk})_{j,k=0}^{n-1}$, consider the integral operator defined by

$$(K_n f)(x) = \int_0^1 a_{[nx],[ny]}\, f(y)\, dy \quad \text{on} \quad L^2(0,1).$$



From matrices to integral operators

We have matrices $A_n = (a_{jk})_{j,k=0}^{n-1}$ and integral operators

$$(K_n f)(x) = \int_0^1 a_{[nx],[ny]}\, f(y)\, dy \quad \text{on} \quad L^2(0,1).$$

$$\|A_n\|_\infty = n\,\|K_n\|_\infty$$

If $(1/n^\alpha)K_n \to K$ in the norm (uniformly),

then $(1/n^\alpha)\,\|K_n\|_\infty \to \|K\|_\infty$, and hence $\|K_n\|_\infty \sim \|K\|_\infty\, n^\alpha$.

$$\|A_n\|_\infty \sim \|K\|_\infty\, n^{\alpha+1}$$

Matrices and integral operators in the Laguerre case

$A_n = T_n$ is an upper-triangular Toeplitz matrix with

$$a_{jk} = \binom{k-j}{\nu-1} \quad \text{for} \quad j < k.$$

We have

$$\begin{aligned}
&\frac{1}{n^{\nu-1}}\, a_{[nx],[ny]} \\
&= \frac{1}{n^{\nu-1}} \binom{[ny]-[nx]}{\nu-1} \\
&= \frac{1}{(\nu-1)!}\, \frac{[ny]-[nx]}{n}\, \frac{[ny]-[nx]-1}{n} \cdots \frac{[ny]-[nx]-\nu+2}{n} \\
&\rightrightarrows \frac{1}{(\nu-1)!}\,(y-x)^{\nu-1} \quad \text{for} \quad x < y.
\end{aligned}$$

Emergence of Volterra operators

$\|T_n\|_\infty = n\|K_n\|_\infty$ and $(1/n^{\nu-1})\, K_n \to K =: L_\nu$ in the norm

Theorem

We have $\lambda_n^{(\nu)} \sim \|L_\nu\|_\infty\, n^\nu$ *where* L_ν *is given on* $L^2(0,1)$ *by*

$$(L_\nu f)(x) = \frac{1}{(\nu-1)!} \int_x^1 (y-x)^{\nu-1} f(y)\, dy.$$

Thus, $\lim_{n\to\infty} \frac{\lambda_n^{(\nu)}}{n^\nu}$ exists and equals $\|L_\nu\|_\infty$. Note that

$$(L_\nu^* f)(x) = \frac{1}{(\nu-1)!} \int_0^x (x-y)^{\nu-1} f(y)\, dy$$

and $\|L_\nu\|_\infty = \|L_\nu^*\|_\infty$.

Norms of Volterra operators

$(L_1 f)(x) = \int_x^1 f(y)\,dy$ and $(L_1^* f)(x) = \int_0^x f(y)\,dy$

$\lambda_n^{(1)} \sim \|L_1\|_\infty\, n$ and we know from Schmidt that $\lambda_n^{(1)} \sim (2/\pi)\, n$.

Consequently,

$$\|L_1\|_\infty = \frac{2}{\pi} = \frac{1}{\omega_0}, \quad \cos\omega = 0.$$

Paul Halmos 1967: Proved

$$\|L_1\|_\infty = 2/\pi$$

in a straightforward way.

The proof by Halmos

We have $\|L_1\|_\infty^2 = \|L_1^* L_1\|_\infty =$ maximal μ such that $L_1^* L_1 f = \mu f$ has a nontrivial solution. We put $\mu = 1/\omega^2$.

We get

$$(L_1^* L_1 f)(x) = \int_0^x \int_y^1 f(t)\,dt\,dy = \frac{1}{\omega^2} f(x) \quad (\Longrightarrow f(0) = 0).$$

Twice differentiating we arrive at the boundary problem

$$y'' + \omega^2 y = 0, \quad y(0) = 0, \quad y'(1) = 0.$$

The smallest $\omega > 0$ for which this problem has a nontrivial solution is $\omega = \pi/2$.

Thus,

$$\|L_1\|_\infty^2 = \mu = \frac{1}{\omega^2} = \frac{4}{\pi^2},$$

implying that

$$\|L_1\|_\infty = \frac{2}{\pi}.$$

The question by Halmos

$$(L_1 f)(x) = \int_x^1 f(y)\,dy \quad \text{and} \quad (L_1^* f)(x) = \int_0^x f(y)\,dy$$

$$(L_\nu f)(x) = \frac{1}{(\nu-1)!}\int_x^1 (y-x)^{\nu-1} f(y)\,dy$$

$$(L_\nu^* f)(x) = \frac{1}{(\nu-1)!}\int_0^x (x-y)^{\nu-1} f(y)\,dy$$

Paul Halmos 1967:

After proving that $\|L_1\|_\infty = 2/\pi$, he asked what the norms of the iterates L_1^ν are.

Note that $L_1^\nu = L_\nu$.

$\nu = 1$ and $\nu = 2$

We know that

$$\|L_1\|_\infty = \frac{2}{\pi} = \frac{1}{\omega_0}, \quad \cos\omega = 0.$$

We know from Bö/Dörfler that $\lambda_n^{(2)} \sim \|L_2\|_\infty \, n^2$, and we know from Shampine that $\lambda_n^{(2)} \sim (1/\omega_0^2)\, n^2$. Thus,

$$\|L_2\|_\infty = \frac{1}{\omega_0^2}, \quad 1 + \cos\omega \, \cosh\omega = 0.$$

Eigenvalues of differential operators

Theorem (Thorpe 1998)

We have

$$\|L_\nu\|_\infty = \frac{1}{\omega_0^\nu}$$

where ω_0 *is the smallest positive number such that*

$$\begin{aligned}
&(-1)^\nu g^{(2\nu)}(x) = \omega^{2\nu}\, g(x),\\
&g(0) = g'(0) = \cdots = g^{(\nu-1)}(0) = 0\\
&g^{(\nu)}(1) = g^{(\nu+1)}(1) = \cdots = g^{(2\nu-1)}(1) = 0
\end{aligned}$$

has a nontrivial solution.

Estimates for the norms of Volterra operators

Bö/Dörfler 2009:

For all $\nu \geq 1$,

$$\frac{1}{(\nu-1)!}\frac{1}{\sqrt{(2\nu+1)(2\nu-1)}} \leq \|L_\nu\|_\infty \leq \frac{1}{(\nu-1)!}\frac{1}{\sqrt{(2\nu)(2\nu-1)}}.$$

We have even better estimates. These yield

ν	$\|L_\nu\|_\infty$
1	$0.6\ldots$
2	$0.284\ldots$
3	$0.09081\ldots$
4	$0.022213\ldots$
10	$1.413169\ldots \times 10^{-7}$
20	$2.0811690\ldots \times 10^{-19}$
60	$6.034043870\ldots \times 10^{-83}$

The indicated digits are correct.

Discrete proof

T_n is a Toeplitz matrix formed by binomial coefficients.

The estimate is

$$\frac{\|T_n^* T_n\|_2^2}{\|T_n\|_2^2} \leq \|T_n\|_\infty^2 \leq \|T_n\|_2^2,$$

where $\|\cdot\|_2$ is the Hilbert-Schmidt norm ($=$ Frobenius norm).
Thus,

- compute $\|T_n\|_2^2$
- and establish a lower bound for $\|T_n^* T_n\|_2^2$.

Discrete proof

Nach dem Binomischen Lehrsatz ist (2)

$$k^{\nu-1}(j-i+k)^{\nu-1} = \sum_{m=0}^{\nu-1}(j-i)^m k^{2\nu-2-m}\binom{\nu-1}{m}.$$

Daraus folgt:

$$\sum_{k=0}^{i-1} k^{\nu-1}(j-i+k)^{\nu-1} = \sum_{m=0}^{\nu-1}\binom{\nu-1}{m}(j-i)^m \sum_{k=0}^{i-1} k^{2\nu-2-m} =$$

$$= \sum_{m=0}^{\nu-1}\binom{\nu-1}{m}(j-i)^m\left[\frac{1}{2\nu-m-1}\, i^{2\nu-m-1} + O\left(i^{2\nu-m-2}\right)\right] =$$

$$= \sum_{m=0}^{\nu-1}\frac{1}{2\nu-m-1}\binom{\nu-1}{m}(j-i)^m i^{2\nu-m-1} + P_3(i,j),$$

wobei $P_3 \in \mathbb{R}[x,y]$, $\deg P_3 \le 2\nu-2$. (3)

Angewendet auf $(j-i)^m$ ergibt der Binomische Lehrsatz: $\sum_{m=0}^{\nu-1}\frac{1}{2\nu-m-1}\binom{\nu-1}{m}(j-i)^m i^{2\nu-m-1} =$

$$= i^{\nu}\sum_{m=0}^{\nu-1}\frac{1}{2\nu-m-1}\binom{\nu-1}{m}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{l} j^l i^{\nu-1-l} =$$

$$= i^{\nu}\sum_{\alpha=0}^{\nu-1}(-j)^{\alpha} i^{\nu-1-\alpha}\cdot S_\nu(\alpha),$$ wobei

$$S_\nu(\alpha) := \sum_{\beta=\alpha}^{\nu-1}\frac{1}{2\nu-1-\beta}(-1)^{\beta}\binom{\nu-1}{\beta}\binom{\beta}{\alpha},\quad 0\le\alpha\le\nu-1.$$

Discrete proof

Daraus folgt, zusammen mit (1), (2) und (3): ③

$$a_{ij} = \frac{1}{[(\nu-1)!]^2} i^\nu \sum_{\alpha=0}^{\nu-1} (-j)^\alpha i^{\nu-1-\alpha} S_\nu(\alpha) + P_4(i,j),$$

wobei $P_4 \in \mathbb{R}[x,y]$, $\deg P_4 \leq 2\nu-2$. ✓

Schließlich ergibt sich:

$$a_{ij}^2 = \frac{1}{[(\nu-1)!]^4} i^{2\nu} \Big[\sum_{\alpha=0}^{\nu-1} j^{2\alpha} i^{2\nu-2-2\alpha} S_\nu^2(\alpha) +$$

$$+ \sum_{\substack{\gamma,\delta=0 \\ \gamma\neq\delta}}^{\nu-1} (-j)^{\gamma+\delta} i^{2\nu-2-\gamma-\delta} S_\nu(\gamma) S_\nu(\delta) \Big] + P_5(i,j),$$

wobei $P_5 \in \mathbb{R}[x,y]$, $\deg P_5 \leq 4\nu-3$. ✓

Daraus folgt für die Norm:

$$\| D_n^{(\nu)t} D_n^{(\nu)} \|^2 = \sum_{j=1}^{n-\nu+1} a_{jj}^2 + 2 \sum_{j=2}^{n-\nu+1} \sum_{i=1}^{j-1} a_{ij}^2 =$$

$$= \frac{1}{[(\nu-1)!]^4} \sum_{j=1}^{n-\nu+1} j^{2\nu} \Big[\sum_{\alpha=0}^{\nu-1} j^{2\nu-2} S_\nu^2(\alpha) + \sum_{\substack{\gamma,\delta=0 \\ \gamma\neq\delta}}^{\nu-1} (-1)^{\gamma+\delta} j^{2\nu-2} S_\nu(\gamma) S_\nu(\delta) \Big] +$$

Discrete proof

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$$+\frac{2}{[(v-1)!]^4}\sum_{j=2}^{n-v+1}\sum_{i=1}^{j-1} i^{2v}\Big[\sum_{\alpha=0}^{v-1} j^{2\alpha}\, i^{2v-2-2\alpha} S_v^2(\alpha) +$$

$$+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{v-1} (-1)^{\gamma+\delta}\, i^{2v-2-\gamma-\delta} S_v(\gamma) S_v(\delta)\Big] + O(n^{4v-1}),$$

da $\sum_{j=1}^{n-v+1} P_5(j,j) = O(n^{4v-2})$ und

$$\sum_{i=1}^{j-1} P_5(i,j) = O(j^{4v-2}).$$

Das ist weiter gleich:

$$\frac{1}{[(v-1)!]^4}\sum_{j=1}^{n-v+1} j^{4v-2}\Big[\sum_{\alpha=0}^{v-1} S_v^2(\alpha) + \sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{v-1} (-1)^{\gamma+\delta} S_v(\gamma) S_v(\delta)\Big] +$$

$$+\frac{2}{[(v-1)!]^4}\sum_{j=2}^{n-v+1}\Big[\sum_{\alpha=0}^{v-1} S_v^2(\alpha)\, j^{2\alpha}\sum_{i=1}^{j-1} i^{4v-2-2\alpha} +$$

$$+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{v-1} (-j)^{\gamma+\delta} S_v(\gamma) S_v(\delta)\sum_{i=1}^{j-1} i^{4v-2-\gamma-\delta}\Big] + O(n^{4v-1}) =$$

$$= \frac{1}{[(v-1)!]^4}\Big[\sum_{\alpha=0}^{v-1} S_v^2(\alpha) + \sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{v-1} (-1)^{\gamma+\delta} S_v(\gamma) S_v(\delta)\Big]\Big[\frac{n^{4v-1}}{4v-1} + O(n^{4v-2})\Big] +$$

Discrete proof

$$+\frac{2}{[(\nu-1)!]^4}\sum_{j=2}^{n-\nu+1}\Big\{\sum_{\alpha=0}^{\nu-1}S_\nu^2(\alpha)\Big[\frac{j^{4\nu-1}}{4\nu-1-2\alpha}+\mathcal{O}(j^{4\nu-2})\Big]+$$ ⑤

$$+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{\nu-1}(-1)^{\gamma+\delta}S_\nu(\gamma)S_\nu(\delta)\Big[\frac{j^{4\nu-1}}{4\nu-1-\gamma-\delta}+\mathcal{O}(j^{4\nu-2})\Big]\Big\}+\mathcal{O}(n^{4\nu-1})=$$

$$=\frac{2}{[(\nu-1)!]^4}\sum_{j=2}^{n-\nu+1}j^{4\nu-1}\Big[\sum_{\alpha=0}^{\nu-1}\frac{S_\nu^2(\alpha)}{4\nu-1-2\alpha}+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{\nu-1}(-1)^{\gamma+\delta}\frac{S_\nu(\gamma)S_\nu(\delta)}{4\nu-1-\gamma-\delta}\Big]+$$

$$+\sum_{j=2}^{n-\nu+1}\mathcal{O}(j^{4\nu-2})+\mathcal{O}(n^{4\nu-1}).$$

Daraus ergibt sich schließlich:

$$\|D_n^{(\nu)+}D_n^{(\nu)}\|^2=\frac{T^*(\nu)}{[(\nu-1)!]^4}\cdot n^{4\nu}+\mathcal{O}(n^{4\nu-1}),\text{ wobei}$$

$$T^*(\nu):=\frac{1}{2\nu}\Big[\sum_{\alpha=0}^{\nu-1}\frac{S_\nu^2(\alpha)}{4\nu-1-2\alpha}+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{\nu-1}(-1)^{\gamma+\delta}\frac{S_\nu(\gamma)S_\nu(\delta)}{4\nu-1-\gamma-\delta}\Big]. \qquad (4)$$

Wir wollen nun die Gestalt der Summen $S_\nu(\alpha)$ vereinfachen.

Wie man leicht nachrechnet, gilt

$$\binom{\nu-1}{\beta}\binom{\beta}{\nu-k}=\binom{\nu-1}{k-1}\binom{k-1}{\beta-\nu+k},\quad \nu-k\le\beta\le\nu-1,\ 1\le k\le\nu.$$

Discrete proof

Daraus folgt mit $\alpha := v-k$, $1 \le k \le v$: (6)

$$S_v(v-k) = \sum_{\beta=v-k}^{v-1} \frac{1}{2v-1-\beta}(-1)^\beta \binom{v-1}{\beta}\binom{\beta}{v-k} =$$

$$= \sum_{\beta=v-k}^{v-1} \frac{1}{2v-1-\beta}(-1)^\beta \binom{v-1}{k-1}\binom{k-1}{\beta-v+k} =$$

$$= (-1)^{v-1}\binom{v-1}{k-1}\sum_{\beta=v-k}^{v-1}\frac{1}{2v-1-\beta}(-1)^{\beta-v+1}\binom{k-1}{\beta-v+k} =$$

$$\binom{k-1}{\beta-v+k} = \binom{k-1}{v-\beta-1}$$

$$= (-1)^{v-1}\binom{v-1}{k-1}\sum_{j=0}^{k-1}\frac{1}{v+j}(-1)^j\binom{k-1}{j} = \qquad \boxed{j = v-\beta-1}$$

$$= (-1)^{v-1}\binom{v-1}{k-1}\cdot\frac{1}{v}\binom{v+k-1}{v}^{-1}$$ nach [PR1, p611]. No. 4.2.2.44

Eine kurze Rechnung ergibt dann:

$$S_v(v-k) = \frac{(-1)^{v+k-1}}{v}\,\frac{(-v)_k}{(v)_k}, \quad 1 \le k \le v,$$

wobei $(z)_k = z(z+1)\dots(z+k-1)$.

Damit lässt sich auch $T^*(v)$ vereinfachen:

Discrete proof

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$$+\frac{2}{[(\nu-1)!]^4}\sum_{j=2}^{n-\nu+1}\Big\{\sum_{\alpha=0}^{\nu-1} S_\nu^2(\alpha)\Big[\frac{j^{4\nu-1}}{4\nu-1-2\alpha}+\mathcal{O}(j^{4\nu-2})\Big]+$$

$$+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{\nu-1}(-1)^{\gamma+\delta}S_\nu(\gamma)S_\nu(\delta)\Big[\frac{j^{4\nu-1}}{4\nu-1-\gamma-\delta}+\mathcal{O}(j^{4\nu-2})\Big]\Big\}+\mathcal{O}(n^{4\nu-1})=$$

$$=\frac{2}{[(\nu-1)!]^4}\sum_{j=2}^{n-\nu+1} j^{4\nu-1}\Big[\sum_{\alpha=0}^{\nu-1}\frac{S_\nu^2(\alpha)}{4\nu-1-2\alpha}+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{\nu-1}(-1)^{\gamma+\delta}\frac{S_\nu(\gamma)S_\nu(\delta)}{4\nu-1-\gamma-\delta}\Big]+$$

$$+\sum_{j=2}^{n-\nu+1}\mathcal{O}(j^{4\nu-2})+\mathcal{O}(n^{4\nu-1}).$$

Daraus ergibt sich schließlich:

Discrete proof

Das ist weiter gleich:

$$\frac{1}{[(v-1)!]^4}\sum_{j=1}^{n-v+1} j^{4v-2}\Big[\sum_{\alpha=0}^{v-1} S_v^2(\alpha)+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{v-1}(-1)^{\gamma+\delta}S_v(\gamma)S_v(\delta)\Big]+$$

$$+\frac{2}{[(v-1)!]^4}\sum_{j=2}^{n-v+1}\Big[\sum_{\alpha=0}^{v-1} S_v^2(\alpha)\, j^{2\alpha}\sum_{i=1}^{j-1} i^{4v-2-2\alpha}+$$

$$+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{v-1}(-j)^{\gamma+\delta}S_v(\gamma)S_v(\delta)\sum_{i=1}^{j-1} i^{4v-2-\gamma-\delta}\Big]+O(n^{4v-1})=$$

$$=\frac{1}{[(v-1)!]^4}\Big[\sum_{\alpha=0}^{v-1} S_v^2(\alpha)+\sum_{\substack{\gamma,\delta=0\\ \gamma\neq\delta}}^{v-1}(-1)^{\gamma+\delta}S_v(\gamma)S_v(\delta)\Big]\Big[\frac{n^{4v-1}}{4v-1}+O(n^{4v-2})\Big]+$$

Discrete proof

$$\ldots = \sum_{m=0} (j-i)^m k^{2v-2-m} \binom{v-1}{m}.$$

Daraus folgt:

$$\sum_{k=0}^{i-1} k^{v-1}(j-i+k)^{v-1} = \sum_{m=0}^{v-1}\binom{v-1}{m}(j-i)^m \sum_{k=0}^{i-1} k^{2v-2-m} =$$

$$= \sum_{m=0}^{v-1}\binom{v-1}{m}(j-i)^m\left[\frac{1}{2v-m-1} i^{2v-m-1} + O\left(i^{2v-m-2}\right)\right] =$$

$$= \sum_{m=0}^{v-1}\frac{1}{2v-m-1}\binom{v-1}{m}(j-i)^m \cdot i^{2v-m-1} + P_3(i,j),$$

wobei $P_3 \in \mathbb{R}[x,y]$, $\deg P_3 \le 2v-2$. (3)

Angewendet auf $(j-i)^m$ ergibt der Binomische

Continuous proof I

We work instead with

$$(L_\nu f)(x) = \frac{1}{(\nu-1)!} \int_x^1 (y-x)^{\nu-1} f(y)\, dy.$$

The estimate is

$$\frac{\|L_\nu^* L_\nu\|_2^2}{\|L_\nu\|_2^2} \leq \|L_\nu\|_\infty^2 \leq \|L_\nu\|_2^2,$$

where $\|\cdot\|_2$ is the Hilbert-Schmidt norm. Thus,

- compute $\|L_\nu\|_2^2$
- and establish a lower bound for $\|L_\nu^* L_\nu\|_2^2$.

Continuous proof II

$$
\begin{aligned}
\|L_\nu\|_2^2 &= \int_0^1 \int_0^1 |k(x,y)|^2 \, dx \, dy \\
&= \frac{1}{[(\nu-1)!]^2} \int_0^1 \int_0^1 (y-x)_+^{2\nu-2} \, dx \, dy \\
&= \frac{1}{[(\nu-1)!]^2} \int_0^1 \int_0^y (y-x)^{2\nu-2} \, dx \, dy \\
&= \frac{1}{[(\nu-1)!]^2} \int_0^1 \left. \frac{(y-x)^{2\nu-1}}{2\nu-1} \right|_y^0 dy \\
&= \frac{1}{[(\nu-1)!]^2} \int_0^1 \frac{y^{2\nu-1}}{2\nu-1} \, dy \\
&= \frac{1}{[(\nu-1)!]^2} \frac{1}{2\nu(2\nu-1)}.
\end{aligned}
$$

Continuous proof III

$$\|L_\nu^* L_\nu\|_2^2 = \int_0^1 \int_0^1 \left| \int_0^1 k(t,x)k(t,y)\, dt \right|^2 dy\, dx$$

$$= \frac{2}{[(\nu-1)!]^2} \int_0^1 \int_0^x \left(\int_0^y (x-t)^{\nu-1}(y-t)^{\nu-1} dt \right)^2 dy\, dx$$

$[t = ys, 0 \leq s \leq 1]$

$$= \frac{2}{[(\nu-1)!]^2} \int_0^1 \int_0^x \left(\int_0^1 (x-ys)^{\nu-1} y^\nu (1-s)^{\nu-1} ds \right)^2 dy\, dx$$

[the big trick : $x - ys \geq x - xs$ for $0 \leq s,\ 0 \leq y \leq x$]

$$\geq \frac{2}{[(\nu-1)!]^2} \int_0^1 \int_0^x \left(\int_0^1 x^{\nu-1}(1-s)^{\nu-1} y^\nu (1-s)^{\nu-1} ds \right)^2 dy\, dx$$

$$= \frac{2}{[(\nu-1)!]^2} \frac{1}{4\nu(2\nu+1)(2\nu-1)^2}.$$

The conjecture by Lao and Whitley

Our estimate

$$\frac{1}{(\nu-1)!}\frac{1}{\sqrt{(2\nu+1)(2\nu-1)}} \le \|L_\nu\|_\infty \le \frac{1}{(\nu-1)!}\frac{1}{\sqrt{(2\nu)(2\nu-1)}}$$

immediately implies that

$$\|L_\nu\|_\infty \sim \frac{1}{(\nu-1)!}\,\frac{1}{2\nu} = \frac{1}{2\,\nu!}.$$

During a discussion with Hermann Brunner, we learned that this asymptotic behavior was

conjectured by *Lao/Whitley 1997*

and that three independent proofs were subsequently given by

Thorpe 1998, Little/Read 1998, Kershaw 1999.

Fortunately, we didn't know this when proving our estimates. Our (continuous) proof is the simplest of all.

Laguerre norm with weight & Bessel functions

$\|f_n^{(\nu)}\| \le \lambda_n^{(\nu)}(\alpha)\,\|f_n\|$ with $\|f\|^2 = \int_0^\infty |f(x)|^2 x^\alpha e^{-x}\,dx$, $\alpha > -1$

Dörfler 2002:

$\lambda_n^{(1)}(\alpha) \sim \frac{1}{j_\alpha}\,n$ where j_α is the smallest positive root of the Bessel function $J_{(\alpha-1)/2}$. Very complicated proof.

$\alpha = 0$: $\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,cosx$ and thus $j_0 = \frac{\pi}{2}$ (Schmidt).

Theorem (Bö/Dörfler 2010)

We have $\lambda_n^{(\nu)}(\alpha) \sim \|L_{\nu,\alpha}\|_\infty\, n^\nu$ *where*

$$(L_{\nu,\alpha}f)(x) = \frac{1}{(\nu-1)!}\int_x^1 x^{\alpha/2} y^{-\alpha/2}(y-x)^{\nu-1} f(y)\,dy.$$

Matrix of D^v no longer Toeplitz, $L_{\nu,\alpha}$ still Volterra, but no longer convolution, sharp estimates and asymptotics of $\|L_{\nu,\alpha}\|_\infty$.

Norm of Volterrra operators and zeros Bessel functions

Proceeding along the lines of the proof by Halmos for $\|L_1\|_\infty = 2/\pi$ we can determine the norm of the Volterra operator

$$(L_{\nu,\alpha}f)(x) = \frac{1}{(\nu-1)!}\int_x^1 x^{\alpha/2}y^{-\alpha/2}(y-x)^{\nu-1}f(y)\,dy$$

for $\nu = 1$, in which case it is

$$(L_{1,\alpha}f)(x) = \int_x^1 x^{\alpha/2}y^{-\alpha/2}f(y)\,dy$$

We arrive at a boundary value problem for the Bessel differential equation

$$x^2y'' + xy' + \left(x^2 - (\alpha-1)^2/4\right)y = 0.$$

This yields $\|L_{1,\alpha}\|_\infty = 1/j_\alpha$ and thus Dörfler's 2002 result $\lambda_n^{(1)}(\alpha) \sim (1/j_\alpha)\,n$ in a straightforward way.

Gegenbauer (= ultraspherical) norm

$\|f_n^{(\nu)}\| \le \gamma_n^{(\nu)}(\alpha)\,\|f_n\|$

with $\|f\|^2 = \int_{-1}^{1} |f(x)|^2 (1-x)^\alpha (1+x)^\alpha\,dx$, $\alpha > -1$

Only known results:

$\gamma_n^{(1)}(0) \sim \frac{1}{\pi}\,n^2$ (*Schmidt 1944*)

$\gamma_n^{(2)}(0) \sim \frac{1}{4\omega_0^2}\,n^4$, $1 + \cos\omega\,\cosh\omega = 0$ (*Shampine 1965*)

Theorem (Bö/Dörfler 2010)

We have $\gamma_n^{(\nu)}(\alpha) \sim \|G_{\nu,\alpha}\|_\infty\, n^{2\nu}$ *where*

$$(G_{\nu,\alpha}f)(x) = \frac{1}{2^\nu(\nu-1)!}\int_x^1 x^{1/2+\alpha} y^{1/2-\alpha} (y^2 - x^2)^{\nu-1} f(y)\,dy.$$

Moreover,

$$\|G_{\nu,\alpha}\|_\infty = \frac{1}{2^\nu}\,\|L_{\nu,\alpha}\|_\infty.$$

In particular: $\|G_{1,\alpha}\|_\infty = 1/(2j_\alpha)$.

Polynomials in several variables

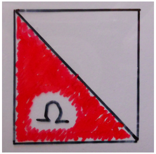
$E \subset [0,1]^2$ closed subset with interior points and such that

We denote by $\mathcal{P}_n(E)$ the linear space of polynomials

$$f(x,y) = \sum_{(j/n,k/n)\in E} f_{jk} x^j y^k.$$

Canonical choices:

$\sum_{j,k\leq n} f_{jk} x^j y^k$

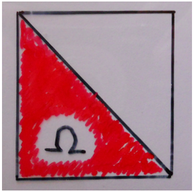


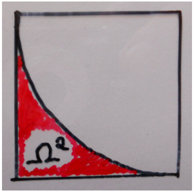
$\sum_{j+k\leq n} f_{jk} x^j y^k$

Powers of the basic set

For $\delta > 0$, we put $E^\delta = \{(s^\delta, t^\delta) : (s,t) \in E\}$.

For example,





Markov inequalities

We equip $\mathcal{P}_n(E)$ with one of the norms

$$\|f\|^2 = \int_0^\infty \int_0^\infty |f(x,y)|^2 x^\alpha y^\beta e^{-x} e^{-y}\, dx\, dy$$

(Laguerre with weight),

$$\|f\|^2 = \int_{-1}^1 \int_{-1}^1 |f(x,y)|^2 (1-x^2)^\alpha (1-y^2)^\beta\, dx\, dy$$

(Gegenbauer),

where $\alpha > -1$ and $\beta > -1$.

Consider the inequality

$$\|\partial_x^\nu \partial_y^\mu f\| \le C\|f\| \text{ for } f \in \mathcal{P}_n(E).$$

Let $\lambda_{E,n}^{(\alpha,\beta)}(\partial_x^\nu \partial_y^\mu)$ and $\gamma_{E,n}^{(\alpha,\beta)}(\partial_x^\nu \partial_y^\mu)$ denote the best constant C in this inequality for the weighted Laguerre norm and the Gegenbauer norm, respectively.

The Volterra integral operators occurring

On $L^2(0,1)$,

$$(L_{\nu,\alpha}f)(x) = \frac{1}{(\nu-1)!}\int_x^1 x^{\alpha/2}y^{-\alpha/2}(y-x)^{\nu-1}f(y)\,dy,$$

$$(G_{\nu,\alpha}f)(x) = \frac{1}{2^\nu(\nu-1)!}\int_x^1 x^{1/2+\alpha}y^{1/2-\alpha}(y^2-x^2)^{\nu-1}f(y)\,dy.$$

The tensor products $L_{\nu,\alpha}\otimes L_{\mu,\beta}$ and $G_{\nu,\alpha}\otimes G_{\mu,\beta}$ are defined on $L^2((0,1)^2)$ in the usual manner. It is easily seen, that these tensor products leave the subspace $L^2(E)$ invariant. We denote the restrictions of the tensor products to $L^2(E)$ by $L_{\nu,\alpha}\otimes L_{\mu,\beta}|L^2(E)$ and $G_{\nu,\alpha}\otimes G_{\mu,\beta}|L^2(E)$.

Asymptotics of the best constants

Theorem (Bö/Dörfler 2011)

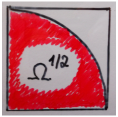
If $\nu \geq 1$ and $\mu \geq 1$, then

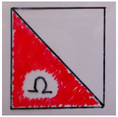
$$\lambda_{E,n}^{(\alpha,\beta)}(\partial_x^\nu \partial_y^\mu) \sim n^{\nu+\mu} \, \|L_{\nu,\alpha} \otimes L_{\mu,\beta} | L^2(E)\|_\infty,$$
$$\gamma_{E,n}^{(\alpha,\beta)}(\partial_x^\nu \partial_y^\mu) \sim n^{2\nu+2\mu} \, \|G_{\nu,\alpha} \otimes G_{\mu,\beta} | L^2(E)\|_\infty,$$

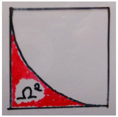
and we also have

$$\|G_{\nu,\alpha} \otimes G_{\mu,\beta} | L^2(E)\|_\infty = \frac{1}{2^{\nu+\mu}} \, \|L_{\nu,\alpha} \otimes L_{\mu,\beta} | L^2(E^2)\|_\infty.$$

Notice that we have E on the left and E^2 on the right. This motivates the consideration of the entire scale Ω^δ instead of the sole triangle Ω.







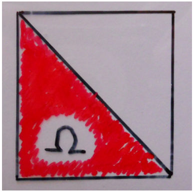
Estimates

We have

$$\begin{aligned}
&\|L_{\nu,\alpha} \otimes L_{\mu,\beta} | L^2(\Omega^\delta)\|_2^2 \\
&= \frac{\delta^2}{\Gamma(2\delta(\nu+\mu)+1)} \, \frac{\Gamma(\alpha+1)\,(2\nu-2)!\,\Gamma(2\delta\nu)}{\Gamma(\alpha+2\nu)\,[(\nu-1)!]^2} \times \\
&\quad \times \frac{\Gamma(\beta+1)\,(2\mu-2)!\,\Gamma(2\delta\mu)}{\Gamma(\beta+2\mu)\,[(\mu-1)!]^2}
\end{aligned}$$

and

$$\begin{aligned}
&\|(L_{\nu,\alpha} \otimes L_{\mu,\beta} | L^2(\Omega^\delta))^*(L_{\nu,\alpha} \otimes L_{\mu,\beta} | L^2(\Omega^\delta))\|_2^2 \\
&\geq \frac{4\delta^2}{\Gamma(4\delta(\nu+\mu)+1)} \, \frac{\Gamma(\alpha+1)^2\,[(2\nu-2)!]^2\,\Gamma(4\delta\nu)}{\Gamma(\alpha+2\nu)^2\,(\alpha+2\nu+1)\,[(\nu-1)!]^4} \times \\
&\quad \times \frac{\Gamma(\beta+1)^2\,[(2\mu-2)!]^2\,\Gamma(4\delta\mu)}{\Gamma(\beta+2\mu)^2\,(\beta+2\mu+1)\,[(\mu-1)!]^4}.
\end{aligned}$$



Thank You!