On Spectral Inclusion Sets and Computing the Spectra and Pseudospectra of Bounded Linear Operators

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...with the help of...

This talk is based on joint work with

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- Brian Davies, KCL
- **Marko Lindner**, TUHH, Germany

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1. Introduction

2. Inclusion Sets and Approximations for the Spectrum and Pseudospectrum

3. Substantial example with lots of pictures! Feinberg-Zee Random Hopping Matrix

4. Extensions

The Spaces

This talk is about a class of bounded linear operators on a space $E = \ell^p$ with $p \in [1, \infty]$.

So $x \in E$ iff $x = (x_k)_{k \in \mathbb{Z}}$, where $x_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$ and

$$\|x\|_E = \sqrt[p]{\sum_{k \in \mathbb{Z}} |x_k|^p}, \qquad p < \infty,$$

$$\|x\|_E = \sup_{k \in \mathbb{Z}} |x_k|, \qquad p = \infty.$$

The Operators

$$L(E) \quad := \quad \text{set of all } \textbf{bounded} \text{ linear operators } E \to E.$$

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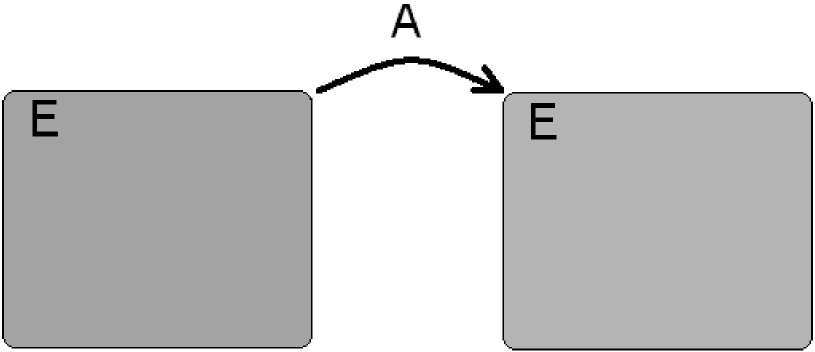
With every bounded linear operator A on E we will associate a matrix

$$\begin{pmatrix} \ddots & \vdots & \iddots \\ \cdots & a_{ij} & \cdots \\ \iddots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ x_j \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ b_i \\ \vdots \end{pmatrix}$$

with indices $i, j \in \mathbb{Z}$ and entries $a_{ij} \in \mathbb{C}$.

For simplicity, we will restrict ourselves to **band matrices**.

Fredholm operators



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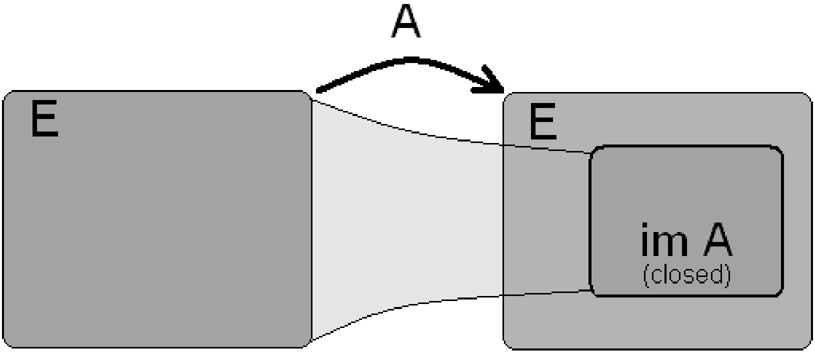

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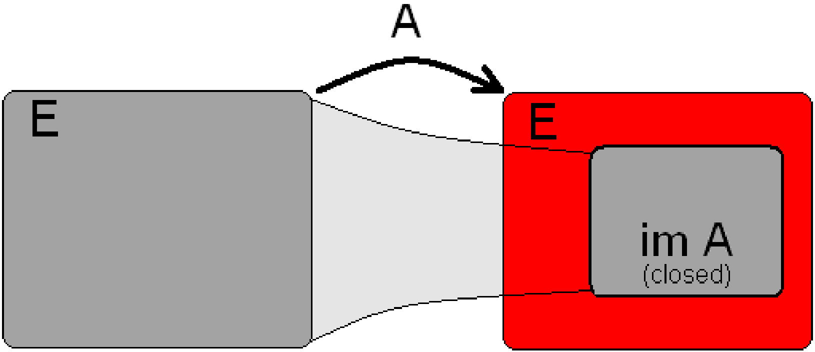

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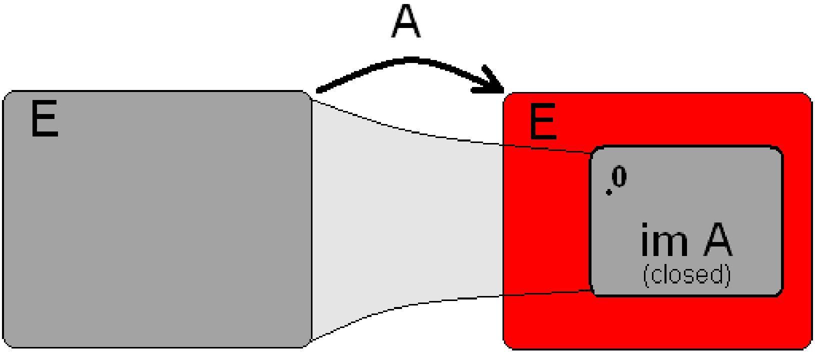

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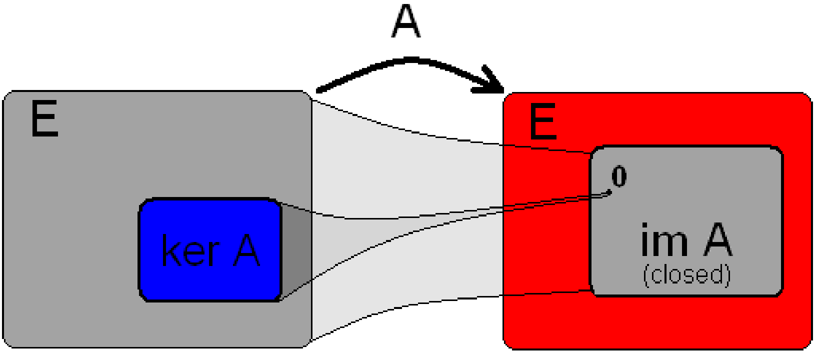

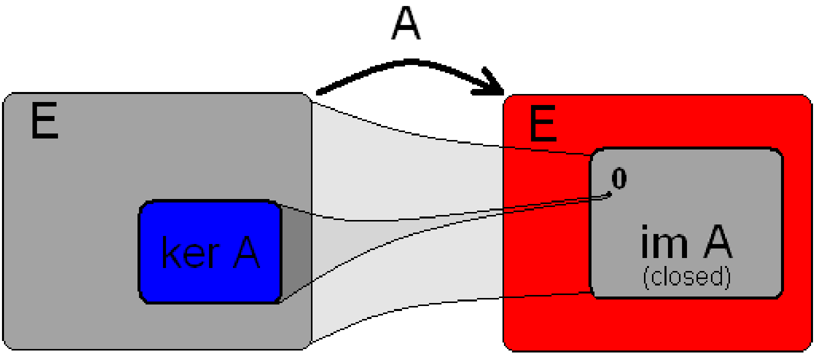

Definition

$A : E \to E$ is a *Fredholm operator*
if $\alpha := \dim(\ker A)$ and $\beta := \text{codim}(\text{im } A)$ are both finite.
The difference $\alpha - \beta$ is then called the *index of A*.

Spectrum, Essential Spectrum, Pseudospectra

For $A \in L(E)$ and $\varepsilon > 0$, we put

$$\operatorname{spec} A = \{\lambda \in \mathbb{C} : A - \lambda I \text{ not invertible on } E\},$$

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&= \bigcup_{\|T\| < \varepsilon} \operatorname{spec}(A + T)\\
&\supseteq \operatorname{spec} A \,+\, \varepsilon\mathbb{D} \qquad (\text{“}=\text{” if } A \text{ is normal}).
\end{aligned}$$

The sets $\operatorname{spec}_\varepsilon A$, $\varepsilon > 0$, are the so-called $\varepsilon-$**pseudospectra** of A. It holds that

$$\operatorname{spec} A \;=:\; \operatorname{spec}_0 A \;\subset\; \operatorname{spec}_{\varepsilon_1} A \;\subset\; \operatorname{spec}_{\varepsilon_2} A, \qquad 0 < \varepsilon_1 < \varepsilon_2.$$

A Bi-Infinite Jacobi Matrix

We study bi-infinite matrices of the form

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \beta_{-2} & \gamma_{-1} & & & & \\ & \alpha_{-2} & \beta_{-1} & \gamma_0 & & & \\ & & \alpha_{-1} & \beta_0 & \gamma_1 & & \\ & & & \alpha_0 & \beta_1 & \gamma_1 & \\ & & & & \alpha_1 & \beta_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix},$$

where $\alpha = (\alpha_i)$, $\beta = (\beta_i)$ and $\gamma = (\gamma_i)$ are bounded sequences of complex numbers.

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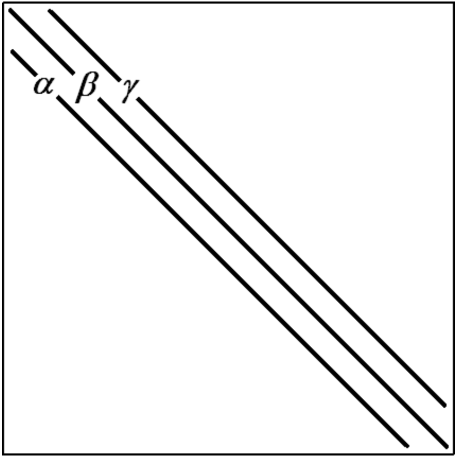
Task

Compute **upper bounds on spectrum and pseudospectra** of A, understood as a bounded linear operator $\ell^2 \to \ell^2$.

Strategy

Look at (pseudo)spectra of the **finite principal submatrices** of A:



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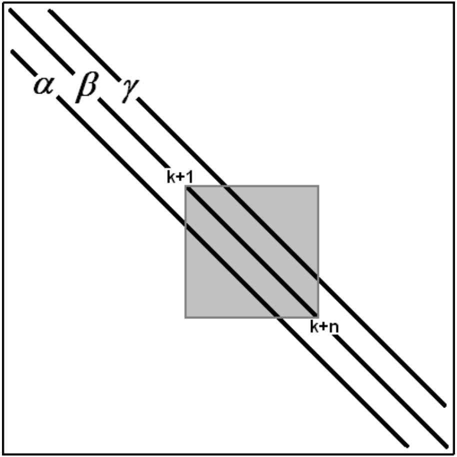

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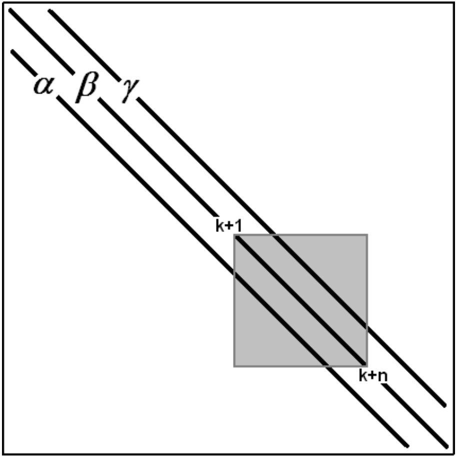

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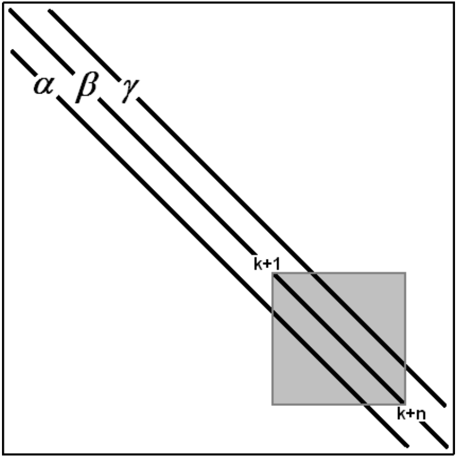
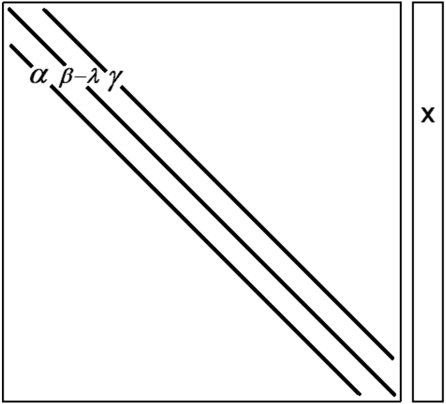


Method 1: Finite principal submatrices

Let $\lambda \in \mathrm{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\, x\| \;<\; \varepsilon\, \|x\|$$

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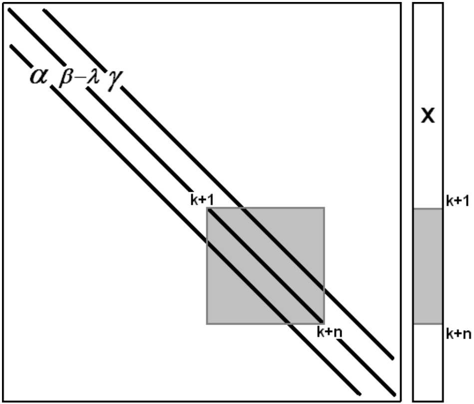

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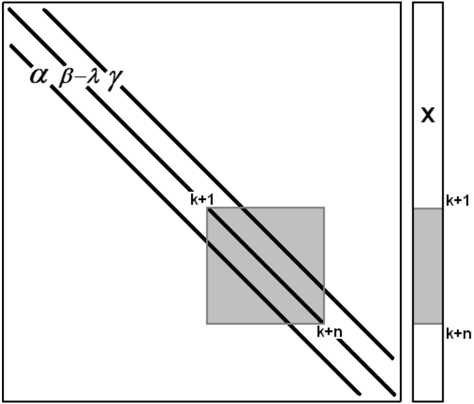

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Claim: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k}-\lambda I_n)\,x_{n,k}\| < (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

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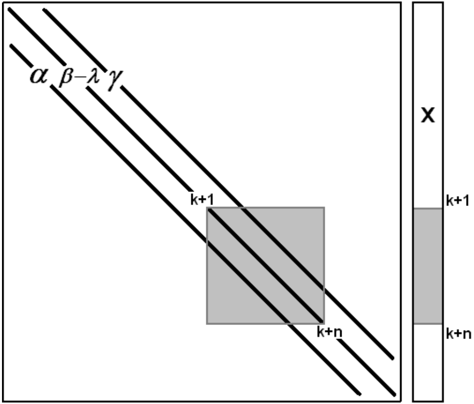

$$\|(A - \lambda I)\,x\| \;<\; \varepsilon\,\|x\|$$

Claim: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| < (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

$$\sum_k \|(A_{n,k} - \lambda I_n)\,x_{n,k}\|^2 < (\varepsilon + \varepsilon_n)^2 \sum_k \|x_{n,k}\|^2$$

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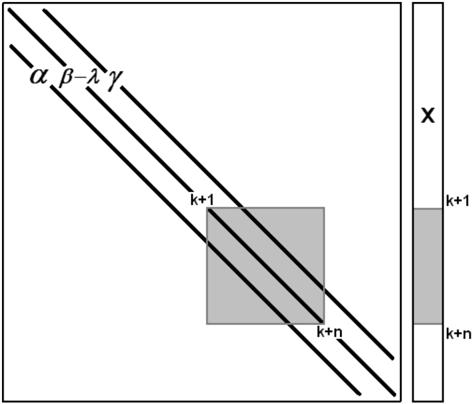

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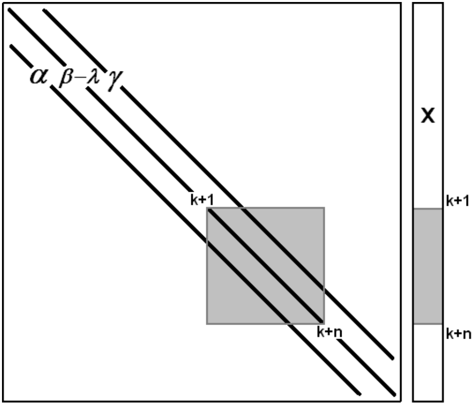

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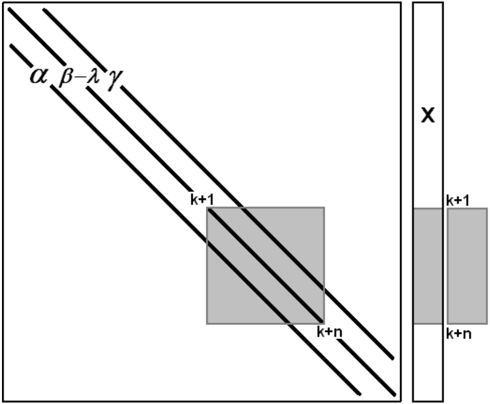

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$$\varepsilon_n \;<\; \frac{1}{\sqrt{n}}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

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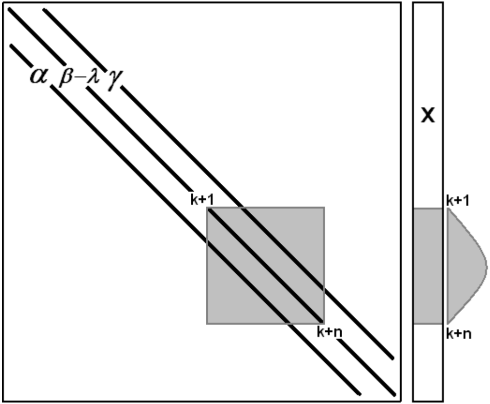

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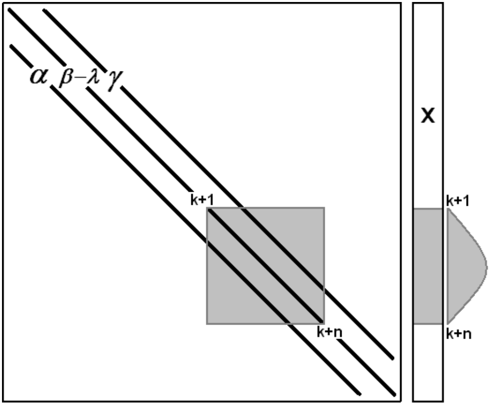

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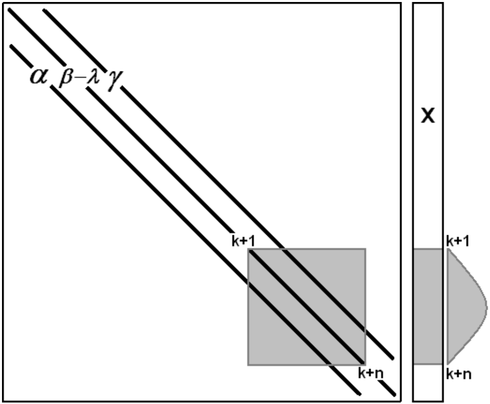

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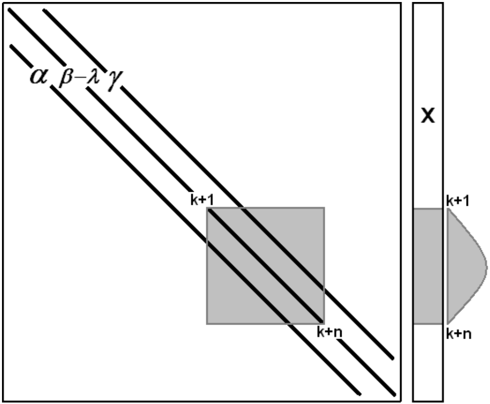

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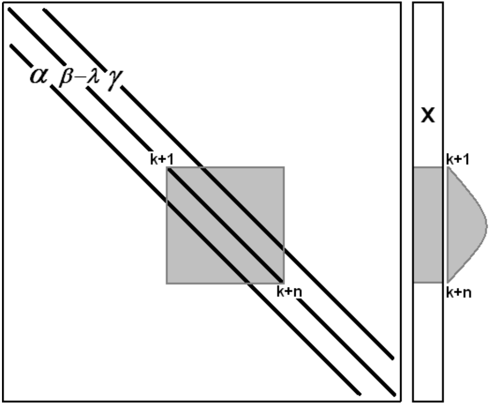

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So one gets

Upper Bound

$$\mathrm{spec}_\varepsilon(A) \quad \subset \quad \bigcup_{k\in\mathbb{Z}} \mathrm{spec}_{\varepsilon+\varepsilon_n}(A_{n,k}),$$

where

$$\varepsilon_n \ = 2(\|\alpha\|_\infty + \|\gamma\|_\infty)\sin\frac{\theta}{2} < \ \frac{\pi}{n+2}(\|\alpha\|_\infty + \|\gamma\|_\infty),$$

and $\dfrac{\pi}{2n+1} \leq \theta \leq \dfrac{\pi}{n+2}$ satisfies

$$2\sin\tfrac{\theta}{2}\cos(n+\tfrac{1}{2})\theta + \tfrac{\|\alpha\|_\infty\,\|\gamma\|_\infty}{(\|\alpha\|_\infty+\|\gamma\|_\infty)^2}\sin(n-1)\theta = 0.$$

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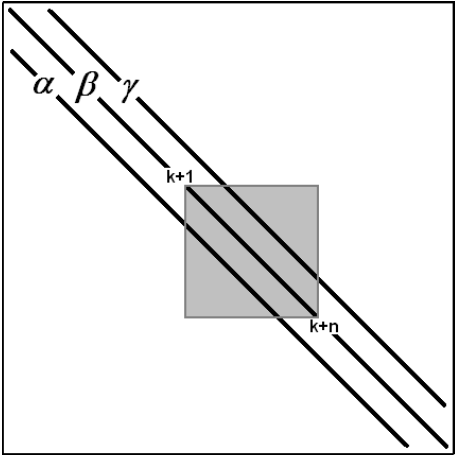
if $\alpha = 0$ or $\gamma = 0$, i.e., A is **bi-diagonal**.

Method 2: **Periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are "periodised",



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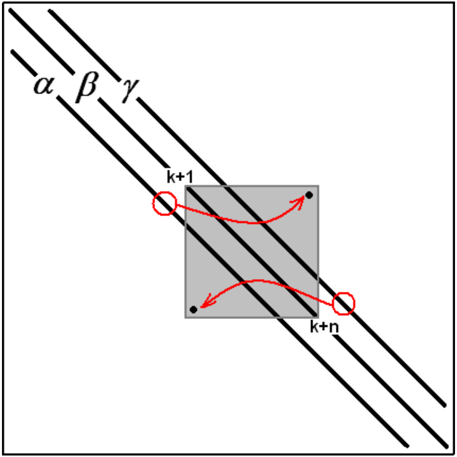

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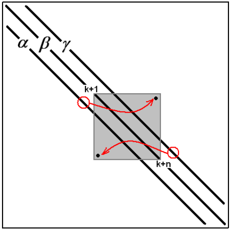

very similar computations show that, again,

$$\mathrm{spec}_\varepsilon(A) \subset \bigcup_{k\in\mathbb{Z}} \mathrm{spec}_{\varepsilon+\varepsilon_n}(A^{\mathrm{per}}_{n,k})$$

with $\quad \varepsilon_n = 2(\|\alpha\|_\infty + \|\gamma\|_\infty) \sin\frac{\pi}{2n} < \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$

and this upper bound on $\mathrm{spec}_\varepsilon(A)$ can be sharper than method 1.

Method 1 vs. Method 2: An Example

Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

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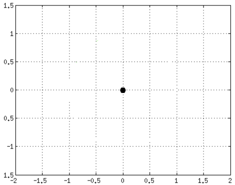

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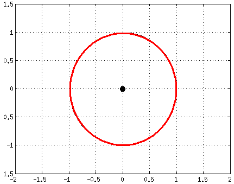

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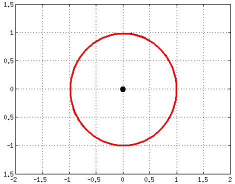

$$A^{\text{per}}_{n,k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}$$

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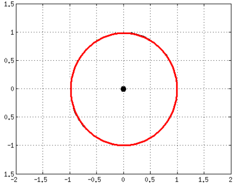
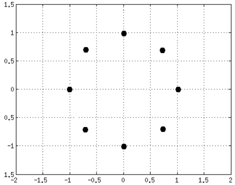


$$A_{n,k}^{\text{per}} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix} \qquad \operatorname{spec} A_{n,k}^{\text{per}} =$$



Method 1 vs. Method 2: An Example

Look at the shift operator

$$(Ax)(i) \;=\; x(i+1), \quad \text{i.e.} \quad A \;=\; \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

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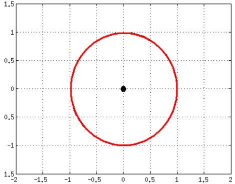

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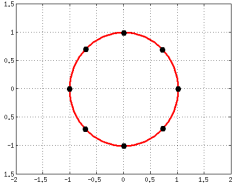

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Look at the shift operator

$$(Ax)(i) = x(i+1), \quad \text{i.e.} \quad A = \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

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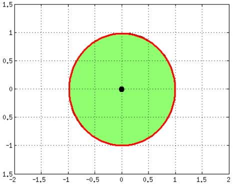

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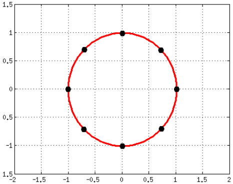

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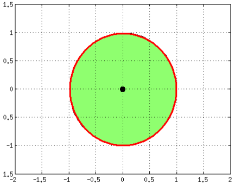

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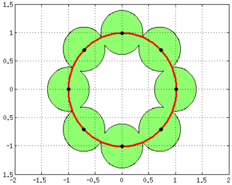

Method 1 vs. Method 2: An Example

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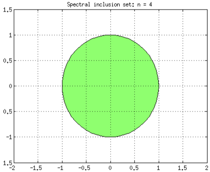

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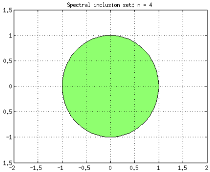

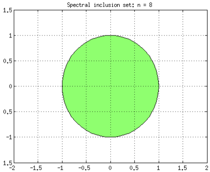

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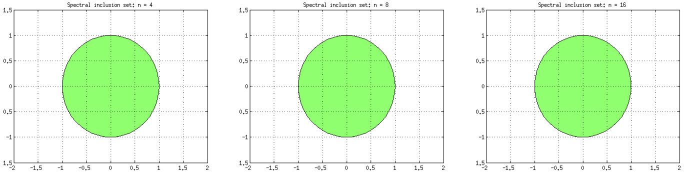

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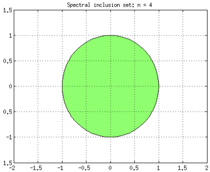

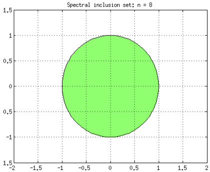

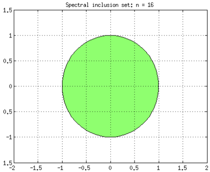

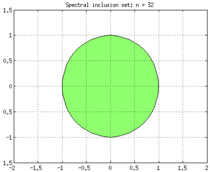

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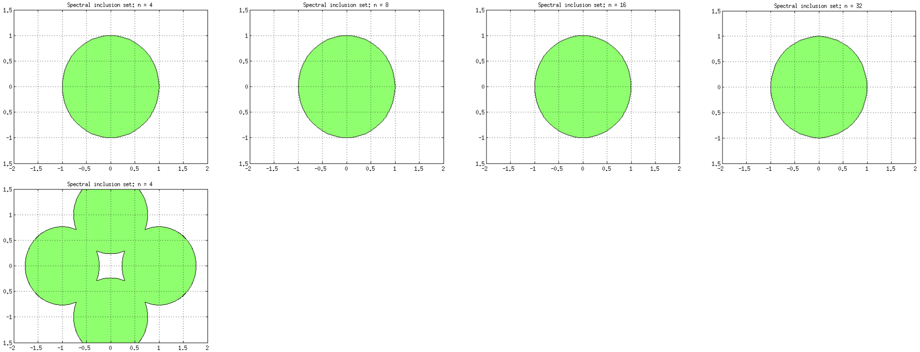

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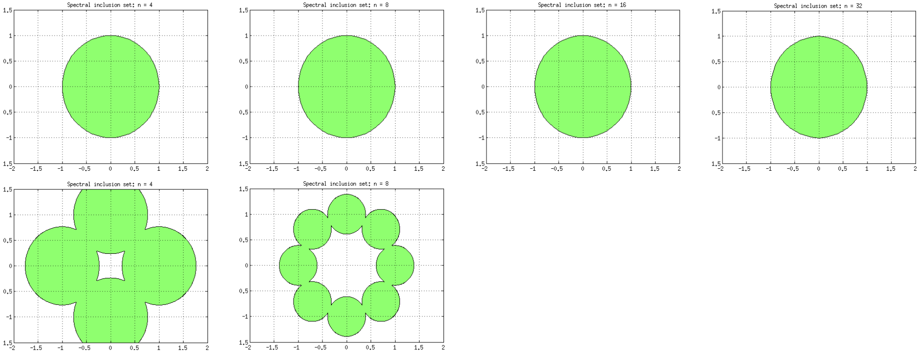

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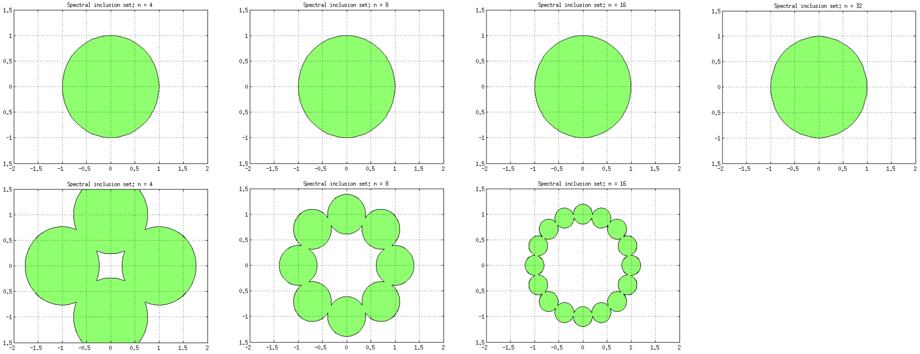

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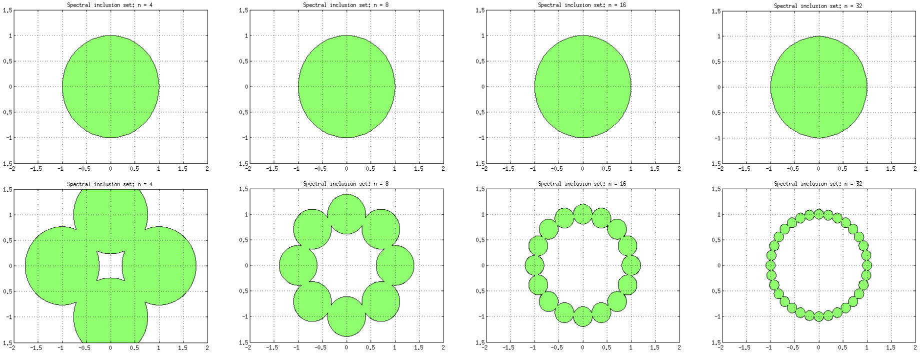


Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give **upper bounds** on spec A and $\text{spec}_\varepsilon A$.

Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give **upper bounds** on $\mathrm{spec}\, A$ and $\mathrm{spec}_\varepsilon A$.
- The bound from Method 2 is often **sharper**.

Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give **upper bounds** on $\operatorname{spec} A$ and $\operatorname{spec}_\varepsilon A$.
- The bound from Method 2 is often **sharper**.
- Conjecture: Method 2 **converges** to $\operatorname{spec}_\varepsilon A$ as $n \to \infty$.

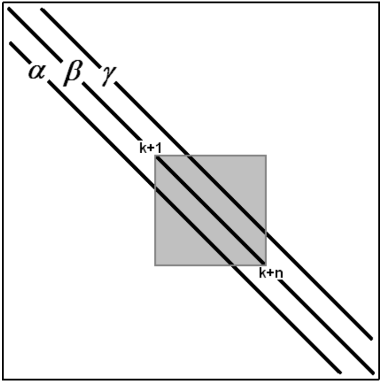
Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give **upper bounds** on spec A and $\mathrm{spec}_\varepsilon A$.
- The bound from Method 2 is often **sharper**.
- Conjecture: Method 2 **converges** to $\mathrm{spec}_\varepsilon A$ as $n \to \infty$.
- Method 1 also works for **semi-infinite** and **finite** matrices A!

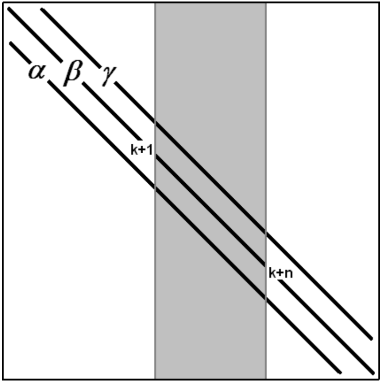
Here is another idea: Method 3

Instead of



Here is another idea: Method 3

We do a **"one-sided"** truncation.



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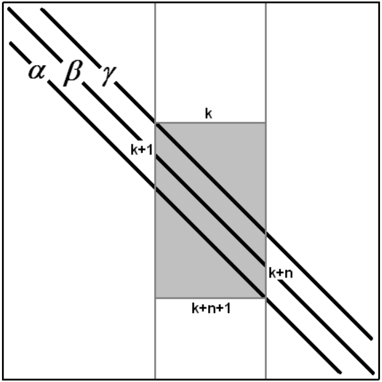


In a sense, we work with **rectangular** finite submatrices.

This is motivated by work of Davies 1998 and Hansen 2008.
(Also see Heinemeyer/Lindner/Potthast [SIAM Num. Anal. 2007].)

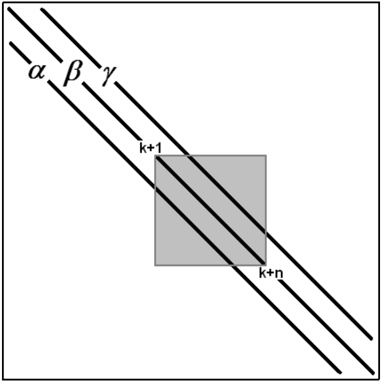
For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $P_{n,k} : \ell^2 \to \ell^2$ denote the projection

$$(P_{n,k}x)(i) := \begin{cases} x(i), & i \in \{k+1, \dots, k+n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Further, we put $X_{n,k} := \operatorname{im} P_{n,k}$ and identify it with $\mathbb{C}^n$ in the obvious way.

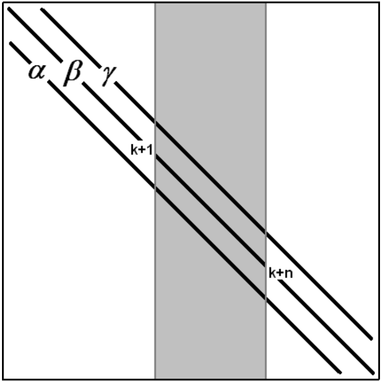
Method 3: Truncations

Method 1:



$$P_{n,k}(A - \lambda I)P_{n,k}|X_{n,k}$$

Method 3:



$$(A - \lambda I)P_{n,k}|X_{n,k}$$

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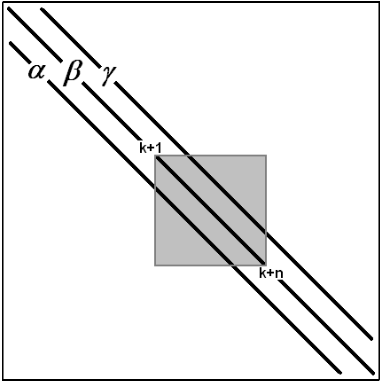
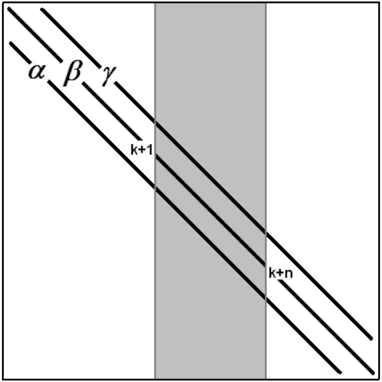


$$\mathbf{P_{n,k}}(A-\lambda I)\mathbf{P_{n,k}}|_{X_{n,k}}$$

Method 3:



$$(A-\lambda I)\mathbf{P_{n,k}}|_{X_{n,k}}$$

Method 1:

$$\lambda \in \mathrm{spec}_\varepsilon(A) \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k}AP_{n,k}|X_{n,k}\right)$$

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i.e. $s_{\min}\left(P_{n,k}(A-\lambda I)P_{n,k}\right) < \varepsilon + \varepsilon_n$

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$$\min \mathrm{spec}\left(\left(P_{n,k}(A-\lambda I)P_{n,k}\right)^*\left(P_{n,k}(A-\lambda I)P_{n,k}\right)\right) < \left(\varepsilon+\varepsilon_n\right)^2$$

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Method 3: Method 1 revisited

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$$\min \operatorname{spec}\left(P_{n,k}(A-\lambda I)^* P_{n,k} P_{n,k}(A-\lambda I)P_{n,k}\right) < (\varepsilon+\varepsilon_n)^2$$

Method 1:

$\lambda \in \text{spec}_\varepsilon(A) \implies$ For some $k \in \mathbb{Z}$:

$$\lambda \in \text{spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k} A P_{n,k}|X_{n,k}\right)$$

i.e. $s_{\min}\left(P_{n,k}(A-\lambda I)P_{n,k}\right) < \varepsilon + \varepsilon_n$

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Idea: $\min \mathrm{spec}\left(P_{n,k}(A-\lambda I)^* P_{n,k}(A-\lambda I)P_{n,k}\right) < \left(\varepsilon+\varepsilon_n\right)^2$ (the second $P_{n,k}$ is struck through in red)

Method 1:

$$\lambda \in \mathrm{spec}_\varepsilon(A) \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

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$$\text{i.e.} \quad s_{\min}\left(\cancel{P_{n,k}}(A-\lambda I)P_{n,k}\right) < \varepsilon + \varepsilon_n$$

Idea: $$\min \mathrm{spec}\left(P_{n,k}(A-\lambda I)^* \cancel{P_{n,k}}(A-\lambda I)P_{n,k}\right) < (\varepsilon+\varepsilon_n)^2$$

Method 3: Method 1 revisited

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$$\lambda \in \mathrm{spec}_\varepsilon(A) \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{spec}_{\varepsilon+\varepsilon_n}\left(\cancel{P_{n,k}} A P_{n,k}|X_{n,k}\right)$$

$$\text{i.e.} \quad s_{\min}\left(\cancel{P_{n,k}}(A-\lambda I)P_{n,k}\right) < \varepsilon + \varepsilon_n$$

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Method 1:

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Method 3

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of all $\lambda \in \mathbb{C}$, for which

$$\min \operatorname{spec}\left(P_{n,k}(A-\lambda I)^*(A-\lambda I)P_{n,k}\right) \;<\; \left(\varepsilon+\varepsilon_n\right)^2$$

Method 3: Method 1 revisited

Method 1:

Idea: $\quad \min \operatorname{spec}\left(P_{n,k}(A-\lambda I)^* \cancel{P_{n,k}}(A-\lambda I)P_{n,k}\right) < (\varepsilon+\varepsilon_n)^2$

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and

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Method 1:

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and $$\min\mathrm{spec}\left(P_{n,k}(A-\lambda I)(A-\lambda I)^*P_{n,k}\right) \;<\; (\varepsilon+\varepsilon_n)^2.$$

Then put

$$\Gamma_\varepsilon^n(A) \;:=\; \bigcup_{k\in\mathbb{Z}} \gamma_\varepsilon^{n,k}(A).$$

Method 3: Spectral bounds

Again we get (as in Methods 1 & 2)

Upper Bound

$$\mathrm{spec}_\varepsilon(A) \quad \subset \quad \bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon+\varepsilon_n}(A) \quad = \quad \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

with $\quad \varepsilon_n = 2(\|\alpha\|_\infty + \|\gamma\|_\infty)\,\sin\dfrac{\pi}{2n+2} < \dfrac{\pi}{n+1}(\|\alpha\|_\infty + \|\gamma\|_\infty)$

and this time the upper bound looks even sharper than before.

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and this time the upper bound looks even sharper than before. But now we also have

Lower Bound

$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{spec}_\varepsilon(A).$$

Method 3: Spectral bounds

From the lower and upper bound

$$\Gamma^n_\varepsilon(A) \subset \mathrm{spec}_\varepsilon(A) \quad \text{and} \quad \mathrm{spec}_\varepsilon(A) \subset \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

we get

Sandwich 1

$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{spec}_\varepsilon(A) \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

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Sandwich 2

$$\mathrm{spec}_\varepsilon(A) \subset \Gamma^n_{\varepsilon+\varepsilon_n}(A) \subset \mathrm{spec}_{\varepsilon+\varepsilon_n}(A).$$

Method 3: Spectral bounds

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$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{spec}_\varepsilon(A) \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon_n}(A)$$

Sandwich 2

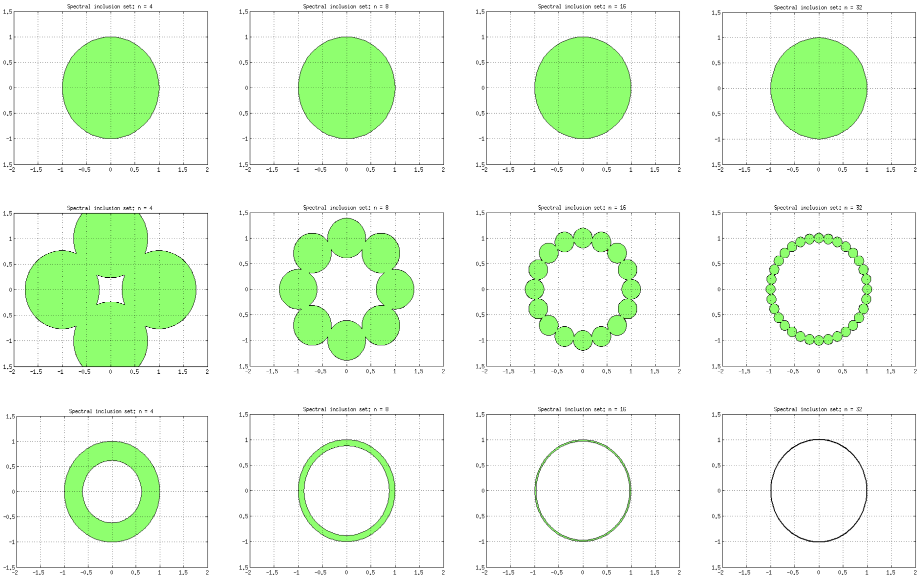
$$\mathrm{spec}_\varepsilon(A) \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon_n}(A) \quad \subset \quad \mathrm{spec}_{\varepsilon+\varepsilon_n}(A).$$

In particular, it follows that

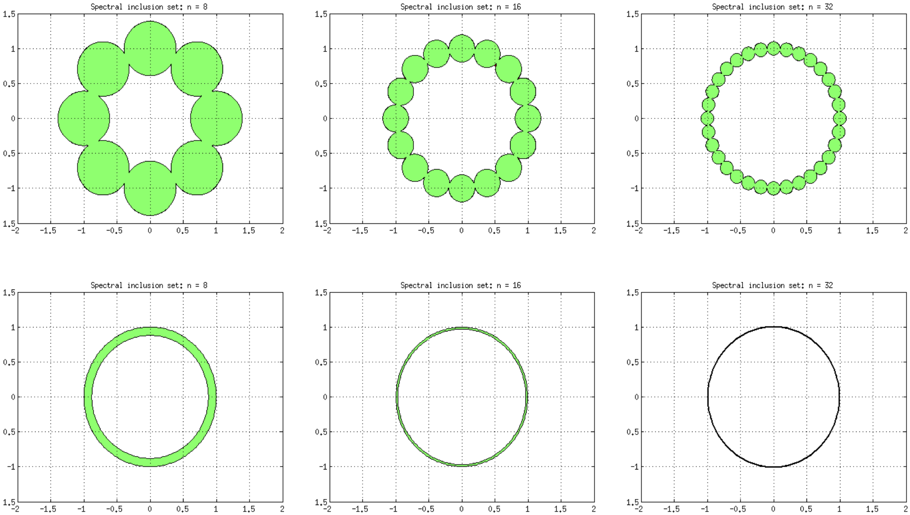
$$\Gamma^n_{\varepsilon+\varepsilon_n}(A) \quad \to \quad \mathrm{spec}_\varepsilon(A), \qquad n \to \infty$$

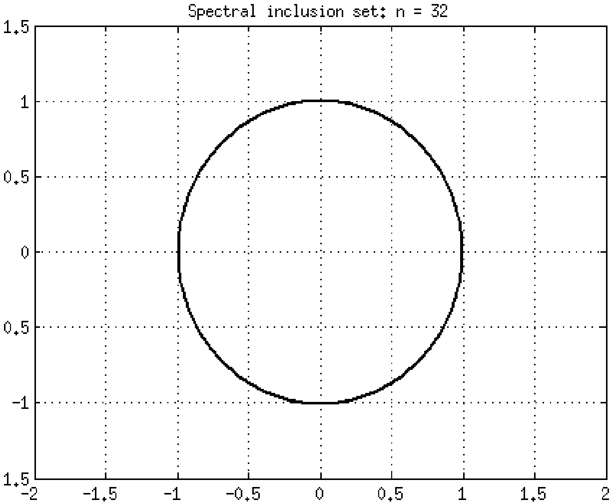
in the Hausdorff metric.

Methods 1, 2 & 3: The Shift Operator



Methods 2 & 3: The Shift Operator

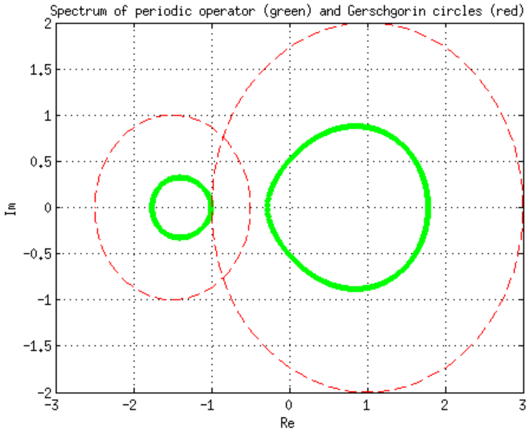



Spectral inclusion set: n = 32
1.5
1
0.5
0
0.5
-1
1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

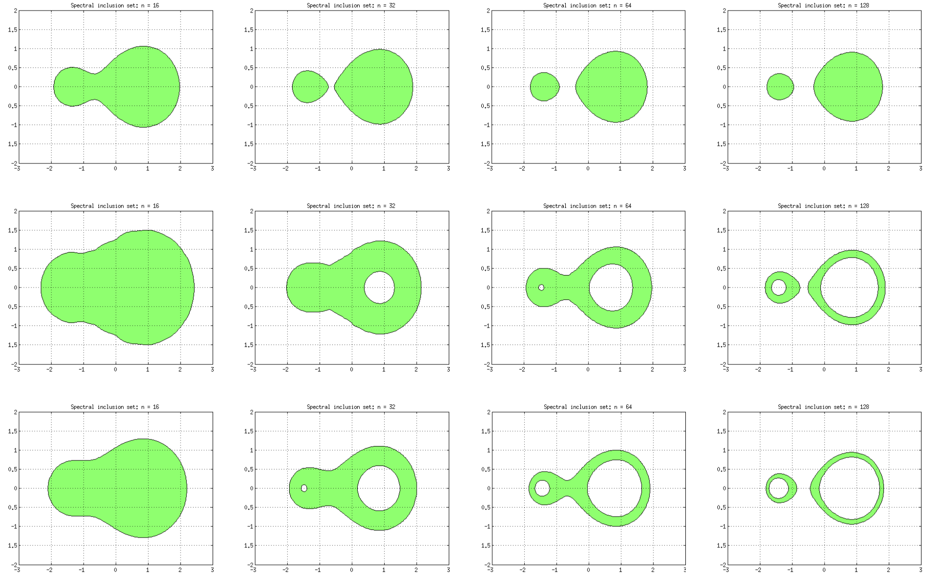
Methods 1, 2 & 3: Second Example

We now look at a matrix A with 3-periodic diagonals:

main diagonal: $\cdots, \ -\frac{3}{2}, \ 1, \ 1, \ \cdots$
super-diagonal: $\cdots, \ 1, \ 2, \ 1, \ \cdots$



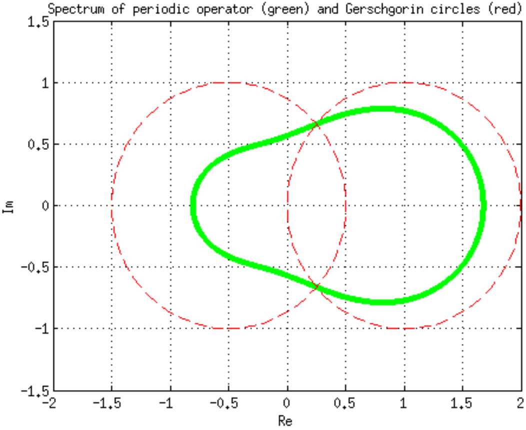
Methods 1, 2 & 3: Second Example



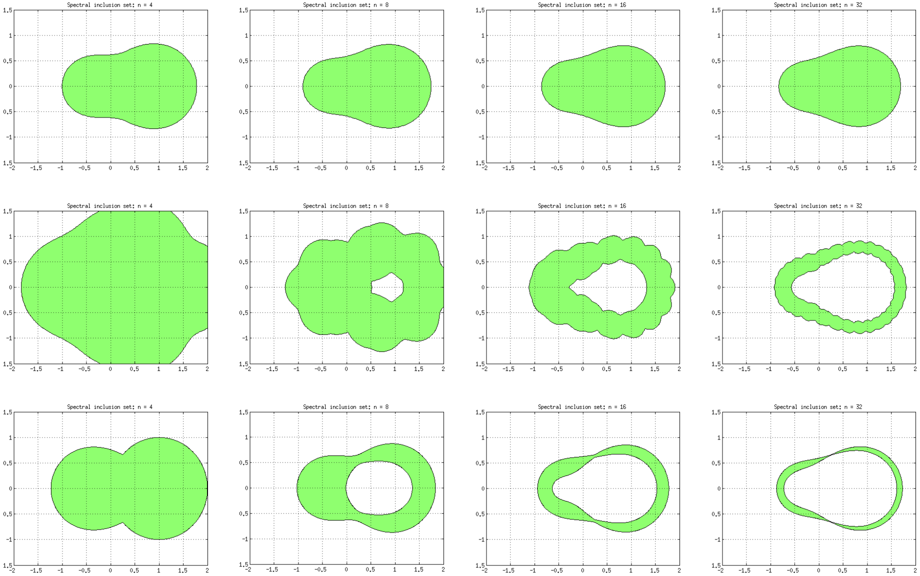
Methods 1, 2 & 3: Third Example

We now look at a matrix A with 3-periodic diagonals:

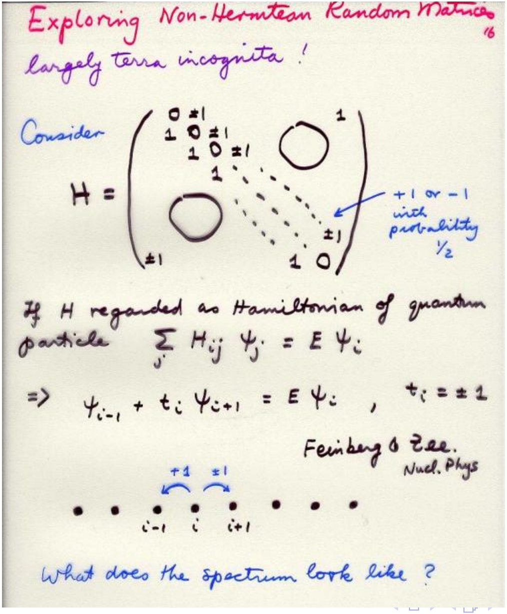
main diagonal: $\cdots,\ -\frac{1}{2},\ 1,\ 1,\ \cdots$
super-diagonal: $\cdots,\ 1,\ 1,\ 1,\ \cdots$



Methods 1, 2 & 3: Third Example



From a talk of Anthony Zee (MSRI Berkeley, 1999)



Final Example [Feinberg/Zee 1999]

Look at the **bi-infinite** matrix

$$A^b = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & b_{-1} & 0 & 1 & & \\ & & b_0 & 0 & 1 & \\ & & & b_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

where $b = (\cdots, b_{-1}, b_0, b_1, \cdots) \in \{\pm 1\}^{\mathbb{Z}}$ is a **pseudoergodic** sequence

Final Example [Feinberg/Zee 1999]

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where $b = (\cdots, b_{-1}, b_0, b_1, \cdots) \in \{\pm 1\}^{\mathbb{Z}}$ is a **pseudoergodic** sequence; that means:

every finite pattern of ± 1's can be found somewhere in the infinite sequence b.

Final Example [Feinberg/Zee 1999]

Spectral Formula C-W, Lindner 2008

If b is pseudoergodic then

$$\operatorname{spec} A^b = \operatorname{spec}_{\mathrm{ess}} A^b = \bigcup_{c\in\{\pm 1\}^{\mathbb{Z}}} \operatorname{spec}^{\infty}_{\mathrm{point}} A^c.$$

(Also see Rabinovich/Roch/Silbermann 1998 and Davies 2001.)

Final Example [Feinberg/Zee 1999]

Spectral Formula C-W, Lindner 2008

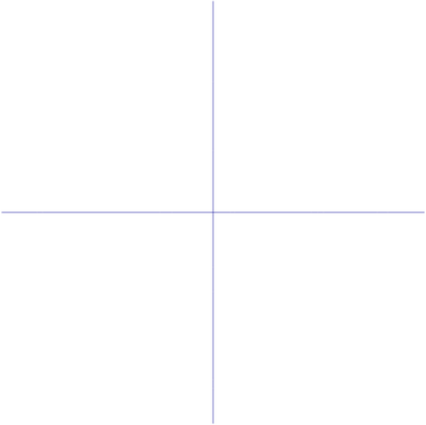
If b is pseudoergodic then

$$\operatorname{spec} A^b \;=\; \operatorname{spec}_{\mathrm{ess}} A^b \;=\; \bigcup_{c\in\{\pm 1\}^{\mathbb{Z}}} \operatorname{spec}^{\infty}_{\mathrm{point}} A^c.$$

(Also see Rabinovich/Roch/Silbermann 1998 and Davies 2001.)

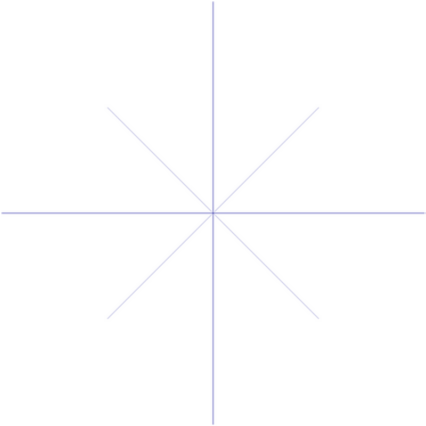
One can try to "exhaust" the RHS by running through all **periodic** ± 1 sequences c.

Final Example [Feinberg/Zee 1999]

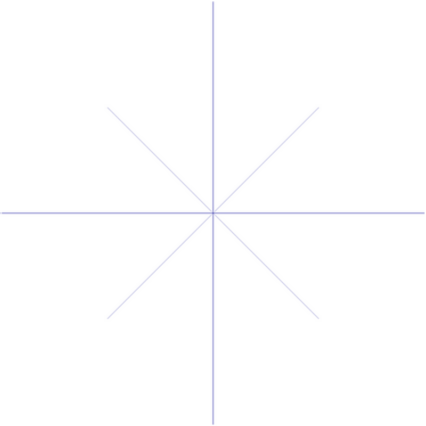


Period 1

Final Example [Feinberg/Zee 1999]

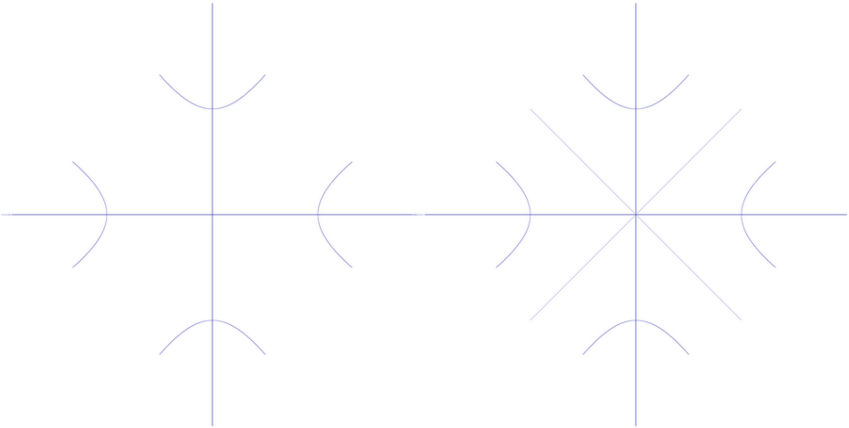


Period 2



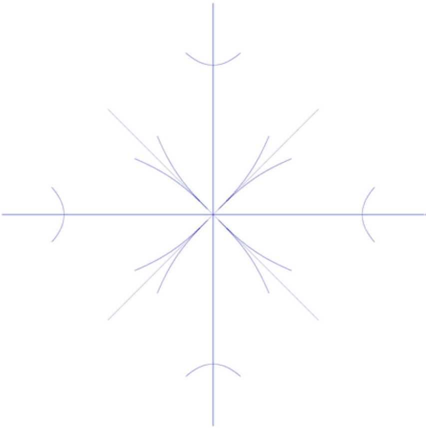
Periods $1, 2$

Final Example [Feinberg/Zee 1999]

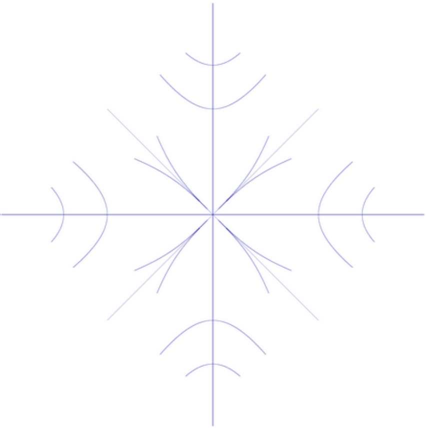


Period 3 Periods 1, ..., 3

Final Example [Feinberg/Zee 1999]

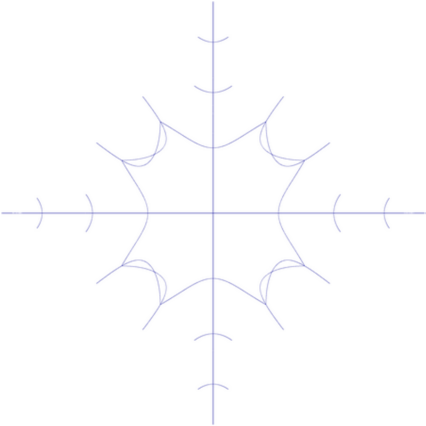


Period 4

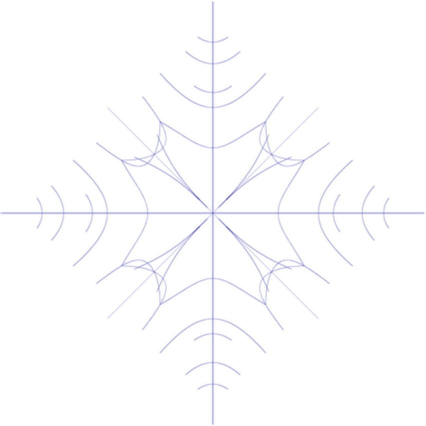


Periods 1, ..., 4

Final Example [Feinberg/Zee 1999]

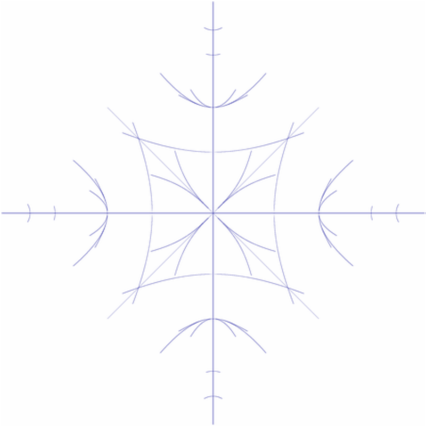


Period 5

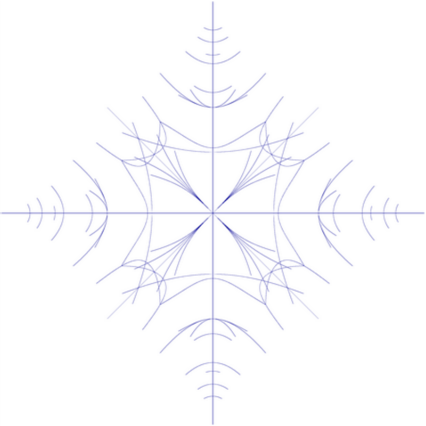


Periods 1, ..., 5

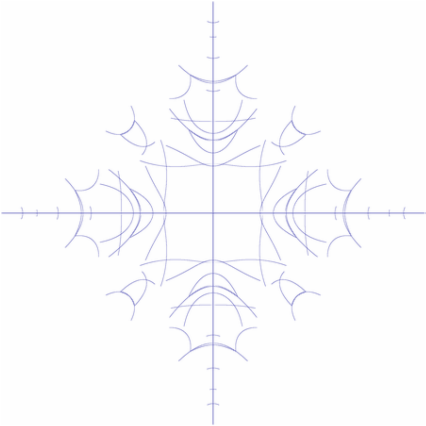
Final Example [Feinberg/Zee 1999]



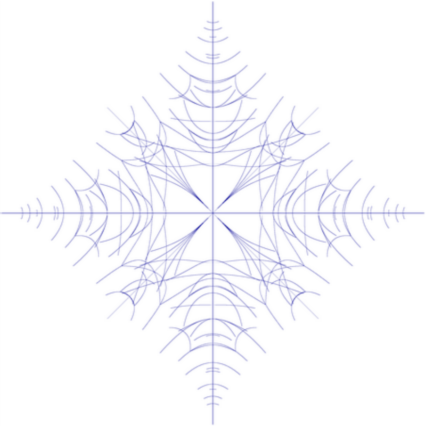
Period 6



Periods 1, ..., 6

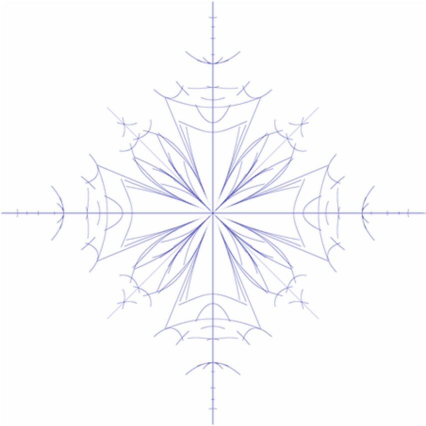


Period 7

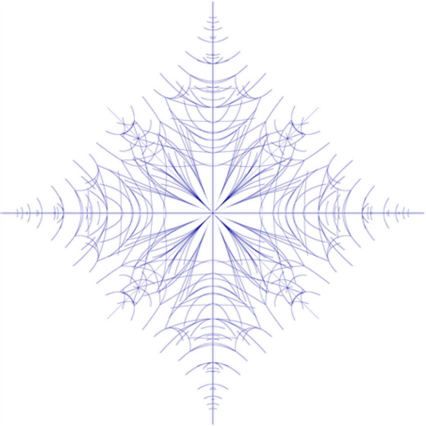


Periods 1, ..., 7

Final Example [Feinberg/Zee 1999]

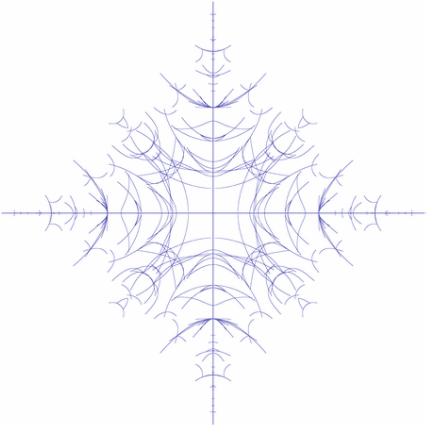


Period 8

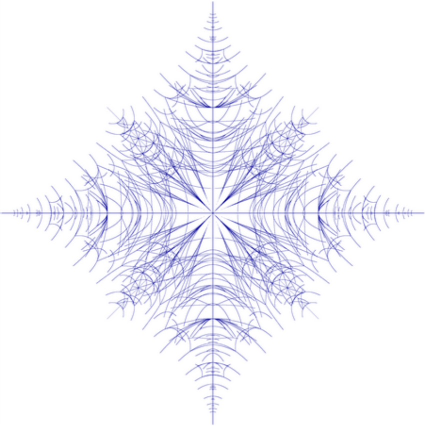


Periods 1, ..., 8

Final Example [Feinberg/Zee 1999]

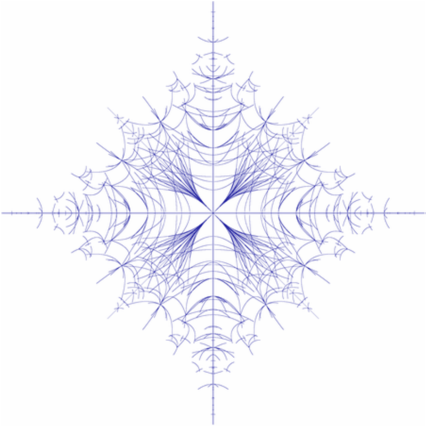


Period 9

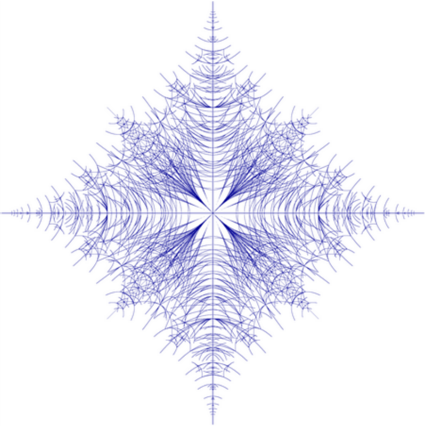


Periods $1, \ldots, 9$

Final Example [Feinberg/Zee 1999]

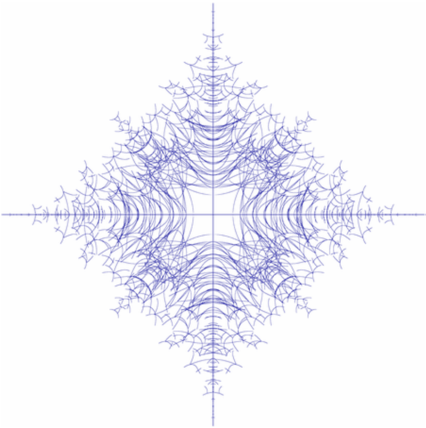


Period 10

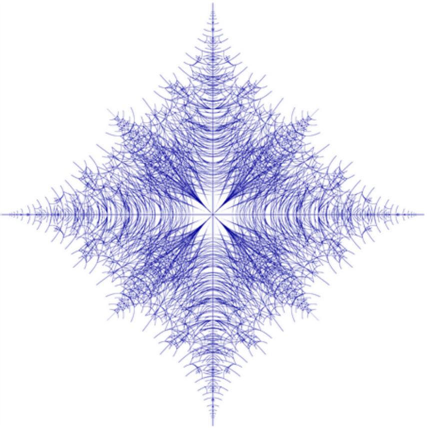


Periods 1, ..., 10

Final Example [Feinberg/Zee 1999]

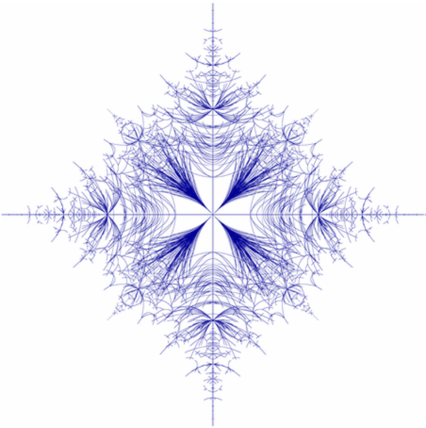


Period 11

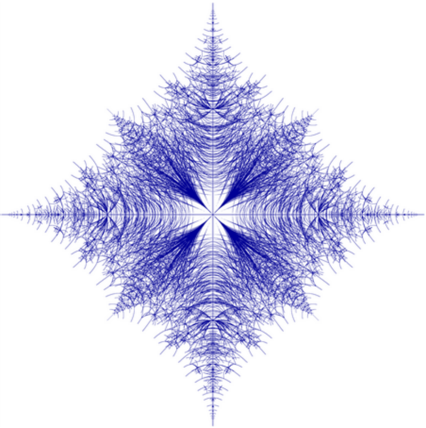


Periods 1, ..., 11

Final Example [Feinberg/Zee 1999]

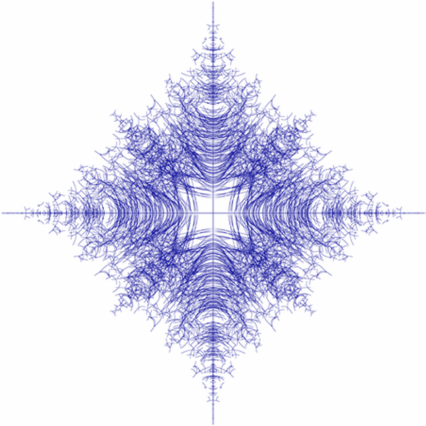


Period 12

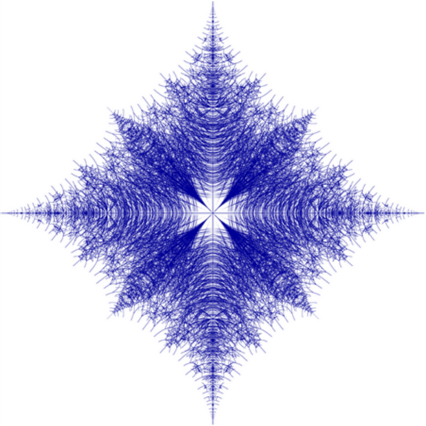


Periods 1, ..., 12

Final Example [Feinberg/Zee 1999]

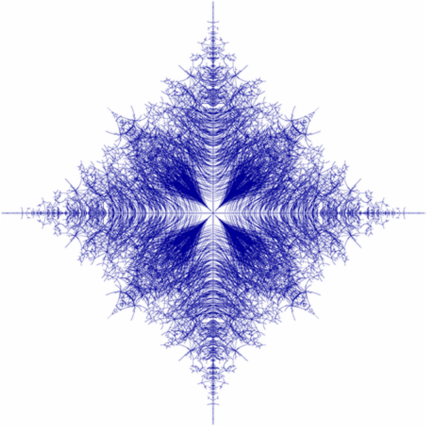


Period 13

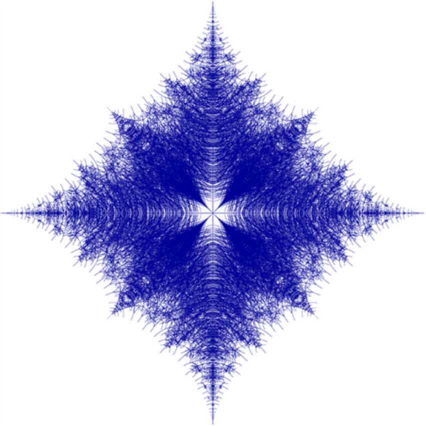


Periods 1, ..., 13

Final Example [Feinberg/Zee 1999]

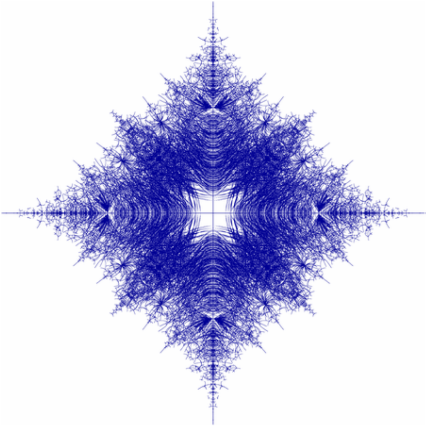


Period 14

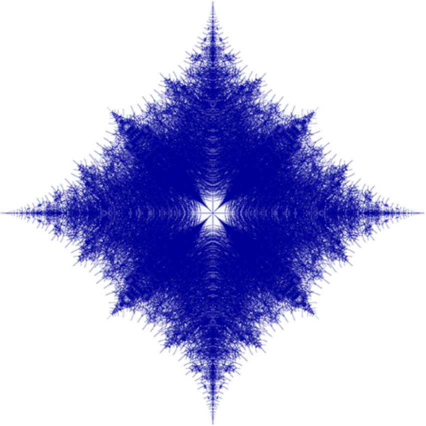


Periods 1, ..., 14

Final Example [Feinberg/Zee 1999]

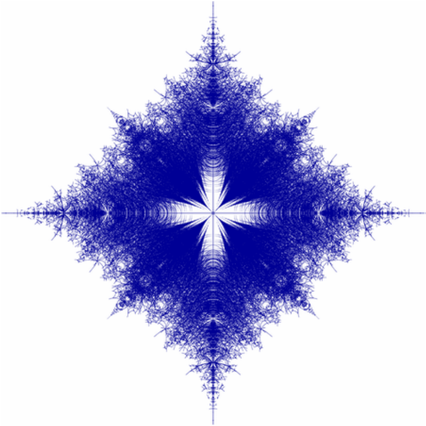


Period 15

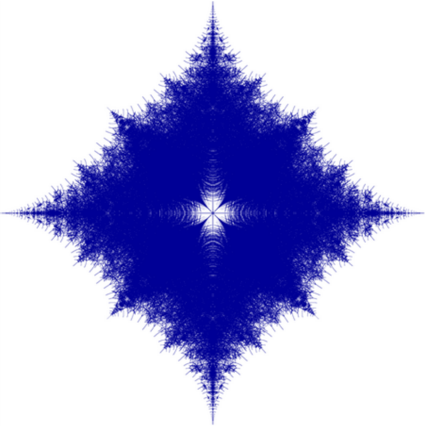


Periods 1, ..., 15

Final Example [Feinberg/Zee 1999]

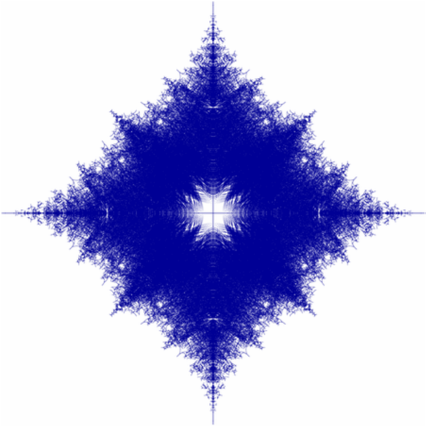


Period 16

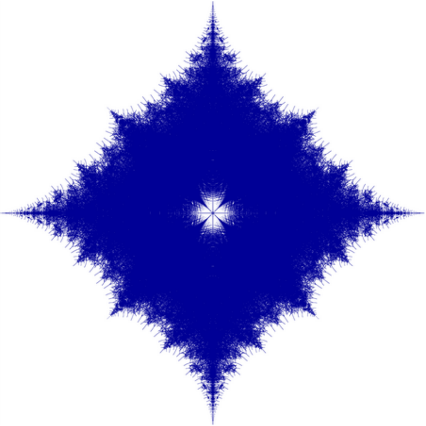


Periods 1, ..., 16

Final Example [Feinberg/Zee 1999]

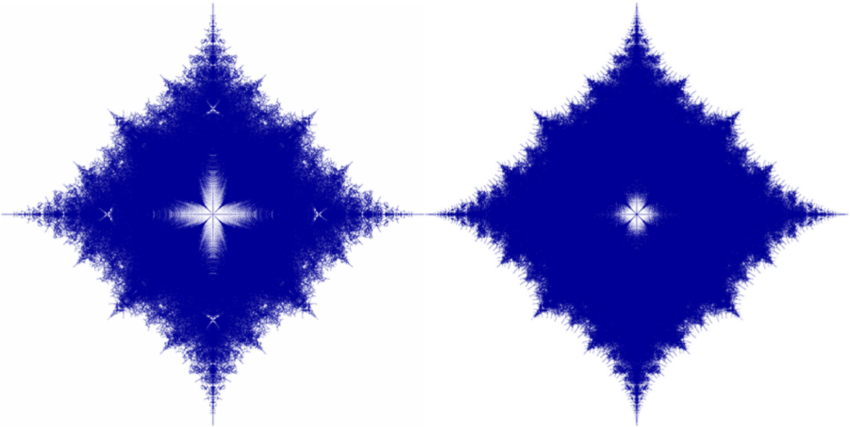


Period 17



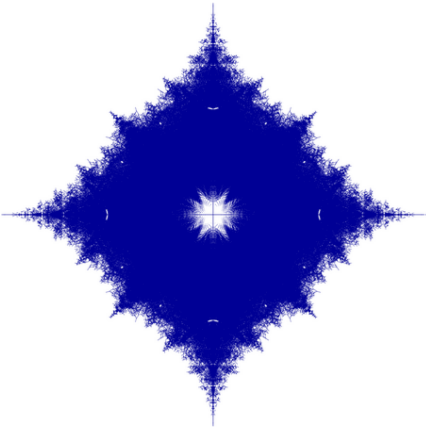
Periods 1, ..., 17

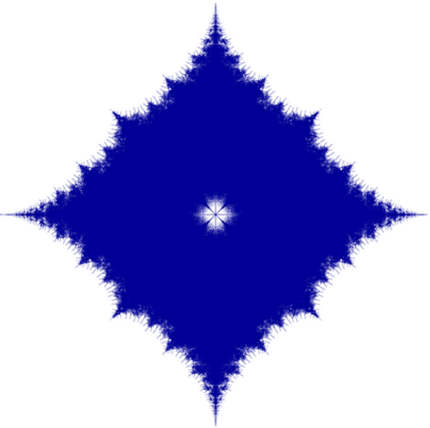
Final Example [Feinberg/Zee 1999]



Period 18 Periods 1, ..., 18

Final Example [Feinberg/Zee 1999]

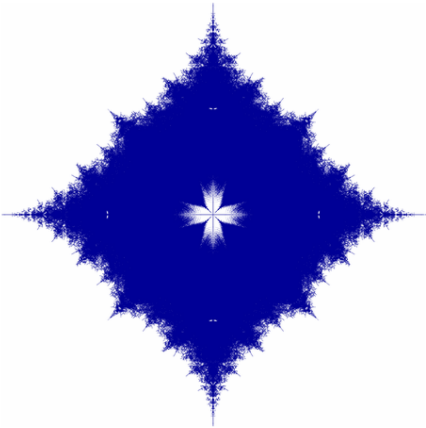




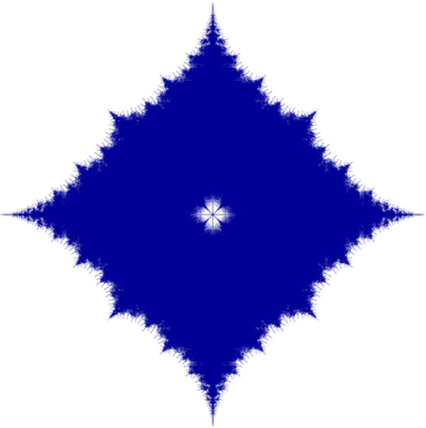
Period 19

Periods 1, ..., 19

Final Example [Feinberg/Zee 1999]

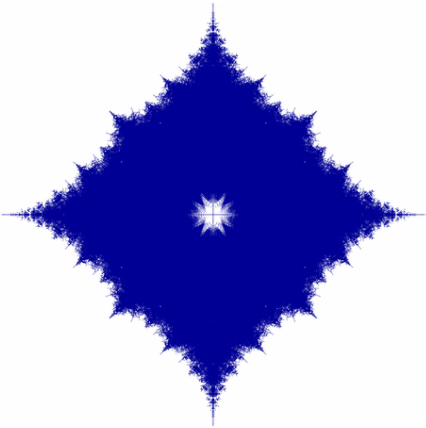


Period 20

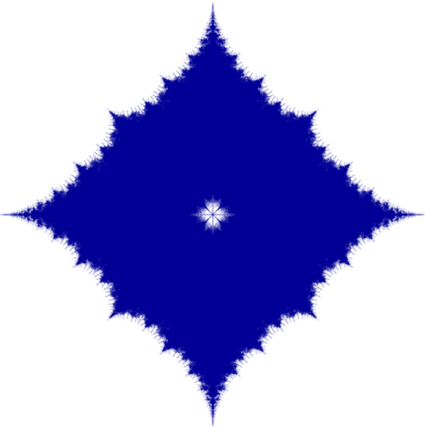


Periods 1, ..., 20

Final Example [Feinberg/Zee 1999]

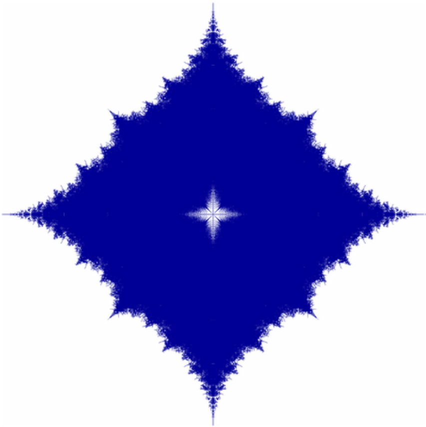


Period 21

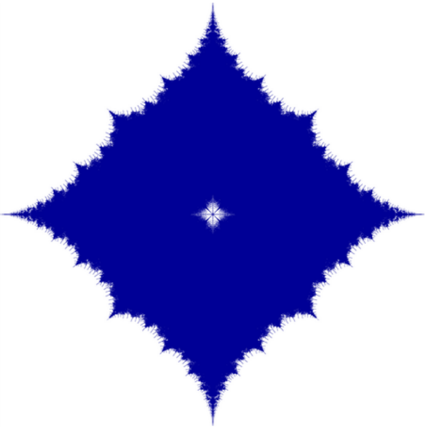


Periods 1, ..., 21

Final Example [Feinberg/Zee 1999]

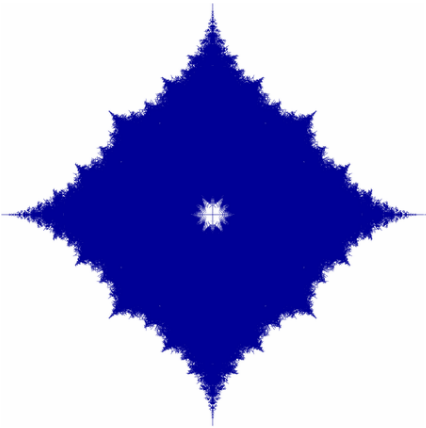


Period 22

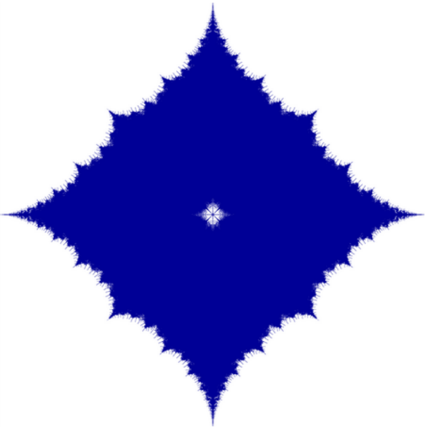


Periods 1, ..., 22

Final Example [Feinberg/Zee 1999]

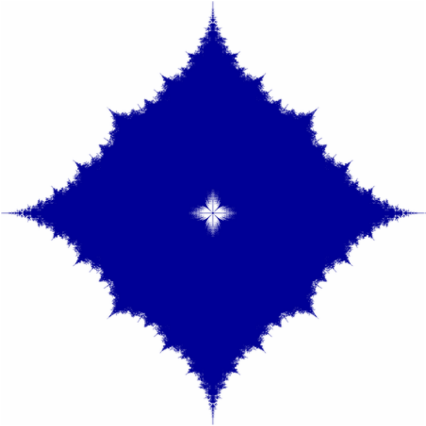


Period 23

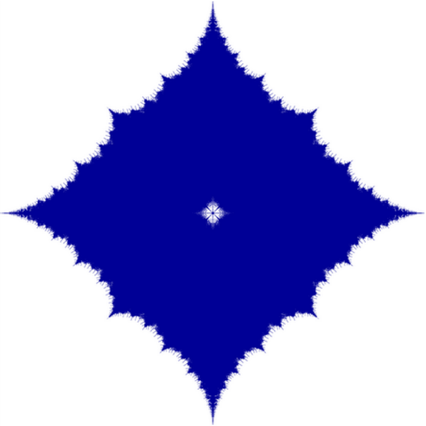


Periods 1, ..., 23

Final Example [Feinberg/Zee 1999]

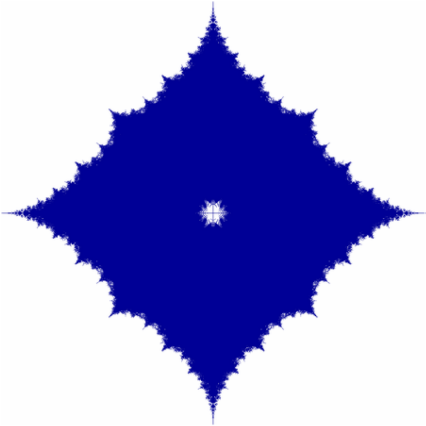


Period 24

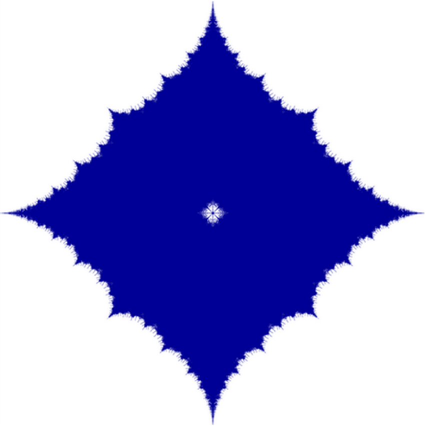


Periods 1, ..., 24

Final Example [Feinberg/Zee 1999]

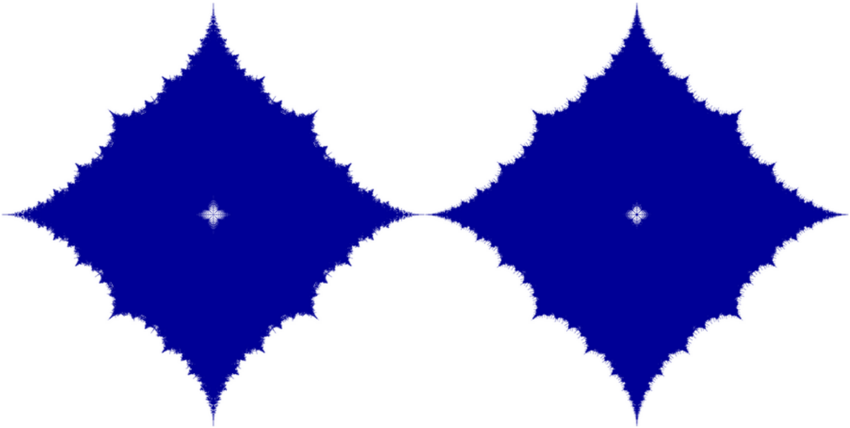


Period 25



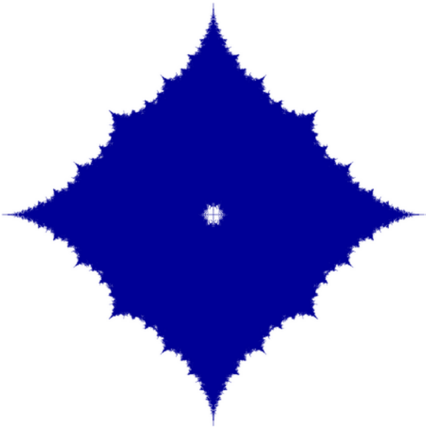
Periods 1, ..., 25

Final Example [Feinberg/Zee 1999]

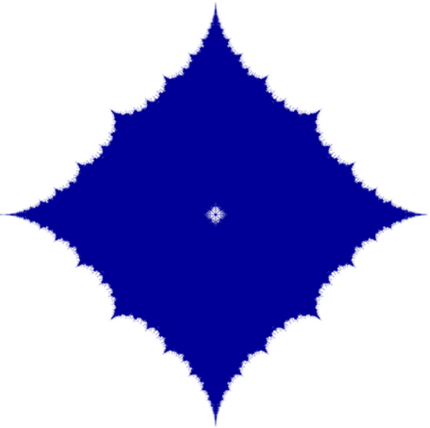


Period 26 Periods 1, ..., 26

Final Example [Feinberg/Zee 1999]

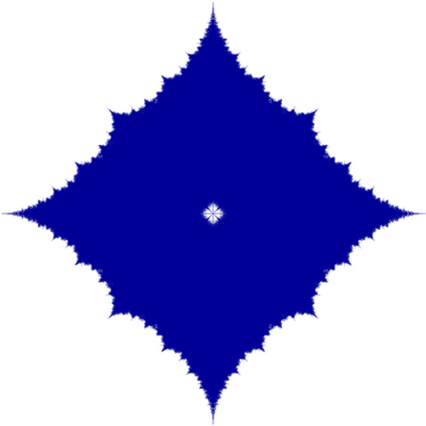


Period 27

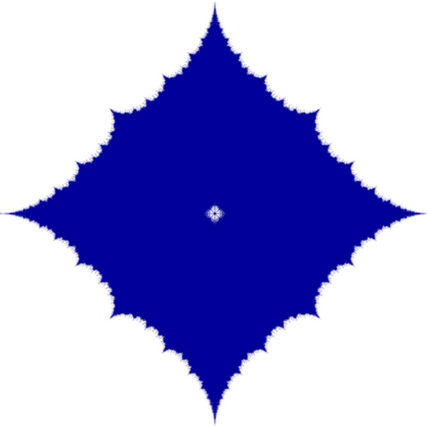


Periods 1, ..., 27

Final Example [Feinberg/Zee 1999]

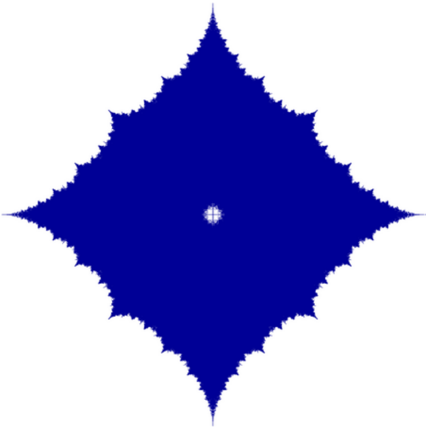


Period 28

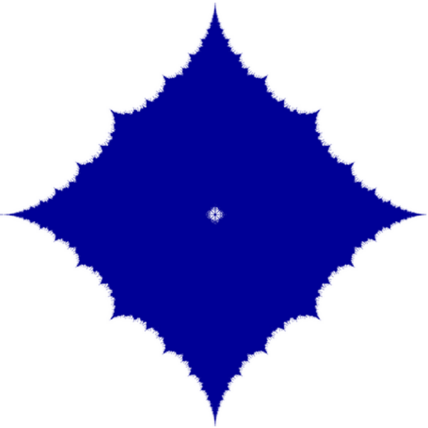


Periods 1, ..., 28

Final Example [Feinberg/Zee 1999]

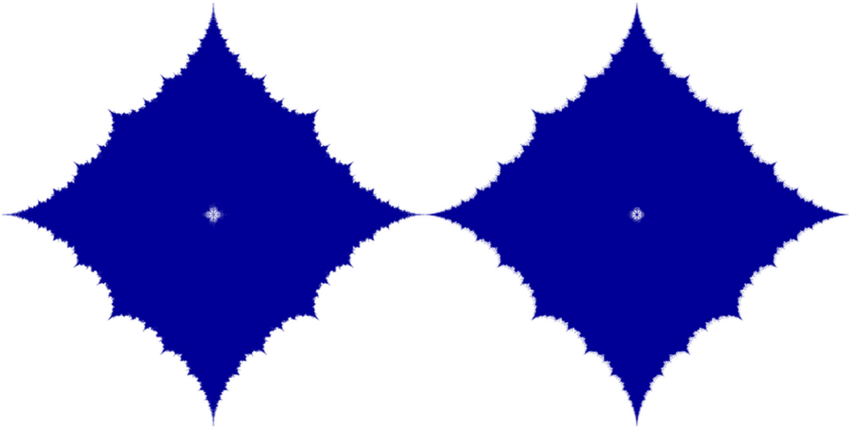


Period 29



Periods 1, ..., 29

Final Example [Feinberg/Zee 1999]



Period 30 | Periods 1, ..., 30

Final Example [Feinberg/Zee 1999]

Recall our "Sandwich 1": In this example, one has

$$\underbrace{\bigcup_{k\in\mathbb{Z}} \mathrm{spec}_\varepsilon(P_{n,k}A^bP_{n,k})}_{=:\ \sigma_n^\varepsilon} \;\subset\; \mathrm{spec}_\varepsilon(A^b) \;\subset\; \underbrace{\bigcup_{k\in\mathbb{Z}} \mathrm{spec}_{\varepsilon+\varepsilon_n}(P_{n,k}A^bP_{n,k})}_{\Sigma_n^\varepsilon\ :=\ \sigma_n^{\varepsilon+\varepsilon_n}}$$

for all $n \in \mathbb{N}$, so let's look at σ_n^ε for $\varepsilon = 0$.

Here are the $n \times n$ matrix eigenvalues

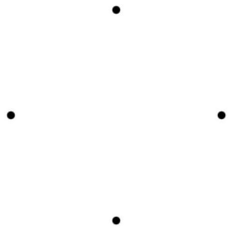
$$\sigma_n^0 \;=\; \bigcup_{k\in\mathbb{Z}} \mathrm{spec}\,(P_{n,k}A^bP_{n,k})$$

for $n = 1, ..., 30$:

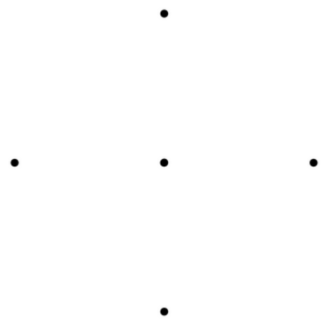
Final Example [Feinberg/Zee 1999]

Size 1

Final Example [Feinberg/Zee 1999]

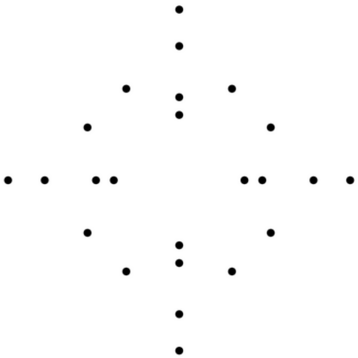


Size 2



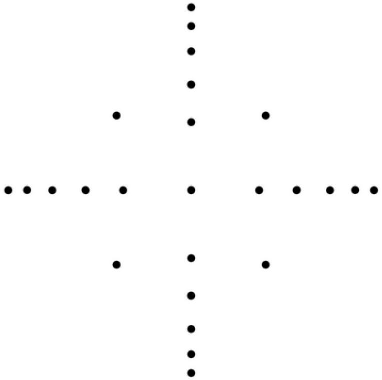
Size 3

Final Example [Feinberg/Zee 1999]



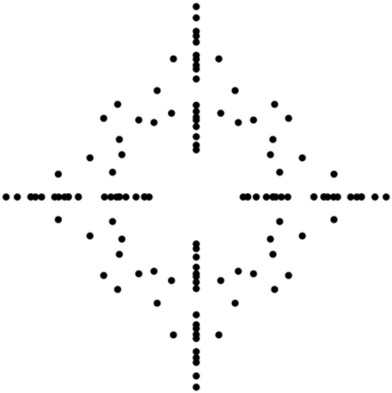
Size 4

Final Example [Feinberg/Zee 1999]



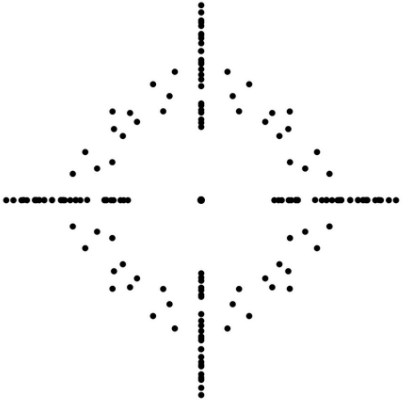
Size 5

Final Example [Feinberg/Zee 1999]



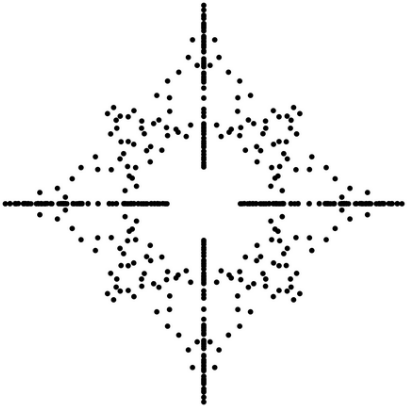
Size 6

Final Example [Feinberg/Zee 1999]



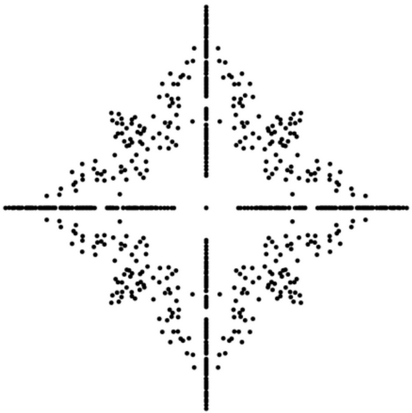
Size 7

Final Example [Feinberg/Zee 1999]



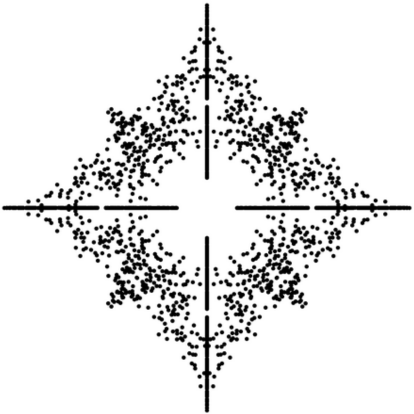
Size 8

Final Example [Feinberg/Zee 1999]



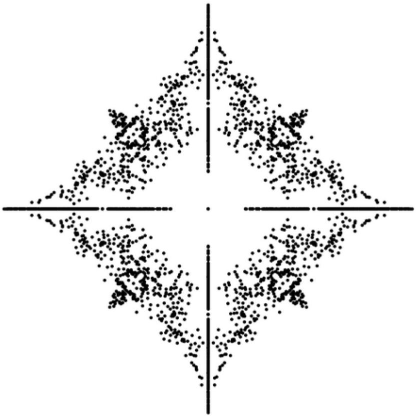
Size 9

Final Example [Feinberg/Zee 1999]



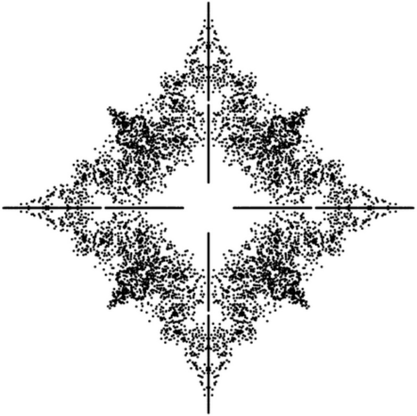
Size 10

Final Example [Feinberg/Zee 1999]



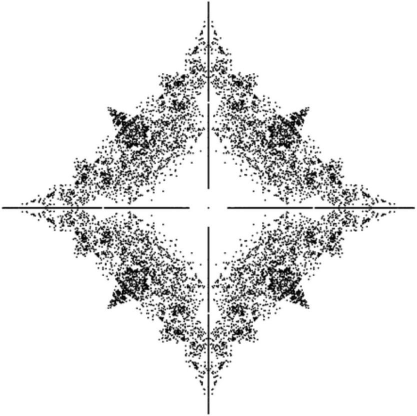
Size 11

Final Example [Feinberg/Zee 1999]



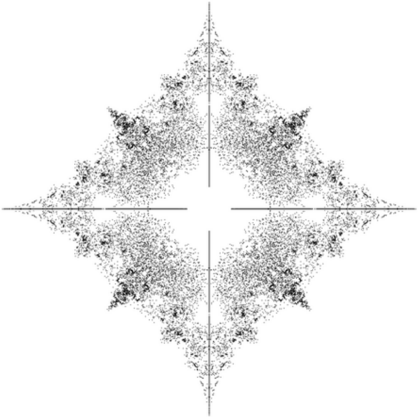
Size 12

Final Example [Feinberg/Zee 1999]



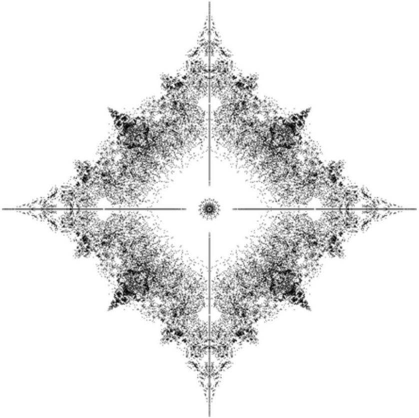
Size 13

Final Example [Feinberg/Zee 1999]



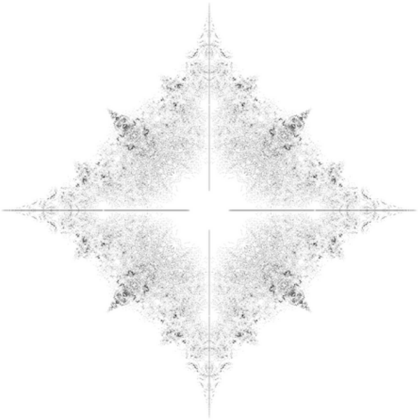
Size 14

Final Example [Feinberg/Zee 1999]



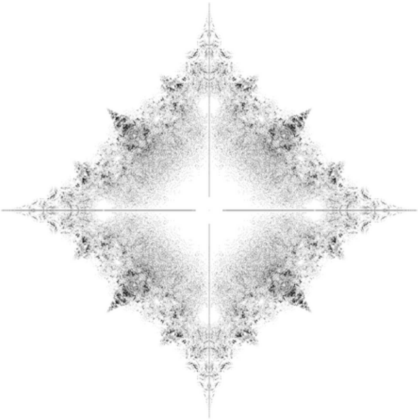
Size 15

Final Example [Feinberg/Zee 1999]



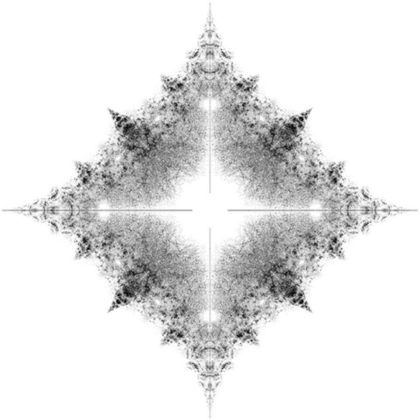
Size 16

Final Example [Feinberg/Zee 1999]



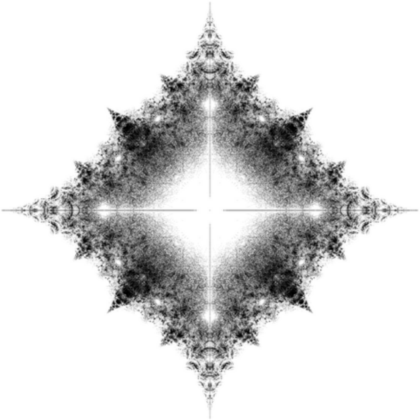
Size 17

Final Example [Feinberg/Zee 1999]



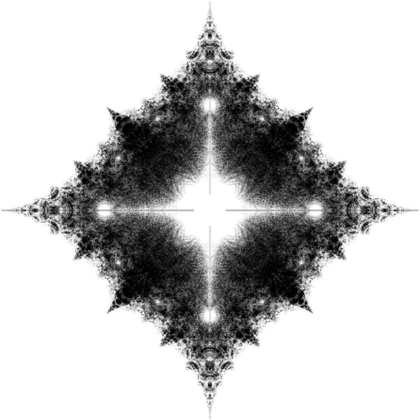
Size 18

Final Example [Feinberg/Zee 1999]



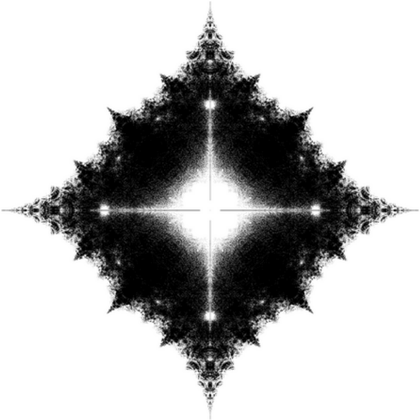
Size 19

Final Example [Feinberg/Zee 1999]



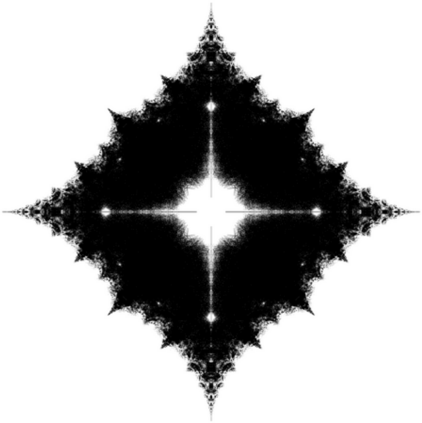
Size 20

Final Example [Feinberg/Zee 1999]



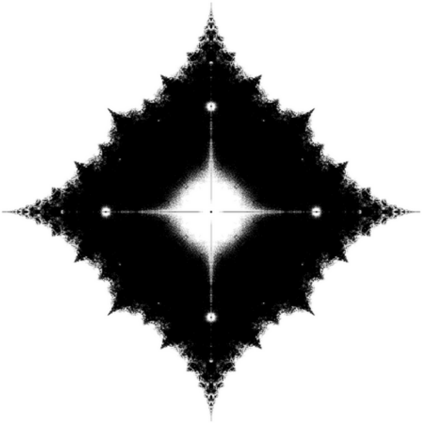
Size 21

Final Example [Feinberg/Zee 1999]



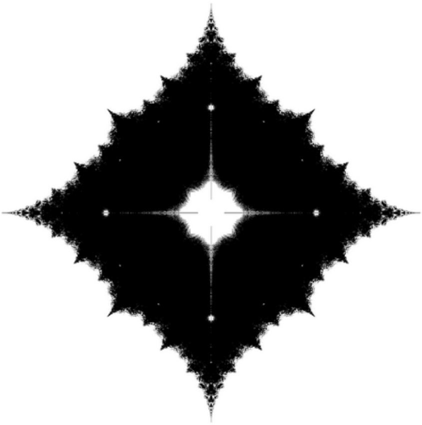
Size 22

Final Example [Feinberg/Zee 1999]



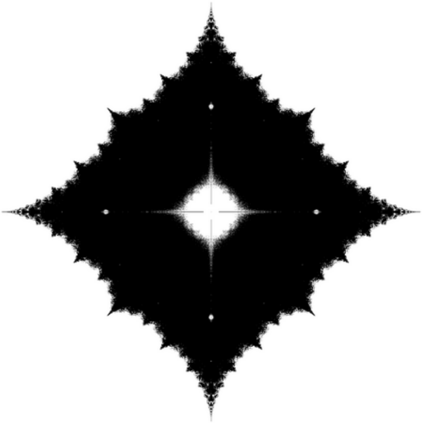
Size 23

Final Example [Feinberg/Zee 1999]



Size 24

Final Example [Feinberg/Zee 1999]

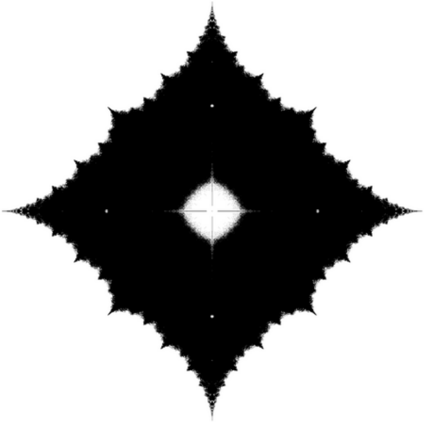


Size 25

Final Example [Feinberg/Zee 1999]

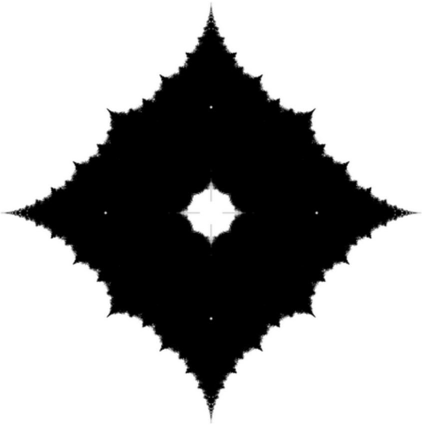
Size 26

Final Example [Feinberg/Zee 1999]



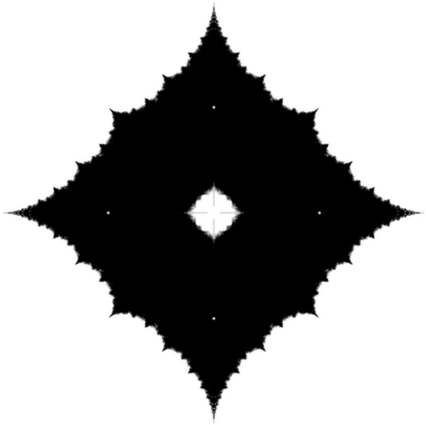
Size 27

Final Example [Feinberg/Zee 1999]



Size 28

Final Example [Feinberg/Zee 1999]

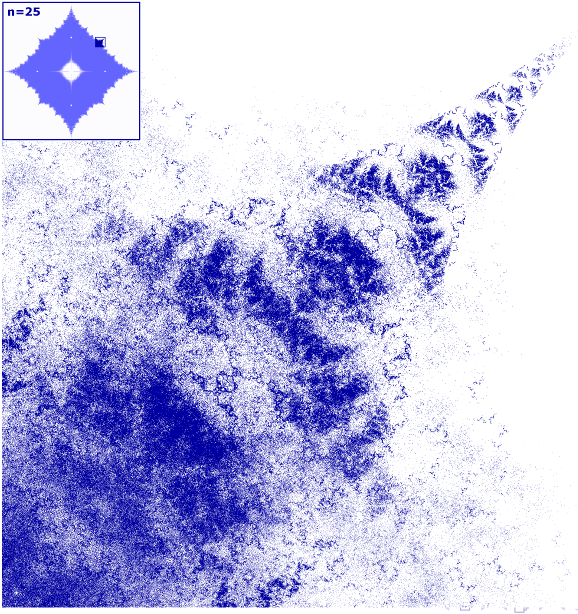


Size 29

Final Example [Feinberg/Zee 1999]

Size 30

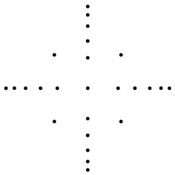
Zoom into Region $1 + i$ of σ_{25}^{0}



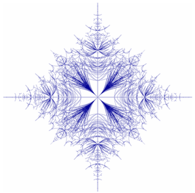
Final Example [Feinberg/Zee 1999]

The **finite** matrix spectra σ_n^0 are even **contained** in the **periodic** (infinite) matrix spectra shown before.

More precisely, the spectra of all $n \times n$ principal submatrices are **contained** in the set of all (2n+2)-periodic matrices:



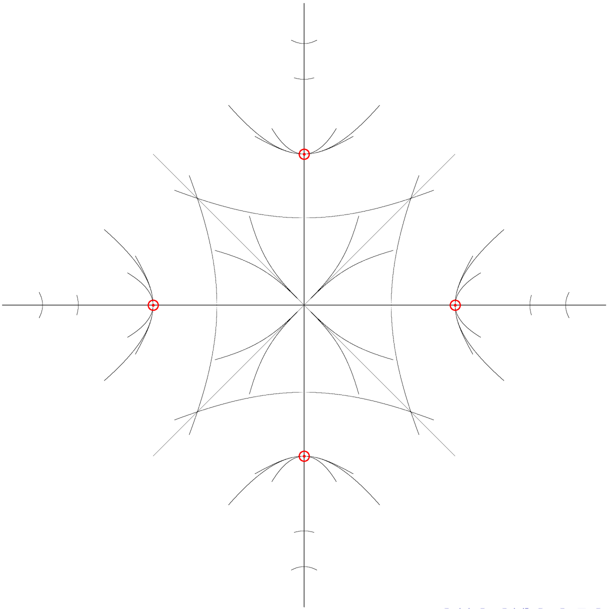
$\subset$



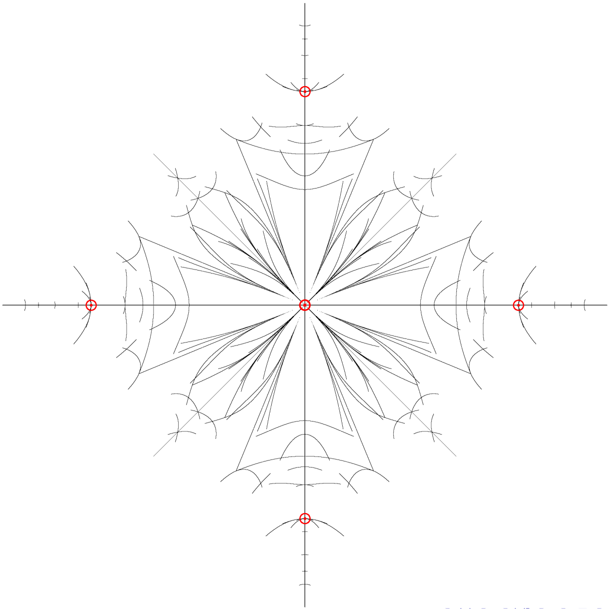
Size $n = 5$ Period $2n + 2 = 12$

Here we demonstrate this inclusion for some values of n.

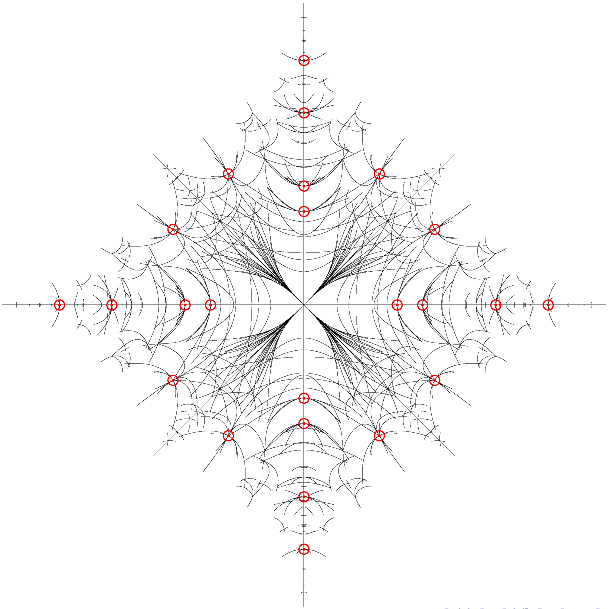
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 2$



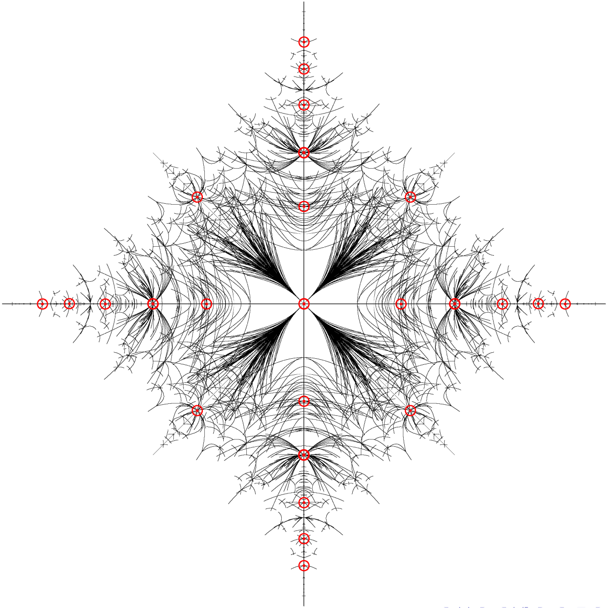
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 3$



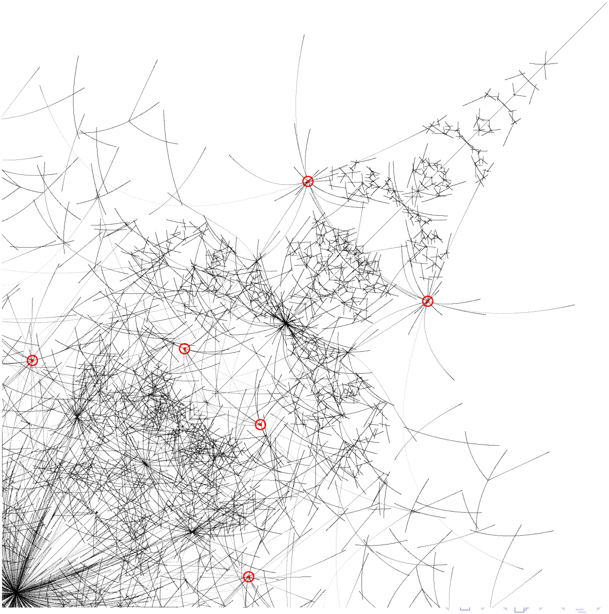
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 4$



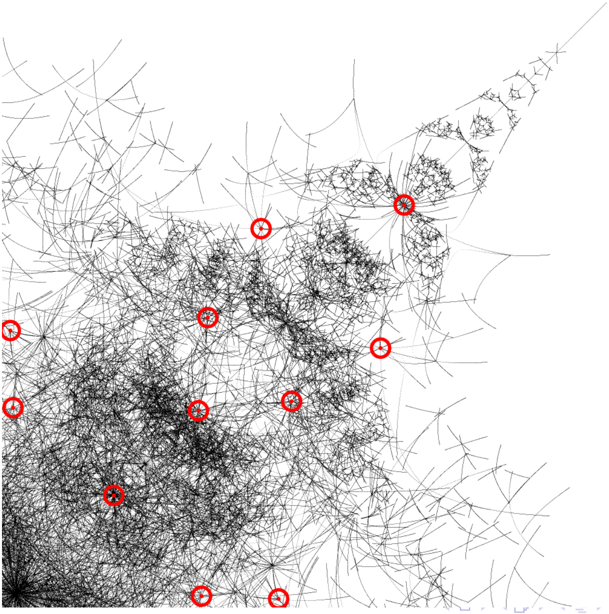
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 5$



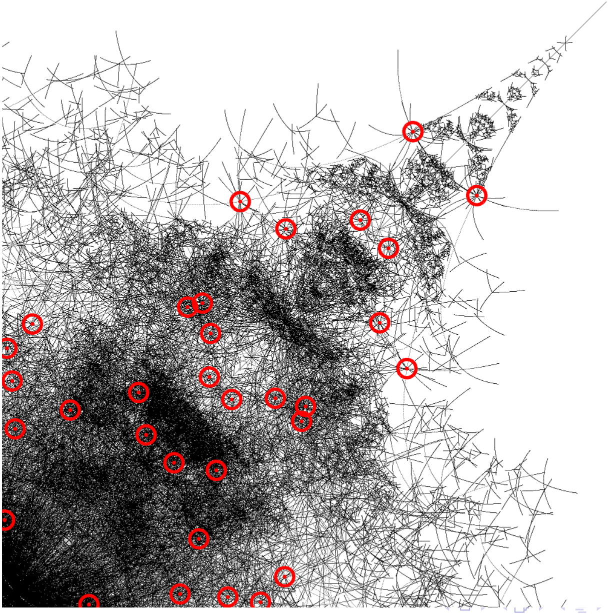
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 8$



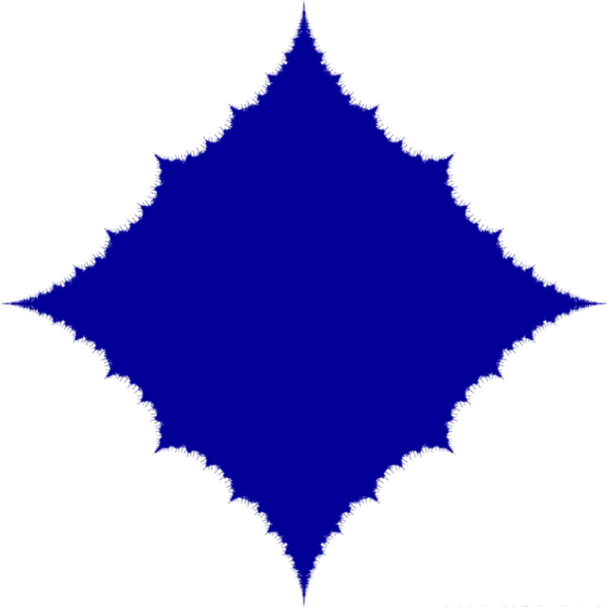
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 9$



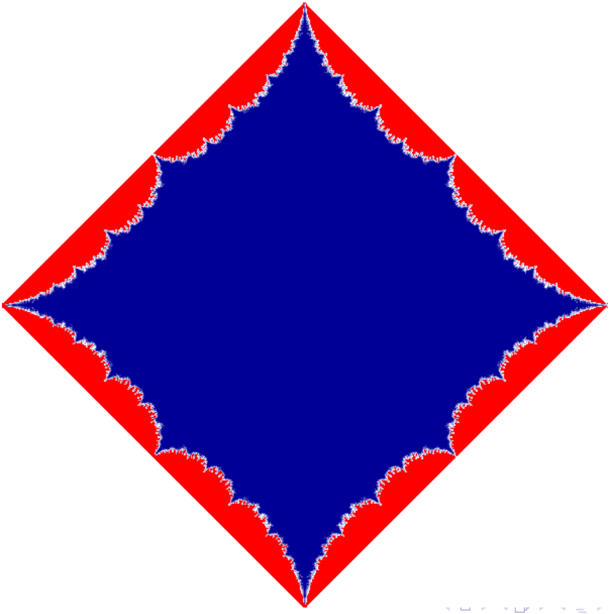
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 10$



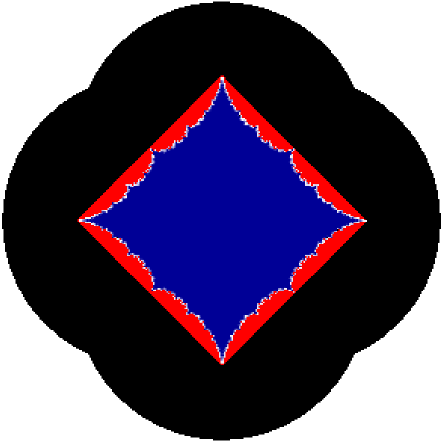
Conjecture: spec A^b if b is pseudoergodic



Upper bound on spec A^b by the closed numerical range

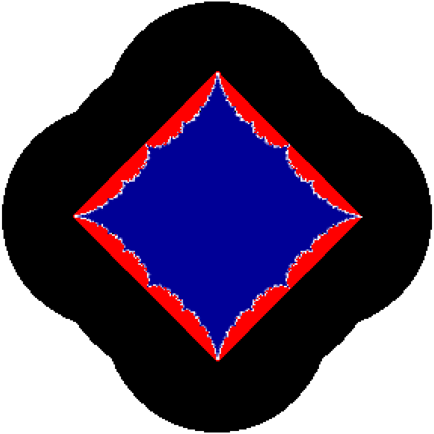


...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 2$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

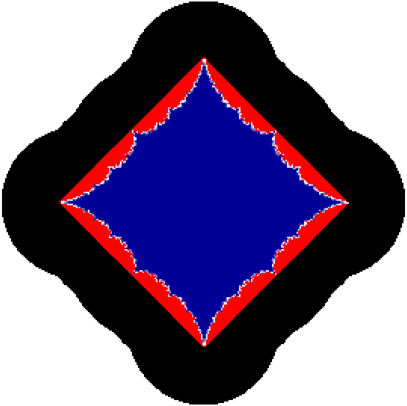


$n = 3$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

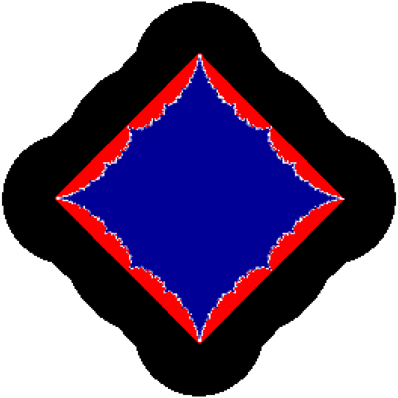
$n = 4$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 5$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 6$

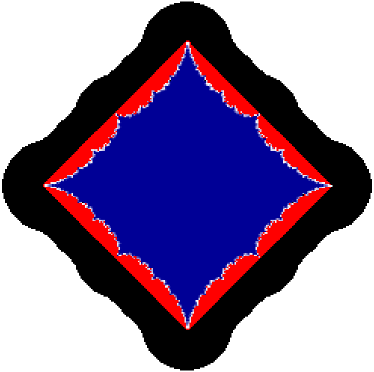
...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 7$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

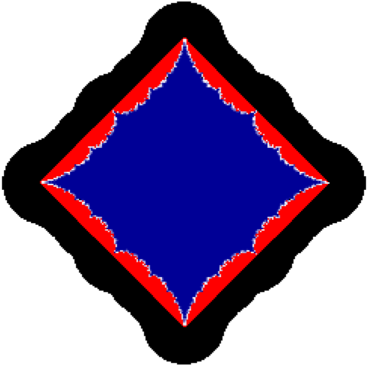
$n = 8$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 9$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 10$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 11$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 12$

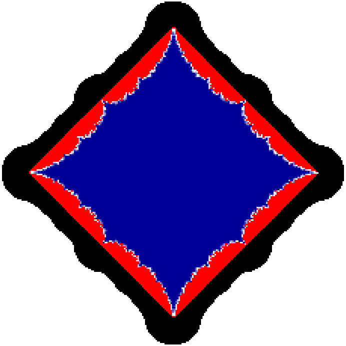
...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 13$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 14$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$



$n = 15$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 16$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 17$

...in comparison with plots of $\Sigma_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 18$

Where does Σ_n^0 go as $n \to \infty$?

Computational cost for these pics: $n \cdot 2^{n-1} \times$ number of pixels.

Where does Σ_n^0 go as $n \to \infty$?

Computational cost for these pics: $n \cdot 2^{n-1} \times$ number of pixels.
So let us focus on just **one** point (pixel) λ:

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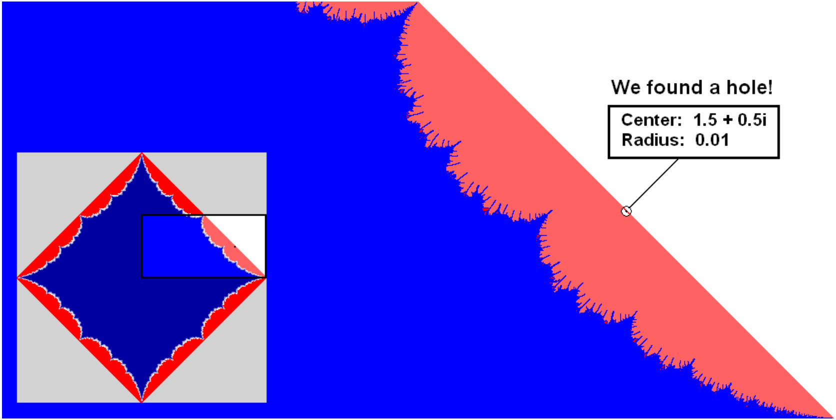
So let us focus on just **one** point (pixel) λ:



$$\lambda = 1.5 + 0.5i \;\notin\; \Sigma_{36}^0 \;\supset\; \operatorname{spec} A^b$$

Extensions

- Everything goes through if $\mathbb{C}$ replaced by $B(X)$
- So everything goes through for matrices with arbitrary bandwidth
- Small perturbations can be handled

S.N. Chandler-Wilde and M. Lindner:
Limit Operators, Collective Compactness & Spectral Theory ...,
Volume 210 (Number 989) of Memoirs of the AMS, 2011.

S.N. Chandler-Wilde and M. Lindner:
Sufficiency of Favard's Condition for a class of ...
Journal of Functional Analysis 2008

C-W, R. Chonchaiya and M. Lindner:
Eigenvalue problem meets Sierpinski triangle ...
Operators and Matrices 2011

S.N. Chandler-Wilde and E.B. Davies:
Spectrum of a Feinberg-Zee random hopping matrix
Journal of Spectral Theory 2012

C-W, R. Chonchaiya and Lindner:
On the spectra and pseudospectra of a class of NSA random ...
Operators and Matrices 2013