Approximation of Bandlimited Functions from Finitely Many Samples

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Bandlimited functions

$$\mathcal{B} = \{f \in L^2(\mathbb{R}^d) : \operatorname{supp} \hat{f} \subseteq [-1/2, 1/2]^d\}$$

Mathematical problem: Recover $f \in \mathcal{B}$ from samples $f(x_j), \in J$.

History: Whittaker-Shannon-Kotelnikov, Paley-Wiener, Duffin-Schaeffer, Beurling, Malliavin, Landau, Pavlov, ...

Facts of life:

- amount of data is finite
- sampling points are non-uniform
- relevant sampling

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Problem

Given samples $f(x_j)$ of $f \in \mathcal{B}$ in region $x_j \in [-M-1, M+1]^d$

Find approximation of f from samples $f(x_j)$.

Alternative Version: Fourier samples
Set $g = \hat{f}$. Recover or approximate g from Fourier samples $\hat{g}(x_j) = f(x_j)$.

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Discrete Model for Numerical Analysis

Discrete sampling problem, use trigonometric polynomials $\mathcal{P}_M$ instead of $\mathcal{B}$

$$\mathcal{P}_M = \{p : p(x) = (2M+1)^{-d/2} \sum_{|k|_\infty \le M} a_k e^{2\pi i k \cdot t/(2M+1)}\}$$

Period $2M+1$, degree M, dimension of $\mathcal{P}_M$ is $(2M+1)^d$.

Observe: If $p \in \mathcal{P}_M$, then

$$\hat{p} = (2M+1)^{-d/2} \sum_{|k|_\infty \le M} a_k \delta_{\frac{k}{2M+1}}$$

$$\text{supp } \hat{p} \subseteq [-1/2, 1/2]^d$$

Solution for Finitely Many Samples

Given samples $f(x_j)$ of $f \in \mathcal{B}$ on region $x_j \in [-M-1, M+1]^d$.
Solve the least square problem

$$p_M = \operatorname{argmin}_{p \in \mathcal{P}_M} \sum_{|x_j|_\infty \leq M+1} |f(x_j) - p(x_j)|^2 w_j$$

Connection to Generalized Sampling of Adcock, Hansen, etc.:

Sampling space $\mathcal{S} = \operatorname{span}\{T_{x_j}\operatorname{sinc} : x_j \in [-M-1, M+1]^d\}$

Reconstruction space $\mathcal{R} = \mathcal{P}_M$ or rather $\mathcal{P}_M \cdot \chi_{[-M-1/2, M+1/2]^d}$

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From Sampling to Linear Algebra

$$y_j = f(x_j) = (2M+1)^{-d/2} \sum_{|k|_\infty \le M} a_k e^{2\pi i k\cdot x_j/(2M+1)}$$

Structure:

1. Sampling matrix V_M with entries

$$(V_M)_{jk} = (2M+1)^{-d/2}\, e^{2\pi i k\cdot x_j/(2M+1)}$$

is *Vandermonde matrix*, and measurements are $\mathbf{y} = V_M \mathbf{a}$.

2. Then $T_M = V_M^* V_M$ is (block) *Toeplitz matrix* with entries

$$(T_M)_{kl} = (2M+1)^{-d} \sum_{x_j \in [-M-1, M+1]^d} e^{2\pi i (k-l)\cdot x_j}$$

3. So

$$\mathbf{a} = (V_M^* V_M)^{-1} V_M^* \mathbf{y} = T_M^{-1} V_M^* \mathbf{y} = V_M^\dagger \mathbf{y}$$

From Sampling to Linear Algebra

$$\sqrt{w_j}y_j = \sqrt{w_j}f(x_j) = (2M+1)^{-d/2}\sqrt{w_j}\sum_{|k|_\infty \le M} a_k e^{2\pi i k\cdot x_j/(2M+1)}$$

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Algorithm

Step 1: Compute $b_M = V_M^* \mathbf{y} \in \mathbb{C}^{(2M+1)^d}$

Step 2: Compute $a_M = T_M^{-1} b_M = V_M^\dagger \mathbf{y} \in \mathbb{C}^{(2M+1)^d}$

Step 3: Compute the trigonometric polynomial $p_M \in \mathcal{P}_M$

$$p_M(x) = \sum_{|k|_\infty \leq M} a_M(k)(2M+1)^{-d/2} e^{2\pi i k \cdot x/(2M+1)}$$

Then

$$p_M = \mathrm{argmin}_{p \in \mathcal{P}_M} \sum_{x_j \in [-M-1, M+1]^d} |p(x_j) - y_j|^2 w_j$$

Remarks

Adaptive weights with Conjugate gradient acceleration and Toeplitz structure (Feichtinger, G., Strohmer)

- Weights are used to improve condition number of T_M (“density compensation factors”)
- NUFFT (nonuniform FFT)
- Operation count: $\mathcal{O}(M^d \log M)$ operations

Fast Toeplitz inversion

From Finite to Infinite Dimension

Q: What does p_M (= output of numerical algorihm) say about f (solution of infinite dimensional problem) ?

Theorem

If

$$\sigma(T_M) \subseteq [\alpha, \beta] \qquad \text{for all } M\,,$$

then

$$\lim_{M\to\infty} \int_{[-M,M]^d} |f(x) - p_M(x)|^2\, dx = 0$$

and $\lim_{M\to\infty} p_M(x) = f(x)$ *uniformly on compact sets for all* $f \in \mathcal{B}$

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Corollary

Sampling inequality for all $f \in \mathcal{B}$

$$\alpha \|f\|_2^2 \leq \sum_{j\in J} |f(x_j)|^2 \leq \beta \|f\|_2^2$$

Accomplishments

- validation of numerical algorithm
- new method for derivation of sampling theorems
- explicit rates of convergence

For $0 < L \leq M$ we have (in dimension 1)

$$\begin{aligned}
\mathrm{err} &= \int_{|x|\leq M+1/2} |f(x) - p_M(x)|^2 \, dx \\
&\leq C(1-\delta)^{-2}\Big(\frac{L^2}{M^3}\|f\|_2^2 + \|f\|_2 \Big(\int_{|x|\geq L} (|f(x)|^2 + |f'(x)|^2)\, dx \Big)^{1/2} \\
&+ \sup_{|x-M-1/2|\leq 1/2} (|f(\pm x)|^2 + |f'(\pm x)|^2) \Big)
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Uniform Condition Numbers

Recall that

$$(T_M)_{kl} = (2M+1)^{-d} \sum_{x_j \in [-M-1, M+1]^d} e^{2\pi i (k-l)\cdot x_j} w_j \qquad |k|_\infty \le M$$

Theorem (Dimension $d = 1$)

Assume that $\sup_{j\in J}(x_{j+1} - x_j) = \delta < 1$ *and choose* $w_j = \frac{1}{2}(x_{j+1} - x_{j-1})$ *as the weights in* T_M. *Then*

$$\sigma(T_M) \subseteq [(1-\delta)^2, 6] \tag{1}$$

REMARK: Similar theorem in higher dimensions with $w_j \approx$ size of Voronoi region at x_j (but: density depends on dimension)

REMARK: Compare to finite section method

Ingredients of the Proof

Let Q_M be orthogonal projection onto sampling space generated by the (orthonormal) basis $\Phi_{k,M}(t) = (2M+1)^{-d/2} e^{2\pi i k\cdot t/(2M+1)}$, thus

$$Q_M f = \sum_{\|k\|_\infty \le M} \langle f, \phi_{k,M}\rangle \phi_{k,M}$$

Step 1. Show that for all $f \in \mathcal{B}$

$$\lim_{M\to\infty} \|f - Q_M f\|_2 = 0\,.$$

Step 2. Show that

$$\lim_{M\to\infty} \sum_{|x_j|_\infty \le M} |f(x_j) - Q_M f(x_j)|^2 w_j = 0\,.$$

Step 3. Show that there exist $\alpha, \beta > 0$ such that

$$\sigma(T_M) \subseteq [\alpha, \beta]$$

(always difficult)

Trends - New Aspects

- Impose additional conditions on $\hat{f}$ and reconstruct with respect to other bases, e.g., wavelet bases (Adcock, Gataric, Hansen; Kutyniok etc.)

- Random sampling (Bass-KG)

- Possible Extensions: model sparsity either by assuming $\operatorname{supp}\hat{f} \subseteq E \subseteq [-1/2, 1/2]^d$ with $\operatorname{meas}(E)$ small or $\operatorname{supp}\hat{f} \subseteq \{\xi \in [-1/2, 1/2]^d : \prod_{k=1}^{d} |\xi_k| \leq \delta << 1\}$

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