The finite section method for dissipative Jacobi matrices and Schrödinger operators

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Schrödinger operators

$$L_0 u = -u'' + q(x)u,$$

where q is real-valued, in limit-point case at infinity, and integrable at 0. We use the domain

$$D(L_0) = \{u \in L^2(0,\infty) \mid -u'' + qu \in L^2(0,\infty),\ u(0) = 0\}$$

so $L_0 = L_0^*$.
The spectrum $\sigma(L_0)$ is any closed, unbounded-above subset of $\mathbb{R}$.
We are interested in the dissipative operator

$$L = L_0 + is(x)\cdot,$$

in which $s \geq 0$, $s \in L^1(0,\infty) \cap L^\infty(0,\infty)$.

Jacobi matrices

We start with $J_0 = J_0^*$ given formally by

$$J_0 = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & \cdots \end{pmatrix}.$$

(a_n) is a sequence of non-zero reals and (b_n) is a real sequence. We are interested in the spectrum of

$$J = J_0 + i\mathrm{diag}(s_1, s_2, \ldots),$$

in which $s = (s_j)_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$.

Finite section method for Schrödinger operators

Replace L by an operator in $L^2(0, M)$, $M \gg 1$:

$$L^M u = -u'' + (q + is)u,$$

$$D(L^M) = \{u \in L^2(0, M) \mid -u'' + qu \in L^2(0, M),\ u(0) = 0 = u(M)\}.$$

How does $\sigma(L^M)$ approximate $\sigma(L)$?

Note that the problem is very well studied in the self-adjoint case.

Finite section method for Jacobi matrices

We replace $J = J_0 + is$ by its leading $M \times M$ sub-matrix J^M and ask, how does $\sigma(J^M)$ approximate $\sigma(J)$, for $M \gg 1$?

Again, the question is well studied in the self-adjoint case and also in some non-selfadjoint cases (e.g. Lindner 2006, Davies and Chandler-Wilde 2012, Chandler-Wilde and Lindner 2013).

Some known results

- In the self-adjoint case, finite section method gives spectral pollution in spectral gaps.
- There is at most one point of spectral pollution in each gap for 1D Schrödinger and Jacobi operators.
- If finitely supported functions/vectors form a core then all points of $\sigma(L_0)$ and $\sigma(J_0)$ will be approximated: both isolated eigenvalues and essential spectrum - BEWZ (1992), Stolz-Weidmann (1994).
- For the dissipative case, if finitely supported functions/vectors form a core then isolated eigenvalues can be approximated - e.g. M (2009), M& Scheichl (2013) for Schrödinger, Strauss (2013) for Jacobi; also Descloux (1981), Pokrzywa (1980).
- For Feinberg-Zee model, finite section always works (Chandler-Wilde and Davies, 2012).
- For pseudo-ergodic Jacobi operators, finite section always works (Chandler-Wilde and Lindner, announced 2013).
- For random Jacobi operators, finite section works with probability 1.

Main results

Theorem (M & Naboko, 2013)

Suppose that $s \in \ell^1(\mathbb{N})$ and $s_j \geq 0$ for all j. Suppose that $\lambda_{\rm ess}$ is a point of essential spectrum of $J = J_0 + is$. Then every open neighbourhood of $\lambda_{\rm ess}$ in $\mathbb{C}$ contains eigenvalues of the leading $M \times M$ submatrix of J, for all sufficiently large M.

Theorem (M & Naboko, 2013)

Suppose $L_0 = L_0^*$, that $\min(q, 0) \in L^\infty(0, \infty)$, $q \in L^2_{loc}$, $s \in L^1(0, \infty) \cap L^\infty(0, \infty)$, $s \geq 0$ and $s(x) \to 0$ as $x \to \infty$. Suppose that $\lambda_{\rm ess}$ is a point of essential spectrum of $L = L_0 + is$. Then every open neighbourhood of $\lambda_{\rm ess}$ in $\mathbb{C}$ contains eigenvalues of the finite-interval operators L^M, for all sufficiently large M.

Further results: error estimates

Theorem (M+Naboko, 2013)

Suppose that $\mathcal{K}$ is any compact set in $\mathbb{C}$. Then

$$\left| \sum_{\lambda_j(J^M)\in\mathcal{K}} \{\lambda_j(J^M)\}^n - \sum_{\lambda_j(J_0^M)\in\mathcal{K}} \{\lambda_j(J_0^M)\}^n \right| \leq C_n, \quad n = 0, 1, 2,$$

where the C_n do not depend on the truncation index M.

This theorem would give an estimate of the quality of approximation of the essential spectrum in the dissipative case, if we had a stability result for a Hausdorff moment problem. Unfortunately a result of De Giorgi (1986) shows that this cannot generally be expected!

Theorem (M & Scheichl, 2013)

Suppose $L_0 = L_0^*$ *and that* $q \in L^\infty(0,\infty)$ *is eventually periodic. Suppose that* $s \in \cap L^\infty(0,\infty)$, $s \geq 0$ *and* s *is compactly supported in* $[0,\infty)$. *If* λ_{ess} *is an interior point of the essential spectrum of* L *then there exist* $\lambda^M \in \sigma(L^M)$ *such that*

$$|\lambda_{\mathrm{ess}} - \lambda^M| = O(M^{-1}).$$

Theorem (M, 2009)

Suppose that q *is real-valued and limit-point at infinity and that* $(\lambda^M)_{M\in\mathbb{N}}$ *is a* *polluting sequence*, *i.e. a sequence with* $\lambda^M \in \sigma(L^M)$ *having a convergent subsequence whose limit does not lie in* $\sigma(L)$. *If* $s(x) \to 0$ *as* $x \to \infty$ *then*

$$\mathrm{Im}\,(\lambda^M) \to 0, \quad M \to \infty \text{ on the subsequence.}$$

Moreover this result also holds for Schrödinger opertors on exterior domains in $\mathbb{R}^d$. *Note that* $s \in L^1$ *is not required.*

Theorem (M& Scheichl, 2013)

Suppose that q *is eventually periodic, and that* $(\lambda^M)_{M\in\mathbb{N}}$ *is a* *polluting sequence*. *If* s *is compactly supported then*

$$\mathrm{Im}\,(\lambda^M) \leq C\exp(-\alpha M)$$

on the subsequence, where $C, \alpha > 0$ *depend on* L *but not on* M.

Proof of Theorem 1

$$J^M = \begin{pmatrix} b_1 + is_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 + is_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 + is_3 & a_3 & 0 & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & a_{M-1} \\ . & . & . & . & a_{M-1} & b_M + is_M \end{pmatrix}$$

Standard calculations show that

$$\lambda \in \sigma(J^M) \iff m_M(\lambda) = f(\lambda),$$

where

$$f(\lambda) = \frac{1}{a_1 a_2}(b_1 + is_1 - \lambda), \quad m_M(\lambda) = \langle (J^M_{red} - \lambda I)^{-1} \mathbf{e}_1, \mathbf{e}_1 \rangle;$$

here

$$J^M_{red} = \begin{pmatrix} b_2 + is_2 & a_2 & 0 & 0 & 0 & \cdots \\ a_2 & b_3 + is_3 & a_3 & 0 & 0 & \cdots \\ 0 & a_3 & b_4 + is_4 & a_3 & 0 & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & a_{M-1} \\ . & . & . & . & a_{M-1} & b_M + is_M \end{pmatrix}$$

Assume $s_1 > 0$ for simplicity. Observe that

$$\left.\begin{array}{c} \operatorname{Im} m_M(\lambda) \leq 0 \\ \operatorname{Im} f(\lambda) \geq s_1/(a_1 a_2) > 0 \end{array}\right\} \quad \operatorname{Im} \lambda \leq 0,$$

so

$$\frac{1}{|m_M(\lambda) - f(\lambda)|} \leq \frac{a_1 a_2}{s_1}, \quad \operatorname{Im} \lambda \leq 0.$$

Let $\lambda_k + i\mu_k$, $k = 1, \ldots, M$ be the eigenvalues of J^M and consider the function

$$\mathcal{F}_M(\lambda) = \frac{B_M(\lambda)}{m_M(\lambda) - f(\lambda)},$$

where B_M is the Blaschke factor

$$B_M(\lambda) = \prod_k \left(1 - \frac{2i\mu_k}{\lambda - \lambda_k + i\mu_k}\right).$$

- The $\mathcal{F}_M$ are holomorphic in $\mathbb{C}^+ \cup \mathbb{R}$ and bounded by their maximum moduli on $\mathbb{R}$.
- The μ_k admit an M-independent trace bound

$$\sum_{k=1}^{M} \mu_k = \sum_{k=1}^{M} s_k \leq \|s\|_{\ell^1}.$$

- If no J^M has eigenvalues in a neighbourhood $\mathcal{U}$ of $\lambda_{\mathrm{ess}} \in \mathbb{R}$ then for $\lambda \in \mathcal{U} \cup \mathbb{C}^+$,

$$\exp(-2\|s\|_{\ell^1} C) \leq |B_M(\lambda)| \leq \exp(2\|s\|_{\ell^1} C), \quad C > 0 \text{ indep. of } M,$$

  so the B_M and $1/B_M$ form normal families on $\mathcal{U}$.
- Hence the $\mathcal{F}_M$ form a normal family in $\mathcal{U} \cap \overline{\mathbb{C}^+}$ with bound $\exp(2\|s\|_{\ell^1} C) s_1/(a_1 a_2)$ indep. of M, attained on $\mathbb{R}$.
- Also $\frac{1}{|m_M(\lambda) - f(\lambda)|} \leq \frac{a_1 a_2}{s_1}$ for $\mathrm{Im}\,\lambda \leq 0$ and so the $\mathcal{F}_M$ form a normal family on all of $\mathcal{U}$.

- Hence the $1/(m_M(\lambda) - f(\lambda))$ form a normal family on $\mathcal{U}$.
- Titchmarsh-Weyl nesting circle analysis shows

$$\lim_{M\to\infty} \left\{ \frac{1}{m_M(\lambda) - f(\lambda)} \right\} = \frac{1}{m(\lambda) - f(\lambda)}, \quad \lambda \in \mathcal{U},$$

where

$m(\lambda) =$ Titchmarsh-Weyl coefficient for the Jacobi operator J.

- Hence $1/(m(\lambda) - f(\lambda))$ is holomorphic in $\mathcal{U}$.
- Hence $m(\lambda)$ is meromorphic in $\mathcal{U}$ and does not see the essential spectrum at $\lambda_{\rm ess}$ of J. This is impossible. □

Remark

We use the fact that $s \in \ell^1$ implies $s_k \longrightarrow 0$ together with a Glazman decomposition trick to prove that the Titchmarsh-Weyl convergence analysis holds off the real axis, away from eigenvalues. This slightly expands the set in which the convergence is proved by Gesztesy and Clark (2004).

Remark

In the Schrödinger case, we replace the M-independent bound $\sum_{k=1}^{M} s_k \le \|s\|_{\ell^1(\mathbb{N})}$ by a Hilbert-Schmidt bound

$$\|\sqrt{s}(L_0^M + \delta I)^{-1}\|_2 \le C.$$

Now

$$\|\sqrt{s}(L_0^M + \delta I)^{-1}\|_2^2 = \sum_{k=1}^{\infty} \int_0^M \frac{s(x)(\phi_k^M(x))^2 dx}{\left(\lambda_k^M + \delta\right)^2}$$

$$\le \sum_{k=1}^{\infty} \int_0^M \frac{s(x)(\phi_k^M(x))^2 dx}{\lambda_k^M + \delta} = \int_0^M s(x) G_M(x,x) dx,$$

where G_M is the kernel of the resolvent $(L_0^M + \delta I)^{-1}$. Some results of Chernyavskaya and Shuster (1994) on $G(x,x)$ can be adapted to give bounds on $G_M(x,x)$:

$$\frac{1}{4} + o(M^{-1}) \le G_M(x,x) \le \frac{3}{2} + o(M^{-1}).$$

Dissipative problems vs. self-adoint: spectral pollution

Example (Perturbed Schrödinger in $\mathbb{R}^2$; Boulton & Levitin (2007))

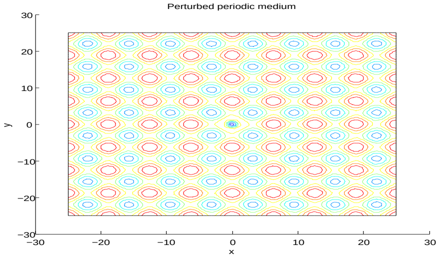
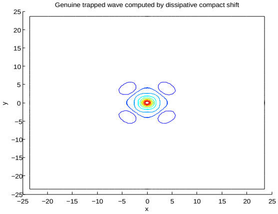


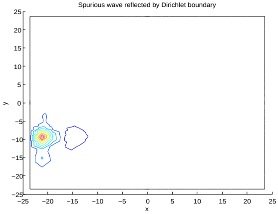
Figure: Contour plot of $q(x, y) = \cos(x) + \cos(y) - 5e^{-x^2-y^2}$.

Add dissipation:

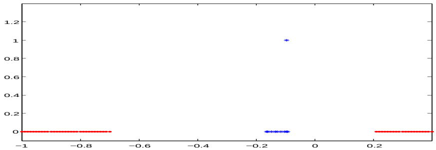
$$q(x, y) \mapsto q(x, y) + i * \frac{1}{4} * (1 - \tanh(|x| - 30))(1 - \tanh(|y| - 30))$$



Genuine trapped wave computed by dissipative compact shift
x
y



Spurious wave reflected by Dirichlet boundary
x
y



Example (Spectral pollution in 1D)

$$-u'' + \left(\sin(x) - \frac{40}{1+x^2} + is(x)\right) u = \lambda u, \quad x \in (0,\infty);$$

$$\cos(\pi/8)y(0) + \sin(\pi/8)y'(0) = 0;$$

here $s(\cdot)$ is the function

$$s(x) = \begin{cases} 1 & (x < 50) \\ 0 & (x \geq 50). \end{cases}$$

Essential spectrum has band-gap structure; first three bands are

$$I_1 = [-0.3785, -0.3477]; \quad I_2 = [0.5948, 0.9181];$$

$$I_3 = [1.2932, 2.2852].$$

Numerics show spurious eigenvalue in one of the spectral gaps:

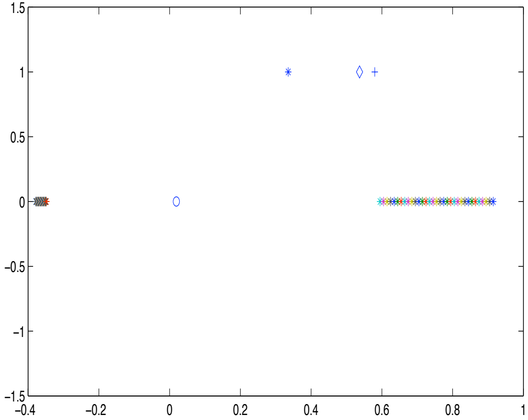


Figure: Numerical results for $M = 100$.

The abstract dissipative barrier method.

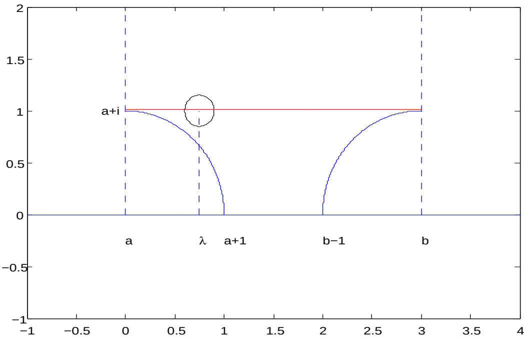
Suppose L is selfadjoint and that Q is a projection which is $|L|^{1/2}$-compact.

Theorem (Strauss, JST to appear)

- *If* $(L + iQ - zI)u = 0$ *and* $\delta = \sqrt{\mathrm{Im}\,(z)(1 - \mathrm{Im}\,(z))}$ *then*

$$[\mathrm{Re}\,(z) - \delta, \mathrm{Re}\,(z) + \delta] \cap \sigma(L) \neq \emptyset.$$

- *Suppose* $\sigma(L) \cap (a, b) = \{\lambda\}$ *where* λ *is an eigenvalue of* L *of multiplicity* d. *Suppose* $\|(I - Q)E(\{\lambda\})\| \leq \varepsilon$. *Then* $L + iQ$ *has* d *eigenvalues* ε*-close to* $\lambda + i$.
- *The other eigenvalues of* $L + iQ$ *are separated from* $\lambda + i$ *and lie in discs* $D(a, 1)$ *and* $D(b, 1)$.
- *When using a projection method, pollution for* $L + iQ$ *occurs in the same sets as for* L.


2
1.5
1
0.5
0
−0.5
−1
a+i
a
λ
a+1
b−1
b
−1
−0.5
0
0.5
1
1.5
2
2.5
3
3.5
4

Example

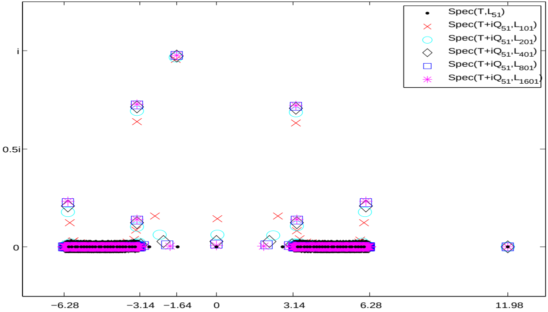
In $H = L^2[-\pi, \pi]$ we consider the multiplication operator given by

$$(Lu)(x) = a(x)u(x) + 10 \int_{-\pi}^{\pi} u(s)ds,$$

where

$$a(x) = \begin{cases} -2\pi - x, & -\pi \le x \le 0, \\ 2\pi - x, & 0 < x \le 2\pi. \end{cases}$$

For the operator Q we use the projection onto a set of Galerkin eigenfunctions whose eigenvalues lie in $(-\pi, \pi)$. (We use a smaller set than the basis used to find the eigenvalues of $L + iQ$.)


Spec(T,L51)
Spec(T+iQ51,L101)
Spec(T+iQ51,L201)
Spec(T+iQ51,L401)
Spec(T+iQ51,L801)
Spec(T+iQ51,L1601)
i
0.5i
0
−6.28
−3.14
−1.64
0
3.14
6.28
11.98

Example: Magnetohydrodynamics Operator

Example

On $\mathcal{H} = (L^2(0,1))^3$, consider the operator $T =$

$$\begin{pmatrix} -\frac{d}{dx}(v_a^2 + v_s^2)\frac{d}{dx} + k^2 v_a^2 & -i(\frac{d}{dx}(v_a^2 + v_s^2) - 1)k_\perp & -i(\frac{d}{dx}v_s^2 - 1)k_\parallel \\ -ik_\perp((v_a^2 + v_s^2)\frac{d}{dx} + 1) & k^2 v_a^2 + k_\perp^2 v_s^2 & k_\perp k_\parallel v_s^2 \\ -ik_\parallel(v_s^2 \frac{d}{dx} + 1) & k_\perp k_\parallel v_s^2 & k_\parallel^2 v_s^2 \end{pmatrix}$$

We have

$$\mathrm{Spec}_{\mathrm{ess}}(T) = \mathrm{Range}(v_a^2 k_\parallel) \ \cup \ \mathrm{Range}\left(\frac{v_a^2 v_s^2 k_\perp}{v_a^2 + v_s^2}\right);$$

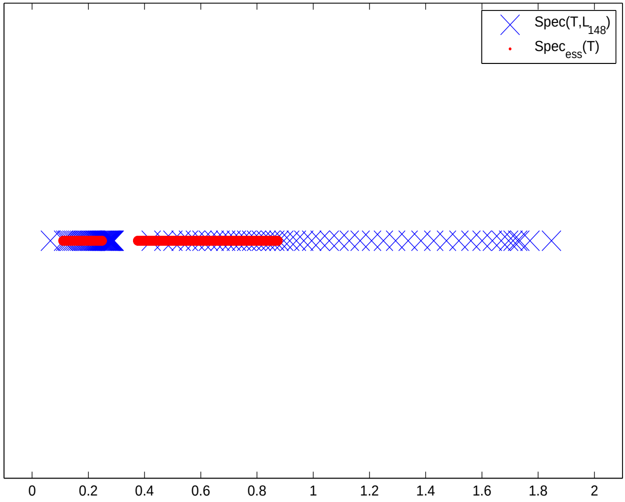
There is an eigenvalue $\lambda \approx 0.27917$ in the gap.

Coefficients: $\rho_0 = 1$, $k_\perp = 1$, $k_\| = 1$, $g = 1$,

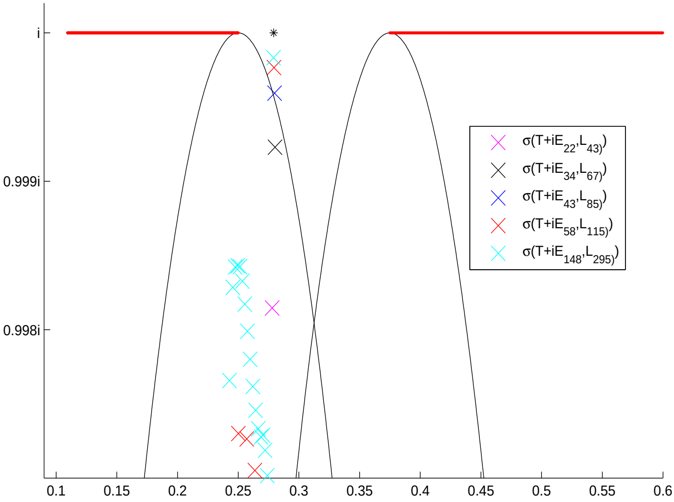
$$v_a(x) = \sqrt{7/8 - x/2}, \quad v_s(x) = \sqrt{1/2 + x/2};$$

essential spectrum:

$$\sigma_{\mathrm{ess}} = \left[\frac{7}{64}, \frac{1}{4}\right] \cup \left[\frac{3}{8}, \frac{7}{8}\right].$$



Spec(T,L148)
Specess(T)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



i
0.999i
0.998i
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
σ(T+iE22,L43))
σ(T+iE34,L67))
σ(T+iE43,L85))
σ(T+iE58,L115))
σ(T+iE148,L295))