Computability concepts in spectral computations

Olavi Nevanlinna
Aalto SCI

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- Is $\sigma(A)$ computable for $A \in B(\ell_2(\mathbb{N}))$

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- J. Ben-Artzi, A. Hansen, O. Nevanlinna , M. Seidel

Matrices first

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- $n \leq 3$: generally convergent rational iteration exists (McMullen 1987)
- $n \leq 5$: a tower of generally convergent rational iterations (Doyle, McMullen 1989)
- $n > 5$: no such towers (Doyle, McMullen 1989)

Matrices continues

radicals, $z \mapsto |z|$ available, then convergent iterations exist for solving roots of polynomials

input finite: the complex coefficients of the polynomial

Computabilities...

"Turing view": problem computable if a computing device exists which solves the problem

Computation in the limit and higher hierarchies

BSS (Blum, Shub, Smale) $\mathbb{R}$-machine model

IBC (infromation based complexity) uses BSS, "tractability"

constructivism, computability on $\mathbb{Z}$ and within computable numbers

Any compact can be spectrum

Represent compact $K \subset \mathbb{C}$ from outside:

$$K = \bigcap K_n$$

where

$$\cdots \subset K_{n+1} \subset K_n \subset \cdots$$

and testing $z \notin K_n$ "easy"

Any compact can be spectrum, so look at Julia sets

We first look at the Julia set $\mathcal{J}$ for the quadratic polynomial $z^2 + 4$.

Consider the question

$$z \in \mathcal{J} \quad ?$$

Then the corresponding question for the spectrum $\sigma(A)$ is

$$\lambda \in \sigma(A) \quad ?$$

The natural formulation of these questions is, can you decide whether the answer is yes or no?

2.1 Julia set $\mathcal{J}$ for $z^2 + 4$

Let

$$p(z) = z^2 + 4$$

Iterate

$$z_{n+1} = p(z_n)$$

If $z_n \to \infty$ then $z_0 \notin \mathcal{J}$.
Note that if $|z_k| > 1 + \sqrt{5}$ for some k, then $|z_{k+1}| > 2|z_k|$ and then $z_n \to \infty$.
For this $p(z)$ the Julia set is homeomorphic to a Cantor set.
Observe that $\mathbb{C} \setminus \mathcal{J}$ is open.

S. Smale and coworkers: $\mathcal{J}$ is not decidable
("semidecidable")

Computation in the limit...

Output as follows:

if $|z_k| \leq 1 + \sqrt{5}$, then $Out(k) = 1$
if $|z_k| > 1 + \sqrt{5}$, then $Out(k) = 0$.

So depending on the initial value we obtain sequences of the form

$$1, 1, \ldots, 1, 0, 0, 0 \ldots$$

and

$$1, 1, 1, \ldots$$

In either case the limit exists; and then you (would) know

Similar question for the spectrum in abstract Banach algebra

Consider the subalgebra generated by just one element a (say, in Banach algebra $\mathcal{A}$). Then the spectrum within the subalgebra is $\mathit{fill}(\sigma(a))$.
If we are allowed to produce polynomials of a and compute their norms but inverting is not allowed, then:

The question

$$\lambda \notin \mathit{fill}(\sigma(a))$$

is semidecidable as follows:

If answer positive: finite termination with sure answer, while

if negative, you will never know (the one you look after does not exist)

What exists is easier to find!

Conclude: Think positive, construct the resolvent

$$\mathbb{C} \setminus \mathit{fill}(\sigma(A)) \to B(X)$$

$$\lambda \mapsto (\lambda - A)^{-1}$$

instead!

Get a multicentric holomorphic calculus - but not during this talk...

Computation in the limit

Example

Let A be diagonal operator in $\ell_2(\mathbb{N})$ such that $a_{ii} \in \{0,1\}$.
Input information: read one diagonal element in time, in a fixed enumeration.
Then

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- $1 \in \sigma(A)$: this cannot be be computed except at the limit
- $1 \in \sigma_{ess}(A)$ this needs "two limits", i.e. a "tower"

How to get the answers

$1 \in \sigma(A)$

- define function for each n

$$\Gamma_n(A) = 1, \text{ if } \sum_{i=1}^{n} a_{ii} > 0,$$
$$0, \text{ otherwise}$$

and set

$$\Gamma(A) = \lim_{n \to \infty} \Gamma_n(A).$$

Then, answer is "yes", when $\Gamma(A) = 1$

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- Using quantifiers: $\exists n \left(\sum_{i=1}^{n} a_{ii} > 0\right)$

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$1 \in \sigma_{ess}(A)$

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$$\Gamma_{m,n}(A) = 1, \text{ if } \sum_{i=1}^{n} a_{ii} > m,$$

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- With two quantifiers: $\forall m\ \exists n\ (\sum_{i=1}^{n} a_{ii} > m)$

Another example

We define $A \in B(\ell_2(\mathbb{N}))$ using diagonal blocks:

$$A = \bigoplus_{j=1}^{\infty} A_{k(j)}$$

where A_k are $k \times k$-matrices with number 1's in the corners, like

$$A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and $k(j) \geq 2$ is some sequence. Thus, A is algebraic, $\sigma(A) = \sigma_{ess}(A) = \{0, 2\}$.

Constructivism, computability

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- then one can "tailor" a computing machine which computes the spectrum in a finite number of operations

Constructivism, computability 2

- The operator

$$B = \bigoplus_{j=1}^{\infty} \beta_j A_{k(j)}$$

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- Then,

- the spectrum is computable.

Constructivism, computability 3

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- In this theory effectively described bounded self-adjoint operators have computable spectra
- but
- there exists an effectively determined bounded non-selfadjoint operator which has a noncomputable real as an eigenvalue.

Computability; towers

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- can be adaptive but only based on what it has already computed
- input enters by reading one element a_{ij} at a time

Example

Then for each such fixed algorithm one can "tailor" a sequence $\{k(j)\}$ such that the algorithm keeps the number 1 as a candidate for the spectrum for the operator

$$A = \bigoplus_{j=1}^{\infty} A_{k(j)}$$

Example continues

In fact, the algorithm would be made to see a finite matrix consisting of diagonal blocks $A_{k(j)}$ and a block having just one nonzero element

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{pmatrix}$$

Thus,

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Thus,

- just one limit would give wrong answer
- but limits on two levels work

Idea of a tower for the example

Let $A = A^* \in B(\ell_2(\mathbb{N}))$ and denote by $\gamma_{m,n}(t)$ the smallest singular value of the $n \times m$- matrix $A_{nm}(t)$ representing

$$P_n(A - tI)$$

when restricted to the range of P_m: $P_m\ell_2(\mathbb{N})$.

Example continues

Applied to

$$A = \bigoplus_{j=1}^{\infty} A_{k(j)}$$

the matrices $A_{nm}(t)$ shall consist of a finite number of square blocks and possibly one rectangle which for fixed m and all large enough n is of the form

$$\begin{pmatrix} 1-t & 0 & 0 & \cdot \\ 0 & -t & 0 & \cdot \\ \cdot & \cdot & -t & \cdot \\ \cdot & & & \\ \boxed{1} & & & \\ 0 & & & \\ \cdot & & & \end{pmatrix}$$

Proto for the tower at the Example

Since $\boxed{1}$ appears, the rectangle has full rank at $t = 1$.

- For example

$$\begin{pmatrix} 1-t & 0 & \boxed{1} \\ 0 & -t & 0 \end{pmatrix} \begin{pmatrix} 1-t & 0 \\ 0 & -t \\ \boxed{1} & 0 \end{pmatrix} = \begin{pmatrix} (1-t)^2 + \boxed{1} & 0 \\ 0 & t^2 \end{pmatrix}$$

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- Denote $\Gamma_{m,n}(A) = \{t \in \mathbb{R} \ : \gamma(t) = 0\}$. Then we have with two quantifiers

$$\forall m \, \exists n_m \, \{n > n_m \implies \Gamma_{m,n}(A) = \{0, 2\}\}$$

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- $\Gamma(A) = \lim_{m\to\infty} \Gamma_m(A) = \{0, 2\} = \sigma(A)$.

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- Limits in the Hausdorff distance between compact sets in $\mathbb{C}$

$$\mathrm{dist}_H(K, M) = \max\{\sup_{z\in K} \inf_{w\in M} |z-w|, \sup_{w\in M} \inf_{z\in K} |z-w|\}$$

Index $= 3$ for bounded operators

- a tower of height 3 works for all $A \in \mathcal{B}(\ell_2(\mathbb{N}))$

Index $= 3$ for bounded operators

- a tower of height 3 works for all $A \in \mathcal{B}(\ell_2(\mathbb{N}))$
- there is an example which shows that three limits are needed in general

References

BSS-model and Julia sets
Blum, Cucker, Shub, Smale

Computability in Analysis Marian B.Pour-el, J.Ian Richards, Computability in Analysis and Physics, Springer 1989
[Second Main Theorem, p 128 and Theorem 5 (Noncomputable eigenvalues, p 132)]