Weighted Hilbert Spaces and Integration of Functions of Infinitely Many Variables

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I. Introduction

Given a probability measure $\rho$ on a set $D$, consider the product measure

$$\mu = \rho^\otimes N$$

on the sequence space $D^N$. Compute

$$I(f) = \int_{D^N} f(x) \mu(dx)$$

for functions $f : D^N \to \mathbb{R}$. 
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on the sequence space \( D^\mathbb{N} \). Compute

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for functions \( f : D^\mathbb{N} \to \mathbb{R} \).

Examples

\( \rho \) uniform distribution on \( D = [0, 1] \),
\( \rho \) standard normal distribution on \( D = \mathbb{R} \).
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for functions $f : D^\mathbb{N} \to \mathbb{R}$.

Motivation  High-dimensional integration

$$I_s(f) = \int_{D^s} f(x_1, \ldots, x_s) (\rho \otimes \cdots \otimes \rho)(dx_1, \ldots, x_s)$$

is well studied. Here, the limit $s \to \infty$. 
Motivation  Random element (stochastic process, random field) $X$ with a representation

$$X = \Gamma(\xi_1, \xi_2, \ldots)$$

based on iid random variables $\xi_j$. Let $\rho$ denote the distribution of $\xi_1$. Then $\mu$ is the joint distribution of $\xi_1, \xi_2, \ldots$. 
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**Examples**

- Series expansion $X = \sum_{j=1}^{\infty} \xi_j \cdot e_j$ with deterministic functions $e_j$.
- SDEs, SPDEs, . . . . Here $\Gamma$ is nonlinear, in general.
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Note

$$\mathbb{E}(\varphi(X)) = I(f)$$

with

$$f(x) = \varphi(\Gamma(x_1, x_2, \ldots)).$$
OUTLINE

II. The Function Classes

III. Algorithms, Error, and Cost

IV. Two Particular Results

V. Embeddings
II. The Function Classes

A particular case

- $\rho$ is the uniform distribution on $D = [0, 1]$,
- $F_\gamma$ consists of functions $f: [0, 1]^N \to \mathbb{R}$ with smooth ANOVA terms in certain weighted Hilbert spaces.

Weights: $\gamma = (\gamma_j)_{j \in \mathbb{N}}$ with

$$\gamma_1 \geq \gamma_2 \geq \cdots > 0$$

and

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$
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Notation: For $u \subseteq \mathbb{N}$ and $x \in D^\mathbb{N}$ let $x_u = (x_j)_{j \in u}$ and $\gamma_u = \prod_{j \in u} \gamma_j$. 
ANOVA decomposition of \( f : [0, 1]^\mathbb{N} \rightarrow \mathbb{R} \). Recursively, for finite sets \( u \subset \mathbb{N} \),

\[
f_u(x_u) = \int_{[0,1]^{\mathbb{N}\setminus u}} f(x) \, dx_{\mathbb{N}\setminus u} - \sum_{v \subset u} f_v(x_v).
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By definition, $f \in F_\gamma$ iff

$$\|f\|_\gamma^2 := \sum_u \gamma_u^{-1} \cdot \|f^{(u)}\|_{L^2([0,1]^u)}^2 \leq 1.$$

Orthogonal decomposition $f = \sum_u f_u$. 
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Example For

$$f(x) = \sum_{j=1}^{\infty} \eta_j \cdot x^2_j$$

we have

$$\|f\|_\gamma < \infty \Leftrightarrow \sum_{j=1}^{\infty} \frac{\eta^2_j}{\gamma_j} < \infty.$$
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ANOVA decomposition of \( f : [0, 1]^N \rightarrow \mathbb{R} \). Recursively, for finite sets \( u \subset \mathbb{N} \),

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- \( \gamma_u \downarrow 0 \): Sloan, Woźniakowski (1998), \( \ldots \), Novak, Woźniakowski (2008), \( \ldots \).
The general setting  Assume that 

(A1) $k \neq 0$ reproducing kernel on $D \times D$ with $D \neq \emptyset$, 

(A2) $H(k) \cap H(1) = \{0\}$, 

(A3) $(\gamma_u)_u$ family of weights $\gamma_u \geq 0$ such that, with $m = \inf_{x \in D} k(x, x)$,

$$\sum_u \gamma_u \cdot m^{|u|} < \infty.$$
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Consider the reproducing kernel

$$K_\gamma(x, y) = \sum_u \gamma_u \prod_{j \in u} k(x_j, y_j)$$

on $X \times X$, where $X = \{x \in D^\mathbb{N} : \sum_u \gamma_u \prod_{j \in u} k(x_j, x_j) < \infty\}$. 
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Finally, let \( F_\gamma \) denote the unit ball in \( H(K_\gamma) \).
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The ANOVA-case: $D = [0, 1]$, $\gamma_u = \prod_{j \in u} \gamma_j$ with $\sum_j \gamma_j < \infty$,

$$k(x, y) = 1/3 + (x^2 + y^2)/2 - \max(x, y).$$
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III. Algorithms, Error, and Cost

The naive Monte Carlo approach

- Truncation, i.e., fix $c \in D$ and consider

$$I^{(s)}(f) = \int_{D^s} f(x_1, \ldots, x_s, c, \ldots) (\rho \otimes \cdots \otimes \rho)(d(x_1, \ldots, x_s)).$$

- With $\xi_{i,j}$ being independent and distributed according to $\rho$,

$$Q(f) = \frac{1}{m} \sum_{i=1}^{m} f((\xi_{i,1}, \ldots, \xi_{i,s}, c, \ldots)).$$
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Note: \( Q \) requires \( m \) independent samples in the \( s \)-dimensional subspace

\[
D_c^{(s)} = \{ x \in D^N : x_{s+1} = \cdots = c \}.
\]

In general, randomized algorithms that evaluate \( f \) in the hierarchy of subspaces \( D_c^{(1)} \subset D_c^{(2)} \subset \ldots \).
Here, for simplicity, **randomized quadrature formulas**

\[ Q(f) = \sum_{i=1}^{m} b_i \cdot f(\xi_i) \]

with \( b_i \in \mathbb{R} \) and \( \bigcup_{s=1}^{\infty} D_c^{(s)} \)-valued random variables \( \xi_i \).
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**Error** and **cost** of \( Q \) on a class \( F \) of functions \( f : D^N \rightarrow \mathbb{R} \)

\[
\text{error}^2(Q, F) = \sup_{f \in F} \mathbb{E}|I(f) - Q(f)|^2
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The \(n\)-th **minimal error**

\[
e_n(F) = \inf\{\text{error}(Q, F) : \text{cost}(Q, F) \leq n\}.
\]
IV. Two Particular Results

As before,

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- $F_\gamma$ consists of functions $f : [0, 1]^N \to \mathbb{R}$ with smooth ANOVA terms in weighted Hilbert spaces.
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Questions

- Do we have \( \lim_{n \to \infty} e_n(F_\gamma) = 0? \)

- If so, sharp upper and lower bound on \( e_n(F_\gamma) \)?
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- If so, sharp upper and lower bound on \( e_n(F_\gamma) \)?

Consider

\[
\lambda_\gamma = \sup \{ r > 0 : \sup_{n \in \mathbb{N}} (e_n(F_\gamma) \cdot n^r) < \infty \}
\]

as a substitute for the order of convergence of the minimal errors.
Assume that $\gamma_j = j^{-\alpha}$ with $\alpha > 1$. Then, for randomized algorithms,

$$\lambda_\gamma = \min((\alpha - 1)/2, 3/2).$$

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Rem.

- Almost optimal algorithms: multi-level plus scrambled lattice rules. The starting point is

$$f_L(x) = f_0(x) + \sum_{\ell=1}^{L} (f_\ell(x) - f_{\ell-1}(x)).$$

with $f_\ell(x) = f(x_1, \ldots, x_{2^\ell}, 1/2, \ldots)$. Use scrambled lattice rules for integration of $f_\ell - f_{\ell-1}$. Apply tractability results.
Assume that $\gamma_j = j^{-\alpha}$ with $\alpha > 1$. Then, for randomized algorithms,

$$
\lambda_\gamma = \min\left(\frac{(\alpha - 1)}{2}, \frac{3}{2}\right).
$$

Rem.

- Almost optimal algorithms: multi-level plus scrambled lattice rules.
- Fixed subspace sampling is suboptimal. More precisely,

$$
\lambda_{\gamma}^{\text{fix}} \leq \frac{3 \cdot (\alpha - 1)}{2 \cdot (\alpha + 2)}
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for the order of convergence of the respective minimal errors.

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Question: Performance of deterministic algorithms?

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Proof, Step 1 Result is known for the unit ball in \( H(L_\gamma) \), where

\[
L_\gamma(x, y) = \sum_u \gamma_u \prod_{j \in u} \ell(x_j, y_j) = \prod_{j=1}^{\infty} \left( 1 + \gamma_j \ell(x_j, y_j) \right)
\]

with

\[
\ell(x, y) = \min(x, y).
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Furthermore, with equivalent norms for every \( \gamma > 0 \),

\[
H(1 + \gamma k) = H(1 + \gamma \ell) = W_2^1([0, 1]).
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**Proof, Step 1** Result is known for the unit ball in $H(L_\gamma)$, where

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**Question:** $H(K_\gamma) = H(L_\gamma)$? **Answer:** No, in general.
V. Embeddings

Thm. Hefter, R (2014)

(i)

\[
\sum_{j=1}^{\infty} \gamma_j^{1/2} < \infty \iff H(K_\gamma) = H(L_\gamma).
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V. Embeddings

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(i)

$$\sum_{j=1}^{\infty} \gamma_j^{1/2} < \infty \iff H(K_{\gamma}) = H(L_{\gamma}).$$

(ii) In general, there exist $0 < c' < 1 < \tilde{c}$ such that

$$H(K_{c'\gamma}) \subseteq H(L_{\gamma}) \subseteq H(K_{\tilde{c}\gamma})$$

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Here \(K_{c\gamma}(x, y) = \prod_{j=1}^{\infty} (1 + c\gamma_j k(x_j, y_j))\), and likewise for \(L_{c\gamma}\).

Rem. General result on embeddings between weighted tensor product spaces available.
Summary

- Integration

\[ I(f) = \int_{D^N} f(x) \rho^N(dx) \]

of functions \( f : D^N \rightarrow \mathbb{R} \).

- Function classes: weighted superpositions of tensor product reproducing kernel Hilbert spaces,

\[ K_{\gamma}(x, y) = \sum_{u} \gamma_u \prod_{j \in u} k(x_j, y_j). \]

- Sharp bound on minimal errors for randomized and deterministic algorithms (here: for a specific example).

- New type of embedding results.