Weighted Hilbert Spaces and Integration of Functions of Infinitely Many Variables

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I. Introduction

Given a probability measure ρ on a set D, consider the product measure

$$\mu = \rho^{\otimes \mathbb{N}}$$

on the sequence space $D^{\mathbb{N}}$. **Compute**

$$I(f) = \int_{D^{\mathbb{N}}} f(\mathbf{x})\, \mu(d\mathbf{x})$$

for functions $f : D^{\mathbb{N}} \to \mathbb{R}$.

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Examples

ρ uniform distribution on $D = [0, 1]$,

ρ standard normal distribution on $D = \mathbb{R}$.

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Motivation High-dimensional integration

$$I_s(f) = \int_{D^s} f(x_1, \ldots, x_s) \, (\rho \otimes \cdots \otimes \rho)(d(x_1, \ldots, x_s))$$

is well studied. Here, the limit $s \to \infty$.

Motivation Random element (stochastic process, random field) X with a representation

$$X = \Gamma(\xi_1, \xi_2, \dots)$$

based on iid random variables ξ_j. Let ρ denote the distribution of ξ_1. Then μ is the joint distribution of $\xi_1, \xi_2, \dots$.

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- Series expansion $X = \sum_{j=1}^{\infty} \xi_j \cdot e_j$ with deterministic functions e_j.
- SDEs, SPDEs, ... Here Γ is nonlinear, in general.

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Note

$$\mathrm{E}(\varphi(X)) = I(f)$$

with

$$f(\mathbf{x}) = \varphi\big(\Gamma(x_1, x_2, \dots)\big).$$

OUTLINE

II. The Function Classes

III. Algorithms, Error, and Cost

IV. Two Particular Results

V. Embeddings

II. The Function Classes

A particular case

- ρ is the uniform distribution on $D = [0, 1]$,
- $F_{\boldsymbol{\gamma}}$ consists of functions $f : [0, 1]^{\mathbb{N}} \to \mathbb{R}$ with smooth ANOVA terms in certain weighted Hilbert spaces.

Weights: $\boldsymbol{\gamma} = (\gamma_j)_{j \in \mathbb{N}}$ with

$$\gamma_1 \geq \gamma_2 \geq \cdots > 0$$

and

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

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Notation: For $u \subseteq \mathbb{N}$ and $\mathbf{x} \in D^{\mathbb{N}}$ let $\mathbf{x}_u = (x_j)_{j \in u}$ and $\gamma_u = \prod_{j \in u} \gamma_j$.

ANOVA decomposition of $f : [0,1]^{\mathbb{N}} \to \mathbb{R}$. Recursively, for finite sets $u \subset \mathbb{N}$,

$$f_u(\mathbf{x}_u) = \int_{[0,1]^{\mathbb{N}\setminus u}} f(\mathbf{x})\, d\mathbf{x}_{\mathbb{N}\setminus u} - \sum_{v \subsetneq u} f_v(\mathbf{x}_v).$$

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By definition, $f \in F_{\boldsymbol{\gamma}}$ iff

$$\|f\|_{\boldsymbol{\gamma}}^2 := \sum_u \gamma_u^{-1} \cdot \|f_u^{(u)}\|_{L_2([0,1]^u)}^2 \le 1.$$

Orthogonal decomposition $f = \sum_u f_u$.

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Example For

$$f(\mathbf{x}) = \sum_{j=1}^{\infty} \eta_j \cdot x_j^2$$

we have

$$\|f\|_{\boldsymbol{\gamma}} < \infty \quad \Leftrightarrow \quad \sum_{j=1}^{\infty} \frac{\eta_j^2}{\gamma_j} < \infty.$$

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  - functions with bounded mixed derivatives: . . . , *Temlyakov (1994)*, ...
- $\gamma_u \searrow 0$: *Sloan, Woźniakowski (1998), . . . , Novak, Woźniakowski (2008)*, ...

The general setting Assume that

(A1) $k \neq 0$ reproducing kernel on $D \times D$ with $D \neq \emptyset$,

(A2) $H(k) \cap H(1) = \{0\}$,

(A3) $(\gamma_u)_u$ family of weights $\gamma_u \geq 0$ such that, with $m = \inf_{x \in D} k(x, x)$,

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Consider the reproducing kernel

$$K_{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \sum_u \gamma_u \prod_{j \in u} k(x_j, y_j)$$

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Finally, let $F_{\boldsymbol{\gamma}}$ denote the unit ball in $H(K_{\boldsymbol{\gamma}})$.

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The ANOVA-case: $D = [0,1]$, $\gamma_u = \prod_{j \in u} \gamma_j$ with $\sum_j \gamma_j < \infty$,

$$k(x,y) = 1/3 + (x^2 + y^2)/2 - \max(x,y).$$

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See *Hickernell, Wang (2001)*, *Hickernell, Müller-Gronbach, Niu, R (2010)*, *Kuo, Sloan, Wasilkowski, Woźniakowski (2010)*, ...

III. Algorithms, Error, and Cost

The naive Monte Carlo approach

- Truncation, i.e., fix $c \in D$ and consider

$$I^{(s)}(f) = \int_{D^s} f(x_1, \ldots, x_s, c, \ldots)\,(\rho \otimes \cdots \otimes \rho)(d(x_1, \ldots, x_s)).$$

- With $\xi_{i,j}$ being independent and distributed according to ρ,

$$Q(f) = \frac{1}{m} \cdot \sum_{i=1}^{m} f\big((\xi_{i,1} \ldots, \xi_{i,s}, c, \ldots)\big).$$

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Note: Q requires m independent samples in the s-dimensional subspace

$$D_c^{(s)} = \{\mathbf{x} \in D^{\mathbb{N}} : x_{s+1} = \cdots = c\}.$$

In general, **randomized algorithms** that evaluate f in the hierarchy of subspaces $D_c^{(1)} \subset D_c^{(2)} \subset \ldots$.

Here, for simplicity, **randomized quadrature formulas**

$$Q(f) = \sum_{i=1}^{m} b_i \cdot f(\xi_i)$$

with $b_i \in \mathbb{R}$ and $\bigcup_{s=1}^{\infty} D_c^{(s)}$-valued random variables ξ_i.

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Error and **cost** of Q on a class F of functions $f : D^{\mathbb{N}} \to \mathbb{R}$

$$\mathrm{error}^2(Q, F) = \sup_{f \in F} \mathrm{E} \Big| I(f) - Q(f) \Big|^2$$

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The n**-th minimal error**

$$e_n(F) = \inf\{\operatorname{error}(Q,F) : \operatorname{cost}(Q,F) \leq n\}.$$

IV. Two Particular Results

As before,

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Questions

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- If so, sharp upper and lower bound on $e_n(F_{\boldsymbol{\gamma}})$?

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Consider

$$\lambda_{\boldsymbol{\gamma}} = \sup\{r > 0 : \sup_{n\in\mathbb{N}} \left(e_n(F_{\boldsymbol{\gamma}}) \cdot n^r\right) < \infty\}$$

as a substitute for the **order of convergence of the minimal errors**.

Thm. *Hickernell, Müller-Gronbach, Niu, R (2010)*, *Baldeaux, Gnewuch (2014)*

Assume that $\gamma_j = j^{-\alpha}$ with $\alpha > 1$. Then, for randomized algorithms,

$$\lambda_{\boldsymbol{\gamma}} = \min((\alpha - 1)/2, 3/2).$$

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Rem.

- Almost optimal algorithms: multi-level plus scrambled lattice rules. The starting point is

$$f_L(\mathbf{x}) = f_0(\mathbf{x}) + \sum_{\ell=1}^{L} \big(f_\ell(\mathbf{x}) - f_{\ell-1}(\mathbf{x})\big).$$

  with $f_\ell(\mathbf{x}) = f(x_1, \ldots, x_{2^\ell}, 1/2, \ldots)$. Use scrambled lattice rules for integration of $f_\ell - f_{\ell-1}$. Apply tractability results.

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- Almost optimal algorithms: multi-level plus scrambled lattice rules.
- Fixed subspace sampling is suboptimal. More precisely,

  $$\lambda_{\boldsymbol{\gamma}}^{\text{fix}} \leq \frac{3 \cdot (\alpha - 1)}{2 \cdot (\alpha + 2)}$$

  for the order of convergence of the respective minimal errors.

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Question: Performance of deterministic algorithms?

Thm. *Hickernell, Müller-Gronbach, Niu, R (2011)*, *Gnewuch (2012)*, *Dick, Gnewuch (2014)*, *Hefter, R (2014)*

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Assume that $\gamma_j = j^{-\alpha}$ with $\alpha > 1$. Then, for deterministic algorithms,

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Proof, Step 1 Result is known for the unit ball in $H(L_{\boldsymbol{\gamma}})$, where

$$L_{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \sum_u \gamma_u \prod_{j \in u} \ell(x_j, y_j) = \prod_{j=1}^{\infty} (1 + \gamma_j \ell(x_j, y_j))$$

with

$$\ell(x, y) = \min(x, y).$$

Furthermore, with equivalent norms for every $\gamma > 0$,

$$H(1 + \gamma k) = H(1 + \gamma \ell) = W_2^1([0, 1]).$$

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Question: $H(K_{\boldsymbol{\gamma}}) = H(L_{\boldsymbol{\gamma}})$? **Answer**: No, in general.

V. Embeddings

Thm. *Hefter, R (2014)*

(i)

$$\sum_{j=1}^{\infty} \gamma_j^{1/2} < \infty \quad \Leftrightarrow \quad H(K_{\boldsymbol{\gamma}}) = H(L_{\boldsymbol{\gamma}}).$$

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$$\sum_{j=1}^{\infty} \gamma_j^{1/2} < \infty \quad \Leftrightarrow \quad H(K_{\boldsymbol{\gamma}}) = H(L_{\boldsymbol{\gamma}}).$$

(ii) In general, there exist $0 < c' < 1 < \tilde{c}$ such that

$$H(K_{c'\boldsymbol{\gamma}}) \subseteq H(L_{\boldsymbol{\gamma}}) \subseteq H(K_{\tilde{c}\boldsymbol{\gamma}})$$

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Rem. General result on embeddings between weighted tensor product spaces available.

Summary

- Integration

$$I(f) = \int_{D^{\mathbb{N}}} f(\mathbf{x})\, \rho^{\otimes \mathbb{N}}(d\mathbf{x})$$

  of functions $f : D^{\mathbb{N}} \to \mathbb{R}$.

- Function classes: weighted superpositions of tensor product reproducing kernel Hilbert spaces,

$$K_{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \sum_{u} \gamma_u \prod_{j \in u} k(x_j, y_j).$$

- Sharp bound on minimal errors for randomized and deterministic algorithms (here: for a specific example).

- New type of embedding results.