BAYESIAN INVERSE PROBLEMS – EXAMPLES 1 Lent 2019

Please email comments, corrections to: hnk22@cam.ac.uk

Return answers to 2. and 4. by Friday 8 Feb. 4pm to DAMTP pigeonhole.

1. Let Σ be a symmetric and positive definite matrix. Show that

$$\int_{\mathbb{R}^d} \exp\Big(-\frac{1}{2}(x-\theta)^T\Sigma^{-1}(x-\theta)\Big)dx = \sqrt{(2\pi)^d\det(\Sigma)}.$$

Hint: use the eigendecomposition of $\Sigma = Q\Lambda Q^\top$.

2. Assume that we observe measurement $m = Au + \eta$, where $A : \mathbb{R}^d \to \mathbb{R}^k$ is a known matrix, $\eta \sim \mathcal{N}(0, \Sigma_\eta)$ and $u \sim \pi = \mathcal{N}(\theta_u, \Sigma_u)$, where Σ_η and Σ_u are both invertible.
i) Show that the posterior covariance Σ and mean $\overline{u}$ can be written as

$$\Sigma = (A^\top \Sigma_\eta^{-1} A + \Sigma_u^{-1})^{-1}$$

and

$$\overline{u} = \Sigma(A^\top \Sigma_\eta^{-1} m + \Sigma_u^{-1}\theta_u).$$

ii) What happens on the small noise limit $\delta \to 0$, $\Sigma_\eta = \delta^2\Sigma_0$, if we assume that $k = n$ and A invertible?
iii) What happens if we only assume that $\text{null}(A) = \{0\}$?

3.* Assume the same model as in 2. but this time $m \in \mathbb{R}^k$ and $u \in \mathbb{R}^d$ with $k < d$, and $\text{rank}(A) = k$. We can then write

$$A = (A_0 \;\; 0)Q^\top,$$

with $Q \in \mathbb{R}^{d\times d}$ being an orthonormal matrix, $Q^\top Q = I$, and $A_0 \in \mathbb{R}^{k\times k}$ an invertible matrix. We denote$L_u = \Sigma_u^{-1}$ and write

$$Q^\top L_u Q = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}^\top & L_{22} \end{bmatrix}, \quad L_{11} \in \mathbb{R}^{k\times k},\; L_{22} \in \mathbb{R}^{(d-k)\times(d-k)}.$$

We also write $Q = (Q_1 \; Q_2)$ with $Q_1 \in \mathbb{R}^{d\times k}$ and $Q_1 \in \mathbb{R}^{d\times(d-k)}$. We define $z \in \mathbb{R}^k$ to be the unique solution of $A_0 z = m$. Let $w \in \mathbb{R}^k$ and $w' \in \mathbb{R}^{d-k}$ be defined via $\Sigma_u^{-1}\theta_u = Q(w \; w')^\top$. Show that on the small noise limit $\delta \to 0$, $\Sigma_\eta = \delta^2\Sigma_0$,

$$\Pi^m \rightharpoonup \mathcal{N}(\overline{\theta}_m, \overline{\Sigma}_m)$$

where

$$\overline{\theta}_m = Q(z \; z')^\top \quad \text{and} \quad \overline{\Sigma}_m = Q_2 L_{22}^{-1} Q_2^\top.$$

Above $z' = -L_{22}^{-1} L_{12}^\top z + L_{22}^{-1} w' \in \mathbb{R}^{d-k}$.

4. Let x be a real valued random variable with probability density $\pi(x)$, such that $\pi(x) = 0$ only at isolated points. We define the cumulative distribution function

$$\Phi(t) = \int_{-\infty}^{t} \pi(x)dx.$$

Define a new random variable $y = \Phi(x)$. Show that $y \sim \mathcal{U}([0,1])$.

5. The Lebesgue measure ν_n on Euclidean space $\mathbb{R}^n$ is countably additive and translation invariant. Show that there is no analogue of Lebesgue measure on infinite-dimensional Banach space X.

6. Let $u \sim \mathcal{N}(0, I)$, where $0 \in \mathbb{R}^d$ and $I \in \mathbb{R}^{d\times d}$ is identity matrix. Where does most of the probability mass lie when d is large? Hint: What happens to the volume of d-dimensional unit ball $B_d(0,1)$ when $d \to \infty$?