Inverse Problems

Michaelmas term 2019

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UNIVERSITY OF CAMBRIDGE

Practicalities

- Lectures take place on Tuesday, Thursday and Saturday, 12-1pm, MR13.
- Course materials (lecture notes, example sheets, etc.) will be provided at `http://www.damtp.cam.ac.uk/research/cia/inverse-problems-michaelmas-2019`
- Four example sheets and example classes.
- Written exam will be held in June.
- For further questions email hnk22@cam.ac.uk or yk362@cam.ac.uk

Example classes

Example classes will be held in MR15 on the following dates:

- Monday 28 October, 2-3.30pm
- Monday 18 November, 2-3.30pm
- Monday 2 December, 2-3.30pm
- Monday 20 January, 2-3.30pm

Return solutions to two questions (specified in advance)

- Example sheets will be made available one week before the deadline.
- Hand in your answers during the lecture on the previous Thursday.

What do we mean by inverse problems?

- Direct problem: Given an object (cause), determine data (effect).
- Inverse problem: Observing (noisy) data, recover the object.

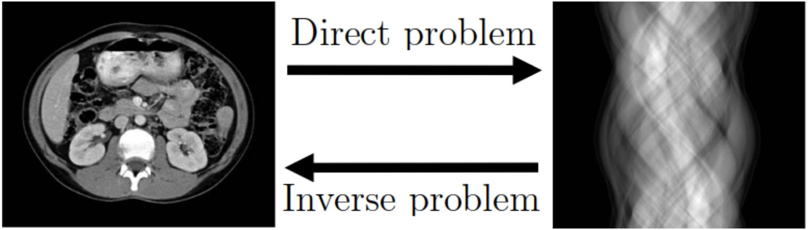
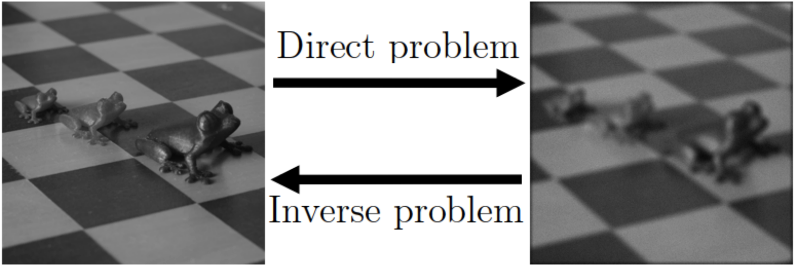


Image processing is a classical example of an inverse problem



Inverse problems are ill-posed

Well-posedness of a problem as defined by Jacques Hadamard (1923):

I) Existence. There should be at least one solution.

II) Uniqueness. There should be at most one solution.

III) Stability. The solution must depend continuously on data.

If any of the above conditions is violated the problem is called ill-posed

We consider linear problems

We consider the linear inverse problem

$$f = Au$$

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Main difficulty: A^{-1} does not exist or is not continuous.

Examples: matrix inversion

I) $A : \mathbb{R}^d \to \mathcal{R}(A) \subsetneq \mathbb{R}^k$, $k > d$, i.e. the system is overdetermined. Because of the noise $f_n \notin \mathcal{R}(A)$
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II) $A : \mathbb{R}^d \to \mathbb{R}^k$, $k < d$, i.e. the system is underdetermined.
$\Rightarrow$ There are several possible solutions.

III) There exists $A^{-1} : \mathbb{R}^d \to \mathbb{R}^d$, but the condition number $\kappa = \lambda_1/\lambda_d$ is very large. Then A is almost singular and $\|A^{-1}n\| \approx \|n\|/\lambda_d$ can be arbitrarily large.
$\Rightarrow$ The naive reconstruction $\widetilde{u} = A^{-1}f_n = u + A^{-1}n$ is dominated by the noise.

Examples: Deblurring (deconvolution)

Figure: The Hubble space telescope had a flaw in its mirror which resulted in the images being blurred.

$$f(x) = (Au)(x) = \int_{\mathbb{R}^2} a(x-y)u(y)dy$$

Signal deblurring for noiseless data

The noiseless data $f(t) = \int_{-\infty}^{\infty} a(t-s)u(s)ds$ has Fourier transform

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and hence by inverse Fourier transform

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\xi} \frac{\hat{f}(\xi)}{\hat{a}(\xi)} d\xi.$$

Signal deblurring for noisy data

We can only observe noisy data and get

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Then the estimate $\widetilde{u}$ given by the Convolution Theorem is

$$\widetilde{u}(t) = u(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{n}(\xi) \exp\left(it\xi + \frac{\alpha^2}{2} \xi^2 \right) d\xi,$$

which may not be even well defined since the Fourier transform of the noise will not decay fast enough.

Examples: Heat distribution in an insulated rod

Consider the problem

$$\begin{aligned} v_t - v_{xx} &= 0 && \text{in } (0,\pi) \times \mathbb{R}_+ \\ v(0,\cdot) = v(\pi,\cdot) &= 0 && \text{on } \mathbb{R}_+ \\ v(\cdot,T) &= f && \text{in } (0,\pi) \\ v(\cdot,0) &= u && \text{in } (0,\pi) \end{aligned}$$

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Forward problem: Determine the final distribution $v(\cdot, T) \in L^2(0,\pi)$, $T > 0$, when the initial distribution $v(\cdot, 0) \in L^2(0,\pi)$ is given.

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Forward problem: Determine the final distribution $v(\cdot,T) \in L^2(0,\pi)$, $T > 0$, when the initial distribution $v(\cdot,0) \in L^2(0,\pi)$ is given.

Inverse problem: Determine the initial distribution $v(\cdot,0) \in L^2(0,\pi)$ from observed (noisy) final distribution $v(\cdot,T) \in L^2(0,\pi)$.

Forward problem

The solution to the forward problem can be given explicitly:

$$v(x,T) = \sum_{j=1}^{\infty} \widehat{u}_j e^{-j^2 T} sin(jx),$$

where $\{\widehat{u}_j\}_{j=0}^{\infty}$ are the Fourier (sine) coefficients of the initial heat distribution u.

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The solution operator

$$A_T : u \mapsto v(\cdot, T) = f, \quad L^2(0, \pi) \to L^2(0, \pi)$$

satisfies the following conditions:

- A_T is injective,
- $\mathcal{R}(A_T)$ is dense in $L^2(0, \pi)$,
- A_T is linear, bounded and **compact** $\Rightarrow$ no continuous inverse.

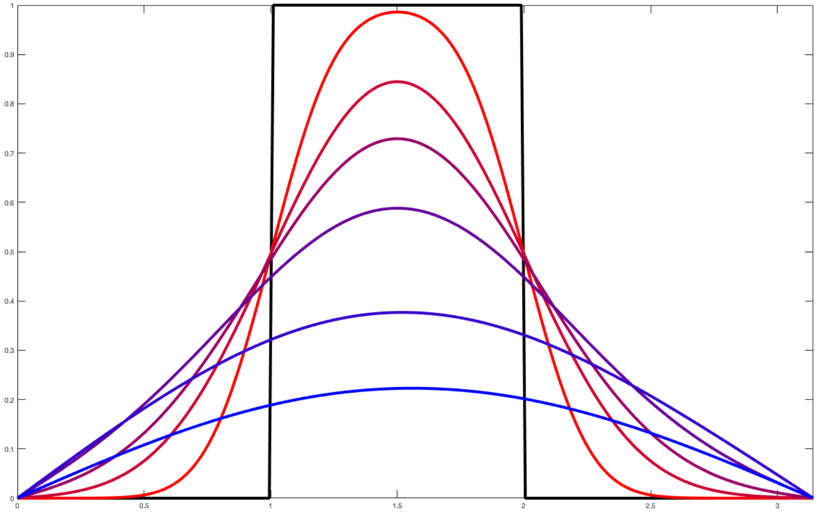
Inverse problem

We notice that, for every $s > 0$,

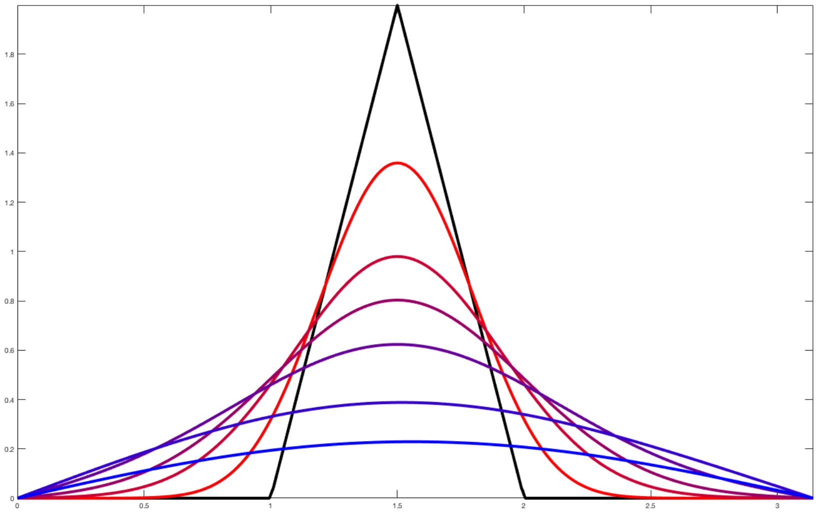
$$\begin{aligned}\|v\|_{H^s}^2 &= \sum_{j=1}^{\infty} j^{2s} e^{-2j^2 T} |\widehat{u}_j|^2 \\ &= T^{-s} \sum_{j=1}^{\infty} (j^2 T)^s e^{-2j^2 T} |\widehat{u}_j|^2 \\ &\leq C T^{-s} \|u\|_{L^2}^2\end{aligned}$$

and hence $\mathcal{R}(A_T) \subset \cap_{s>0} H^s$. However, noise is not smooth and hence $f_n = v(\cdot, T) + n \notin \mathcal{R}(A_T)$.

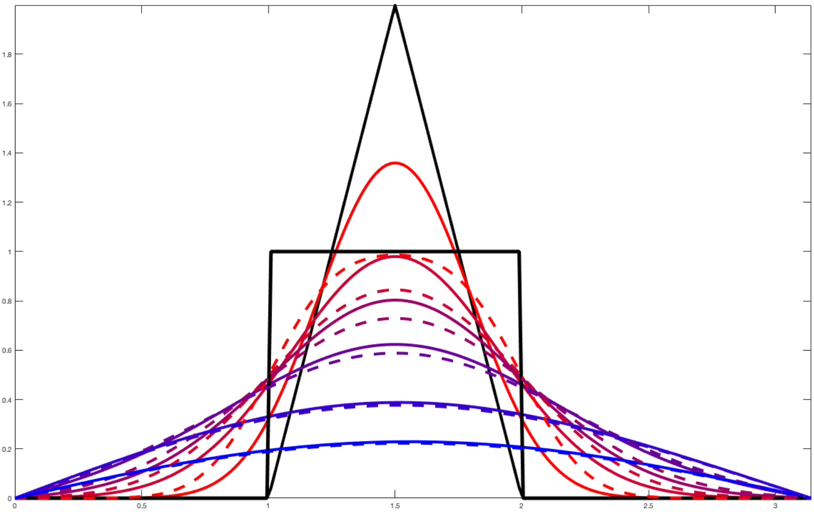
Heat distribution at $t = 0.02, 0.06, 0.1, 0.2, 0.5$ and $t = 1$



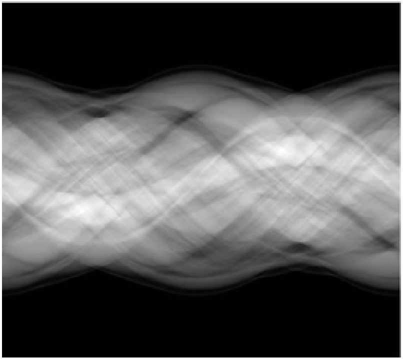
Another heat distribution at $t = 0.02, 0.06, 0.1, 0.2, 0.5$ and $t = 1$



Comparison of the two heat distributions



Examples: Computerised tomography (CT)



$$f(\theta, s) = (Au)(\theta, s) = \int_{x \cdot \theta = s} u(x)dx$$

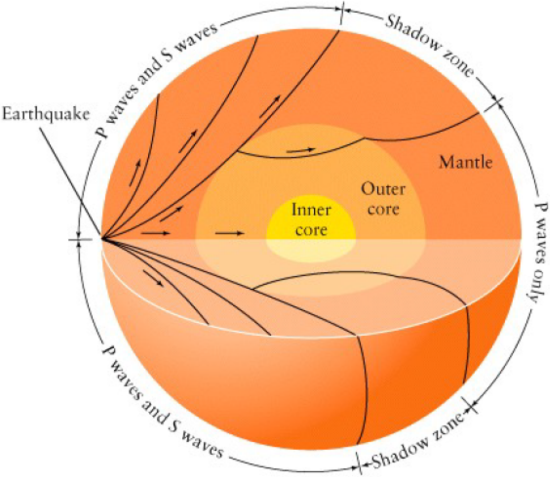
The data are collected by rotating the X-ray source and detectors around the object

Video by Samuli Siltanen. For more videos see
www.siltanen-research.net/IPexamples/xray_tomography

Examples: Ozone layer tomography

Figure: Given star occultation measurements, what is the ozone profile?

Examples: Geodesic X-ray transform



$$f(\gamma) = (Au)(\gamma) = \int u(\gamma(t))dt$$

Examples: Photo-acoustic tomography

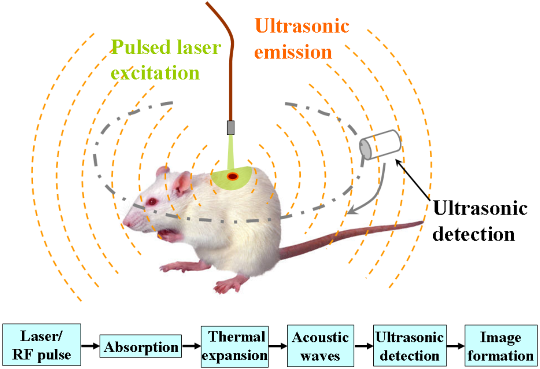


Figure: Photo-acoustic tomography is an example of a hybrid inverse problem that aims to combine high contrast and resolution of different imaging methods.

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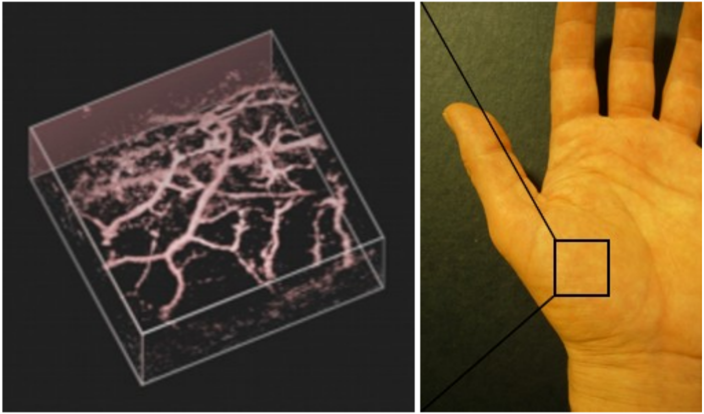

Figure: Images of superficial blood vessels. Image by UCL Photoacoustic Imaging Group

Solving an inverse problem: Deterministic approach

We want to approximate u from a measurement

$$f_n = Au + n,$$

where $A : X \to Y$ is linear and bounded, X and Y are Hilbert spaces and $n \in Y$ is noise.

One approach is to use the least squares method

$$\widetilde{u} = \arg\min_{u \in X} \left\{ \|Au - f_n\|_Y^2 \right\}.$$

Problem: Multiple minima (if A is not injective) and no stability with respect to the data f_n.

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To regularise the problem we add a regularisation term and define

$$\widetilde{u} = \arg\min_{u \in X} \left\{ \|Au - f_n\|_Y^2 + \alpha \mathcal{R}(u) \right\}$$

Regularisation gives a stable approximate solution for the inverse problem.

Solving an inverse problem: Stochastic approach

We consider linear measurement model

$$F = AU + N,$$

where F, U, N are now treated as random variables.

Bayes' formula

Using Bayes' theorem the prior distribution can be updated to a posterior distribution

$$\mathbb{P}(u \in A \,|\, f_n \in B) = \frac{\mathbb{P}(f_n \in B \,|\, u \in A)\mathbb{P}(u \in A)}{\mathbb{P}(f_n \in B)}.$$

The solution is a probability distribution

Finite dimensional Gaussian example

- $X = \mathbb{R}^d$ and $Y = \mathbb{R}^k$
- N is white Gaussian noise
- We choose Gaussian prior

Posterior has density

$$\pi^f(u) = \pi(u \mid f_n) \propto \exp\Big(-\frac{1}{2}\|f_n - Au\|_{\mathbb{R}^k}^2 - \frac{1}{2}\|u\|_\Sigma^2 \Big)$$

We notice that, in this case, solving the mode of the posterior leads to similar optimisation problem as regularisation (with $\mathcal{R}(u) = \|u\|_\Sigma^2$).