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Inverse Problems

Example sheet **2** Presentation **18 Nov. 2019**, **2-3:30pm**, **MR15**.
Please submit after the lecture on **14 November 2019**.

Please submit **Exercises 3 and 5**.

Exercise 1 (Subdifferential)
Let $\mathcal{U}$ be a Banach space and $J : \mathcal{U} \to \overline{\mathbb{R}}$ be a functional. We define the *subdifferential* of J at any $v \in \mathcal{U}$ as

$$\partial J(v) := \left\{ p \in \mathcal{U}^* \,\middle|\, J(u) \geq J(v) + \langle p, u - v \rangle \text{ for all } u \in \mathcal{U} \right\}.$$

Characterise the subdifferential for the

(a) absolute value function: $\mathcal{U} = \mathbb{R}$, $J(v) = |v|$,

(b) ℓ^1-norm: $\mathcal{U} = \ell^2$,

$$J(u) = \|u\|_{\ell^1} := \begin{cases} \sum_{j=1}^\infty |u_j| & \text{if } u \in \ell^1 \\ \infty & \text{else.} \end{cases}$$

(c) characteristic function of the unit ball in $\mathbb{R}$: $\mathcal{U} = \mathbb{R}$, $J(u) = \chi_C(u)$, $C := \{u \in \mathbb{R} \,|\, |u| \leq 1\}$.

Exercise 2 (Proximal operators)
Let $\mathcal{U}$ be a Hilbert space and $J : \mathcal{U} \to \overline{\mathbb{R}}$ be a l.s.c., convex and proper functional. The *proximal operator* of J at any $z \in \mathcal{U}$ and step size $\alpha \geq 0$ is defined as $\operatorname{prox}_{\alpha J} : \mathcal{U} \to \mathcal{U}$ with

$$\operatorname{prox}_{\alpha J}(z) := \arg\min_{u \in \mathcal{U}} \Phi_{\alpha,z}(u)$$

and $\Phi_{\alpha,z}(u) := \frac{1}{2}\|u - z\|_{\mathcal{U}}^2 + \alpha J(u)$. It can be shown that $\partial \Phi_{\alpha,z}(u) = u - z + \alpha \partial J(u)$.

(a) Compute the proximal operators for the following functionals

(i) absolute value function: $\mathcal{U} = \mathbb{R}$, $J(u) = |u|$;

(ii) squared ℓ^2-norm: $\mathcal{U} = \ell^2$, $J(u) = \frac{1}{2}\|u\|_{\ell^2}^2 := \frac{1}{2}\sum_{j=1}^\infty u_j^2$;

(iii) ℓ^1-norm: $\mathcal{U} = \ell^2$,

$$J(u) = \|u\|_{\ell^1} := \begin{cases} \sum_{j=1}^\infty |u_j| & \text{if } u \in \ell^1 \\ \infty & \text{else.} \end{cases}$$

(b) For a subset $C \subset \mathcal{U}$ of the Hilbert space $\mathcal{U}$ we consider the characteristic function

$$\chi_C(u) := \begin{cases} 0 & \text{if } u \in C \\ \infty & \text{else.} \end{cases}$$

(i) For which subsets C is the proximal operator of χ_C well-defined?

(ii) Compute the proximal operators for

- $C = [0, \infty) \subset \mathbb{R}$,
- $C = \{u \in \mathbb{R}^n \mid \|u\|_2 \leq 1\}$, and
- $C = \{u \in \mathbb{R}^n \mid \|u\|_\infty \leq 1\}$.

Exercise 3 (Convex conjugate – submit)
Let $\mathcal{U}$ be a Banach space and let $E\colon \mathcal{U} \to \overline{\mathbb{R}}$ be proper, lower semi-continuous and convex. Then the *Fenchel conjugate* or *convex conjugate* of E is defined to be the mapping $E^*\colon \mathcal{U}^* \to \overline{\mathbb{R}}$ with

$$E^*(v) := \sup_{u \in \mathcal{U}} \Big\{ \langle v, u \rangle - E(u) \Big\} .$$

(a) Compute the convex conjugates of the following functionals.

(i) $E\colon \mathbb{R} \to \mathbb{R}, E(u) = \frac{1}{p}|u|^p$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) $E(u) = \frac{1}{2}\|u\|^2$ for a Hilbert space $\mathcal{U}$.

(iii) $E(u) = \|u\|_{\mathcal{U}}$ for a Banach space $\mathcal{U}$.

(b) Let $\mathcal{U}$ be a Hilbert space and $E : \mathcal{U} \to \overline{\mathbb{R}}$ a proper, lower semi-continuous and convex functional. Show that

$$p \in \partial E(u) \quad \Leftrightarrow \quad u \in \partial E^*(p)$$

for all $u, p \in \mathcal{U}$.
Hint: You may exploit the fact that under the stated assumptions $E = E^{**}$ holds true.

Exercise 4 (Bregman distances)
Let $u, v \in \mathcal{U}$ and $p \in \partial J(v)$ be an element of the subdifferential. Then the *Bregman distance* of J at u, v is defined as

$$D_J^p(u, v) := J(u) - J(v) - \langle p, u - v \rangle .$$

In this exercise, we will investigate the properties of the Bregman distance for convex J.

(a) Show that Bregman distances are non-negative, i.e. for all $u, v \in \mathcal{U}, p \in \partial J(v)$ it holds

$$D_J^p(u, v) \geq 0 .$$

(b) Show that Bregman distances may not be symmetric, i.e. there exists a J and $u, v \in \mathcal{U}$ with $p \in \partial J(v), q \in \partial J(u)$ so that

$$D_J^p(u, v) \neq D_J^q(v, u) .$$

(c) Show that a vanishing Bregman distance may not imply that the two arguments are already the same, i.e. $D_J^p(u, v) = 0 \not\Rightarrow u = v$? What if J is strictly convex?

Exercise 5 (Absolute one-homogeneous functionals – submit)
Recall that a functional $J\colon \mathcal{U} \to \overline{\mathbb{R}}$ is called absolutely one-homogeneous if

$$J(\lambda u) = |\lambda| J(u) \quad \forall \lambda \in \mathbb{R},\ \forall u \in \mathcal{U}.$$

Let J be convex, proper, l.s.c. and absolute one-homogeneous.

(a) Show that $p \in \partial J(v)$ if and only if $J(v) = \langle p, v\rangle$ and for all $u \in \mathcal{U}$ there is $J(u) \geq \langle p, u\rangle$. Thus,

$$D_J^p(u, v) = J(u) - \langle p, u\rangle\,.$$

(b) Show that Bregman distances associated with absolute one-homogeneous functionals fulfill the triangle inequality in the first argument, i.e. for all $u, v, w \in \mathcal{U}$ and $p \in \partial J(w)$ there is

$$D_J^p(u + v, w) \leq D_J^p(u, w) + D_J^p(v, w)\,.$$

(c) Show that the Fenchel conjugate $J^*(\cdot)$ is the characteristic function of the convex set $\partial J(0)$. Compare this to the results of Exercise 3 (a-iii).