Exercise 1 (Subdifferential)
Let $\mathcal{U}$ be a Banach space and $J : \mathcal{U} \to \mathbb{R}$ be a functional. We define the subdifferential of $J$ at any $v \in \mathcal{U}$ as

$$\partial J(v) := \left\{ p \in \mathcal{U}^* \middle| J(u) \geq J(v) + \langle p, u - v \rangle \text{ for all } u \in \mathcal{U} \right\}.$$ 

Characterise the subdifferential for the

(a) absolute value function: $\mathcal{U} = \mathbb{R}, J(v) = |v|$, 
(b) $\ell^1$-norm: $\mathcal{U} = \ell^2$, 
(c) characteristic function of the unit ball in $\mathbb{R}$: $\mathcal{U} = \mathbb{R}, J(u) = \chi_C(u), C := \{ u \in \mathbb{R} \mid |u| \leq 1 \}$.

Exercise 2 (Proximal operators)
Let $\mathcal{U}$ be a Hilbert space and $J : \mathcal{U} \to \mathbb{R}$ be a l.s.c., convex and proper functional. The proximal operator of $J$ at any $z \in \mathcal{U}$ and step size $\alpha \geq 0$ is defined as

$$\text{prox}_{\alpha J}(z) := \arg\min_{u \in \mathcal{U}} \Phi_{\alpha, z}(u)$$

and $\Phi_{\alpha, z}(u) := \frac{1}{2} \| u - z \|_{\mathcal{U}}^2 + \alpha J(u)$. It can be shown that $\partial \Phi_{\alpha, z}(u) = u - z + \alpha \partial J(u)$.

(a) Compute the proximal operators for the following functionals

(i) absolute value function: $\mathcal{U} = \mathbb{R}, J(u) = |u|$;
(ii) squared $\ell^2$-norm: $\mathcal{U} = \ell^2, J(u) = \frac{1}{2} \| u \|_{\ell^2}^2 := \frac{1}{2} \sum_{j=1}^{\infty} u_j^2$;
(iii) $\ell^1$-norm: $\mathcal{U} = \ell^2, J(u) = \| u \|_{\ell^1} := \left\{ \begin{array}{ll} \sum_{j=1}^{\infty} |u_j| & \text{if } u \in \ell^1 \\ \infty & \text{else.} \end{array} \right.$

(b) For a subset $C \subset \mathcal{U}$ of the Hilbert space $\mathcal{U}$ we consider the characteristic function

$$\chi_C(u) := \left\{ \begin{array}{ll} 0 & \text{if } u \in C \\ \infty & \text{else.} \end{array} \right.$$
(i) For which subsets $C$ is the proximal operator of $\chi_C$ well-defined?
(ii) Compute the proximal operators for
- $C = [0, \infty) \subset \mathbb{R}$,
- $C = \{u \in \mathbb{R}^n \mid \|u\|_2 \leq 1\}$, and
- $C = \{u \in \mathbb{R}^n \mid \|u\|_{\infty} \leq 1\}$.

Exercise 3 (Convex conjugate – submit)
Let $U$ be a Banach space and let $E: U \to \mathbb{R}$ be proper, lower semi-continuous and convex. Then the Fenchel conjugate or convex conjugate of $E$ is defined to be the mapping $E^*: U^* \to \mathbb{R}$ with

$$E^*(v) := \sup_{u \in U} \left\{ \langle v, u \rangle - E(u) \right\}.$$  

(a) Compute the convex conjugates of the following functionals.
(i) $E: \mathbb{R} \to \mathbb{R}, E(u) = \frac{1}{p} |u|^p$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.
(ii) $E(u) = \frac{1}{2} \|u\|^2$ for a Hilbert space $U$.
(iii) $E(u) = \|u\|_U$ for a Banach space $U$.

(b) Let $U$ be a Hilbert space and $E : U \to \mathbb{R}$ a proper, lower semi-continuous and convex functional. Show that $p \in \partial E(u) \iff u \in \partial E^*(p)$

for all $u, p \in U$.

Hint: You may exploit the fact that under the stated assumptions $E = E^{**}$ holds true.

Exercise 4 (Bregman distances)
Let $u, v \in U$ and $p \in \partial J(v)$ be an element of the subdifferential. Then the Bregman distance of $J$ at $u, v$ is defined as

$$D^p_J(u, v) := J(u) - J(v) - \langle p, u - v \rangle.$$  

In this exercise, we will investigate the properties of the Bregman distance for convex $J$.

(a) Show that Bregman distances are non-negative, i.e. for all $u, v \in U, p \in \partial J(v)$ it holds

$$D^p_J(u, v) \geq 0.$$  

(b) Show that Bregman distances may not be symmetric, i.e. there exists a $J$ and $u, v \in U$ with $p \in \partial J(v), q \in \partial J(u)$ so that

$$D^p_J(u, v) \neq D^q_J(v, u).$$  

(c) Show that a vanishing Bregman distance may not imply that the two arguments are already the same, i.e. $D^p_J(u, v) = 0 \not\Rightarrow u = v$? What if $J$ is strictly convex?
Exercise 5 (Absolute one-homogeneous functionals – submit)

Recall that a functional \( J : U \to \mathbb{R} \) is called absolutely one-homogeneous if
\[
J(\lambda u) = |\lambda| J(u) \quad \forall \lambda \in \mathbb{R}, \forall u \in U.
\]

Let \( J \) be convex, proper, l.s.c. and absolute one-homogeneous.

(a) Show that \( p \in \partial J(v) \) if and only if \( J(v) = \langle p, v \rangle \) and for all \( u \in U \) there is \( J(u) \geq \langle p, u \rangle \). Thus,
\[
D^p_J(u, v) = J(u) - \langle p, u \rangle.
\]

(b) Show that Bregman distances associated with absolute one-homogeneous functionals fulfill the triangle inequality in the first argument, i.e. for all \( u, v, w \in U \) and \( p \in \partial J(w) \) there is
\[
D^p_J(u + v, w) \leq D^p_J(u, w) + D^p_J(v, w).
\]

(c) Show that the Fenchel conjugate \( J^*(\cdot) \) is the characteristic function of the convex set \( \partial J(0) \). Compare this to the results of Exercise 3 (a-iii).