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Inverse Problems

Example sheet **1** Presentation **31 January 2018**, **2-3pm**, **MR15**.

Exercise 1 (Integral operators - submit)
For $\Omega = [0,1]^2$ and $\mathcal{X} = L^2(\Omega)$, we consider the integral operator $K : \mathcal{X} \to \mathcal{X}$ with

$$(Ku)(y) \coloneqq \int_\Omega k(x,y)u(x)\,dx,$$

for $k \in L^2(\Omega \times \Omega)$. Show that

(a) K is linear with respect to u.,

(b) K is a bounded linear operator, i.e. $\|Ku\|_{\mathcal{X}} \leq \|K\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}\|u\|_{\mathcal{X}}$. Give also an estimate for $\|K\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}$,

(c) the adjoint K^* is given via

$$(K^*v)(y) = \int_\Omega k(y,x)v(x)\,dx.$$

Exercise 2 (Generalised inverse - submit)

(a) Let $m, n \in \mathbb{N}$ with $m \geq n \geq 2$. Compute the Moore-Penrose inverses of the following matrices:

  (i) $K = (1, 1, \ldots, 1) \in \mathbb{R}^{1\times n}$

  (ii) $K = \operatorname{diag}(a_1, \ldots, a_n) \in \mathbb{R}^{n\times n}$ with $a_j \in \mathbb{R}$ for $j \in \{1, \ldots, n\}$

  (iii) $K \in \mathbb{R}^{m\times n}$ with $K^T K = I_n$

(b) Let $a, b \in \mathbb{R}$ with $a < b$. Compute the Moore-Penrose inverse of the operator $K : L^2([a,b]) \to \mathbb{R}$ with

$$Ku = \int_a^b u(x)\,dx.$$

Please turn over!

Exercise 3 (Heat equation)
The heat equation in $\mathbb{R}^n$ is defined as the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u, \tag{1}$$

where Δ denotes the Laplace operator $\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$. Let us consider the case for $n = 2$.

(a) Compute the solution u of the heat equation for the initial value $u(x, 0) = f(x)$ with

$$f(x_1, x_2) = \sum_{p=0}^{m_1-1} \sum_{q=0}^{m_2-1} c_{pq}\, \varphi_{pq}(x_1, x_2), \tag{2}$$

for $x = (x_1, x_2)^\top \in \Omega = [0, 1]^2$, coefficients $c_{pq} \in \mathbb{R}$ and functions $\varphi_{pq} \in C^2(\Omega)$ that are eigenfunctions of the (negative) Laplace operator with eigenvalue λ_{pq}, i.e. $-\lambda_{pq}\varphi_{pq} = \Delta\varphi_{pq}$.

(b) Show that the functions $\varphi_{pq} \in C^\infty(\Omega)$ with

$$\varphi_{pq}(x_1, x_2) \coloneqq v_p w_q \cos(\pi q x_1) \cos(\pi p x_2)$$

and $p \in \{0, \ldots, m_1 - 1\}$, $q \in \{0, \ldots, m_2 - 1\}$, are eigenfunctions of the (negative) Laplace operator, and compute the corresponding eigenvalues λ_{pq}. Here, the weights v_p and w_q are defined as

$$v_p = \begin{cases} \frac{1}{\sqrt{m_1}} & p = 0, \\ \sqrt{\frac{2}{m_1}} & 1 \le p \le m_1 - 1, \end{cases} \quad \text{and} \quad w_q = \begin{cases} \frac{1}{\sqrt{m_2}} & q = 0, \\ \sqrt{\frac{2}{m_2}} & 1 \le q \le m_2 - 1. \end{cases}$$

(c) Use MATLAB and the results of Exercises 3(a) and (b) to compute the solution u of the heat equation for f given as the discrete image `'trees.tif'`, evaluated at the points $x_1 = (2i + 1)/(700)$, $i \in \{0, \ldots, 349\}$, and $x_2 = (2j + 1)/(516)$, $j \in \{0, \ldots, 257\}$. Visualise your results for suitable choices of t.
Hint: Make use of the MATLAB commands `imread('trees.tif')`, `dct2`, and `idct2`.

Please turn over!

Exercise 4 (Inverse heat equation)
We now want to consider the inverse problem of (1). Instead of an initial value $u(x,0)$ we are given the accumulated value $f(x)$ at time $t = T$ with $T > 0$, i.e. $u(x,T) = f(x)$.

(a) Compute the solution u of the heat equation for $t \in [0,T]$ and $u(x,T) = f(x)$, with f being defined as in Exercise 3(a), equation (2).

(b) Show that for $f \in L^\infty(\Omega)$ the inverse problem of the heat equation is ill-posed in the sense of Hadamard.

(c) Use Matlab and the results of Exercise 4(a) and 3(b) to compute the solution of the inverse problem of the heat equation for f given as the discrete image `'moon.tif'`, evaluated at the points $x_1 = (2i+1)/(716)$, $i \in \{0,\dots,357\}$, and $x_2 = (2j+1)/(1074)$, $j \in \{0,\dots,536\}$. Visualise your results for suitable choices of t.

    In addition, use these results to invert the heat equation. Generate a solution of the heat equation for a time point T as in Exercise 3(c). Use this solution as the initial image for the inversion at time T and compute the solution of the heat equation at time $t = 0$. Run the inversion again with a tiny distortion (e.g. add Gaussian distributed noise via `randn`) of the initial image.

Please turn over!

Exercise 5 (Convolution)
Many forward problems are either modelled as convolutions or they are modelled as the composition of several components one of which is a convolution. Therefore convolutions play an important role in inverse problems. As in Exercise 1, let $\Omega = [0,1]^2$ be the unit square and let $\mathcal{X} = L^2(\Omega)$. A convolution is the special case of an integral operator $K : \mathcal{X} \to \mathcal{X}$ where the kernel has a simple structure:

$$(Ku)(y) \coloneqq \int_\Omega k(y-x)u(x)\,dx,$$

for $k \in L^2(\Omega)$. It follows easily from Exercise 1 that K is linear and bounded.

(a) Although being shown in general in Exercise 1, give an explicit form of the adjoint of the convolution.

(b) Let $f = Ku$. It follows from the convolution theorem that a convolution can be inverted by means of the Fourier transform

$$u = (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1}\left(\frac{\mathcal{F}(f)}{\mathcal{F}(k)}\right), \tag{3}$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ its inverse. Implement this formula in MATLAB to deblur (deconvolve) the blurry tree image f generated by the script `ex5b_generate_data.m`, which is provided online[1]. Note that the script also outputs $\mathcal{F}(k)$. Add some noise to the data and show that the inversion formula is ill-conditioned.
Hint: Make use of the MATLAB commands `fft2` and `ifft2`.

(c) Reformulate equation (3) so that the denominator is non-negative and give a stable approximation of this formula. Implement this formula in MATLAB and empirically show that it is stable.
Hint: Make use of the MATLAB command `conj`.

[1] `http://store.maths.cam.ac.uk/DAMTP/ll542/teaching/2018inverseproblems/ex5b_generate_data.zip`