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Inverse Problems

Example sheet **3** Presentation **28 February 2018, 2-3pm, MR15**
Submission **26 February 2018, 12pm, MR14**

Please submit exercises 1, 2a and 4a.

Exercise 1 (Subdifferential)
Let $\mathcal{U}$ be a Banach space and $J : \mathcal{U} \to \mathbb{R}_\infty$ be a functional. We define the *subdifferential* of J at any $v \in \mathcal{U}$ as

$$\partial J(v) := \left\{ p \in \mathcal{U}^* \;\middle|\; J(u) \geq J(v) + \langle p, u - v \rangle \text{ for all } u \in \mathcal{U} \right\}.$$

Characterise the subdifferential for the

a) quadratic function: $\mathcal{U} = \mathbb{R}, J(u) = \frac{1}{2}u^2$,

b) absolute value function: $\mathcal{U} = \mathbb{R}, J(u) = |u|$,

c) squared ℓ^2-norm: $\mathcal{U} = \ell^2, J(u) = \frac{1}{2}\|u\|_{\ell^2}^2 := \frac{1}{2}\sum_{j=1}^\infty |u_j|^2$,

d) ℓ^1-norm: $\mathcal{U} = \ell^2$,

$$J(u) = \|u\|_{\ell^1} := \begin{cases} \sum_{j=1}^\infty |u_j| & \text{if } u \in \ell^1 \\ \infty & \text{else} \end{cases}, \text{ and}$$

e) characteristic function of the unit ball in $\mathbb{R}$: $\mathcal{U} = \mathbb{R}, J(u) = \chi_C(u), C := \{u \in \mathbb{R} \,|\, |u| \leq 1\}$.

Exercise 2 (Proximal operators)
Let $\mathcal{U}$ be a Hilbert space and $J : \mathcal{U} \to \mathbb{R}_\infty$ be a l.s.c., coercive, convex and proper functional. The *proximal operator* of J at any $z \in \mathcal{U}$ and step size $\alpha \geq 0$ is defined as $\text{prox}_J^\alpha : \mathcal{U} \to \mathcal{U}$ with

$$\text{prox}_J^\alpha(z) := \arg\min_{u \in \mathcal{U}} \Phi_{\alpha,z}(u)$$

and $\Phi_{\alpha,z}(u) := \frac{1}{2}\|u - z\|_{\mathcal{U}}^2 + \alpha J(u)$. It can be shown that $\partial \Phi_{\alpha,z}(u) = u - z + \alpha \partial J(u)$.

a) Compute the proximal operators for the functionals defined in Exercise 1.

b) For a subset $C \subset \mathcal{U}$ of the Hilbert space $\mathcal{U}$ we consider the characteristic function

$$\chi_C(u) := \begin{cases} 0 & \text{if } u \in C \\ \infty & \text{else} \end{cases}.$$

i. For which subsets C is the proximal operator of χ_C well-defined?
Hint: You may use any standard results from linear / functional analysis.

ii. Compute the proximal operators for
- $C = [0, \infty) \subset \mathbb{R}$,
- $C = \{u \in \mathbb{R}^n \mid \|u\|_2 \leq 1\}$, and
- $C = \{u \in \mathbb{R}^n \mid \|u\|_\infty \leq 1\}$.

Exercise 3 (Bregman distances)
Let $u, v \in \mathcal{U}$ and $p \in \partial J(v)$ be an element of the subdifferential. Then the *Bregman distance* of J at u, v is defined as

$$D_J^p(u, v) := J(u) - J(v) - \langle p, u - v \rangle .$$

a) In this exercise, we will investigate the properties of the Bregman distance for convex J.

i. Show that Bregman distances are non-negative, i.e. for all $u, v \in \mathcal{U}, p \in \partial J(v)$ it holds

$$D_J^p(u, v) \geq 0 .$$

ii. Let J be absolutely one-homogeneous, i.e. for all $\lambda \in \mathbb{R}, u \in \mathcal{U}$ we have $J(\lambda u) = |\lambda| J(u)$. Show that $p \in \partial J(v)$ if and only if $J(v) = \langle p, v \rangle$ and for all $u \in \mathcal{U}$ there is $J(u) \geq \langle p, u \rangle$. Thus,

$$D_J^p(u, v) = J(u) - \langle p, u \rangle .$$

iii. Let J be absolutely one-homogeneous. Show that Bregman distances fulfill the triangle inequality in the first argument, i.e. for all $u, v, w \in \mathcal{U}$ and $p \in \partial J(w)$ there is

$$D_J^p(u + v, w) \leq D_J^p(u, w) + D_J^p(v, w) .$$

iv. Show that Bregman distances may not be symmetric, i.e. there exists a J and $u, v \in \mathcal{U}$ with $p \in \partial J(v), q \in \partial J(u)$ so that

$$D_J^p(u, v) \neq D_J^q(v, u) .$$

v. Show that a vanishing Bregman distance may not imply that the two arguments are already the same, i.e. $D_J^p(u, v) = 0 \not\Rightarrow u = v$? What if J is strictly convex?

b) Compute the Bregman distances for the functions and functionals defined in Exercise 1.

Exercise 4 (Variational regularisation)
We analyse some of the motivating examples for variational regularisation models which are of the form $R_\alpha f := \arg\min_{u\in\mathcal{U}} \Phi_{\alpha,f}(u)$ with

$$\Phi_{\alpha,f}(u) := \frac{1}{2}\|Ku - f\|_{\mathcal{V}}^2 + \alpha J(u)$$

and $J : \mathcal{U} \to \mathbb{R}_\infty$ being a regularization functional.

a) On ℓ^2 we can define the regularization operators

$$R_\alpha^1 f := \arg\min_{u\in\mathcal{U}} \left\{ \frac{1}{2}\|Ku - f\|_{\mathcal{V}}^2 + \alpha\|u\|_1 \right\}$$

with $\|u\|_1 := \infty$ if $u \in \ell^2 \setminus \ell^1$ and

$$R_\alpha^2 f := \arg\min_{u\in\mathcal{U}} \left\{ \frac{1}{2}\|Ku - f\|_{\mathcal{V}}^2 + \frac{\alpha}{2}\|u\|_2^2 \right\}$$

which promote different properties in the regularized solution. For simplicity let $K = I$ be the identity operator. Show that

i. $R_\alpha^1 f$ is always sparse if $\alpha > 0$, i.e. $|\operatorname{supp}(R_\alpha^1 f)| < \infty$.
ii. $R_\alpha^2 f$ may not be sparse.

However, the price to pay is linearity. Show that

iii. R_α^1 is non-linear.
iv. R_α^2 is linear.

Hint: You may use the results of Exercise 2.

b) For $\Omega = [-1, 1]$, we consider total variation regularization that is naturally defined on the space of functions of bounded variation $\mathrm{BV}(\Omega)$ which provides regularity while still containing discontinuous functions. In general one can prove that

$$\mathrm{H}^{1,1}(\Omega) \subset \mathrm{BV}(\Omega) \subset \mathrm{L}^1(\Omega)\,.$$

Show that these inclusions are strict. In particular, show that

i. $\mathrm{BV}(\Omega)$ is larger than $\mathrm{H}^{1,1}(\Omega)$ as it contains discontinuous functions.
ii. $\mathrm{BV}(\Omega)$ is smaller than $L^1(\Omega)$ as it provides more regularity.

Exercise 5 (Denoising with basis representations)
In this exercise we are considering the task of denoising with the help of different basis representations. Let $f \in \mathbb{R}^n$ be a given image and $f^\delta \in \mathbb{R}^n$ a noisy observation of it, the task is to find f. Obviously the task is ill-posed in the sense that there is not even a deterministic model that maps f to f^δ which we could invert. Instead we approach this problem by assuming that f is regular and that $f^\delta = f + \eta$ where η is an instance of a multi-variate Gaussian distribution with zero mean and uniform standard deviation. Then this task can be modelled by minimising the cost functional

$$\Phi_{\alpha,f^\delta}(u) = \frac{1}{2}\|u - f^\delta\|_2^2 + \alpha J(u)\,,$$

where the first term penalises a mismatch to the data f^δ and the second term J is designed such that desirable solutions have a low value. It is reasonable to assume that for a regular f there exists a basis $\{b_1, \dots, b_n\}$ of $\mathbb{R}^n$ such that an energy

$$J(f) = \frac{1}{p}\|Bf\|_p^p = \frac{1}{p}\sum_{j=1}^{n} |\langle f, b_j\rangle|^p \qquad p \in \{1, 2\},$$

or

$$J(f) = \|Bf\|_0 := \sum_{j=1}^{n} |\langle f, b_j\rangle|^0 = |\{j \in \{1, \dots, n\} \mid \langle f, b_j\rangle \neq 0\}|$$

of the basis representation Bf is small. Here $B : \mathbb{R}^n \to \mathbb{R}^n$ denotes the basis transformation

$$Bf = \sum_{j=1}^{n} \langle f, b_j\rangle b_j\,.$$

a) Does Φ_{α,f^δ} have minimisers and are they unique?

b) Compute explicit formulas that minimise Φ_{α,f^δ}. Do all solutions depend on B?

c) Consider the Fourier basis and the wavelet bases `'haar'` and `'db10'`. Write a function in MATLAB that computes the solution of 2.
Hint: You can use the MATLAB functions `fft2` and `wavedec2` for the transformation into the Fourier and wavelet basis.

d) Load your favourite picture (if you do not have one, take `'cameraman.tif'` which is built into MATLAB) and generate several noisy data sets (e.g. with MATLAB's `randn`) and denoise them. Choose the regularisation parameter by

   i. by the eye-ball-metric (select a few parameters and choose the solution you like) and

   ii. by Morozov's discrepancy principle.