UNIVERSITY OF CAMBRIDGE

Inverse Problems

Lent term 2018

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Organisation

- Lectures take place Mondays, Wednesdays, and Fridays, 11am–12pm, MR14.
- Course materials (lecture notes, example sheets, etc.) will be provided at

  `http://www.damtp.cam.ac.uk/research/cia/teaching/201718lentinvprob.html`
- Four example sheets and example classes (details to follow).
- Revision class will be held on 30 May 2018, 2–3pm, MR14.
- Written exam will be held on 11 June 2018, 1.30–4.30pm, location TBA.
- For further questions email either `m.j.ehrhardt@damtp.cam.ac.uk` or `ll542@cam.ac.uk`.

Example classes

- Example sheets will be discussed/presented on the following dates:
  - 31 January 2018, 2–3pm, MR15.
  - 14 February 2018, 2–3pm, MR15.
  - 28 January 2018, 2–3pm, MR15.
  - 14 March 2018, 2–3pm, MR15.
- Hand in solutions to at least two questions which are specified in advance:
  - See course website.
  - Made available one week before deadline.
  - Hand in two days before respective example class (i.e. lecture on Monday 11am).

What are inverse problems?

- Engl, Hanke, Neubauer (1996):

  "Inverse problems are concerned with determining causes for a desired or an observed effect."

- Direct problem:

  Cause (Parameter, Unkown, etc.) $\Rightarrow$ Effect (Data, Measurements, etc.)

- Inverse problem:

  Cause (Parameter, Unkown, etc.) $\Leftarrow$ Effect (Data, Measurements, etc.)

What are inverse problems?

- Inverse problems $\approx$ ill-posed/ill-conditioned problems.
- Well-posedness in the sense of Hadamard (1923):
  - Existence of a solution (for all admissible data),
  - Uniqueness of a solution,
  - Continuous dependence of the solution on the data.
- If any of these conditions is violated the problem is called ill-posed.

What are inverse problems?

- Given a (physical/mathematical) model $K : \mathcal{U} \to \mathcal{V}$,
- Given measurements $f \in \mathcal{V}$,
- Recover $u \in \mathcal{U}$ such that

$$Ku = f.$$

- Main difficulty: K^{-1} does not exist or is not continuous.
- Applications in:
  - physics, biology, medicine,
  - engineering, finance, machine learning,
  - imaging (e.g. computed tomography), computer vision, image processing,
  - and many more...

Examples: deblurring

Observed image f.

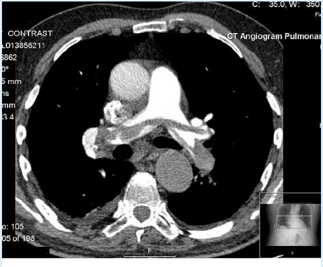
$$\xrightarrow{K^{-1}}$$

Original image u.

$$f(y) = (Ku)(y) := \int_{\mathbb{R}^2} k(y-x)u(x)\,dx$$

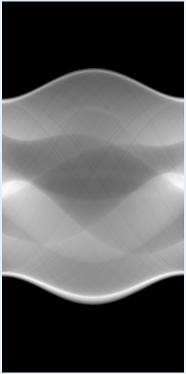
Examples: computerised tomography (CT)

CT scanner, Wikipedia, CC BY 2.0, by daveynin.



CT scan, Wikipedia, public domain.

Examples: computerised tomography (CT)



Measured data f.

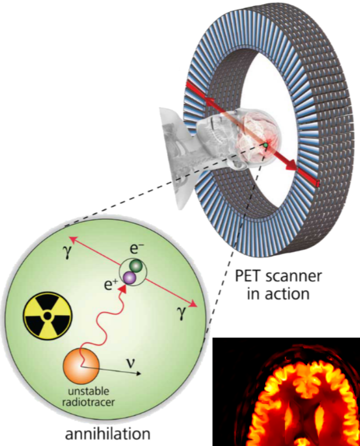
$$\xrightarrow{K^{-1}}$$

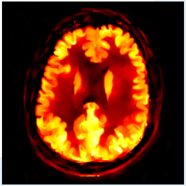
Original image u.

$$f(\theta, s) = (Ku)(\theta, s) := \int_{x\cdot\theta = s} u(x)\, dx$$

See videos at `http://www.siltanen-research.net/IPexamples/xray_tomography`.

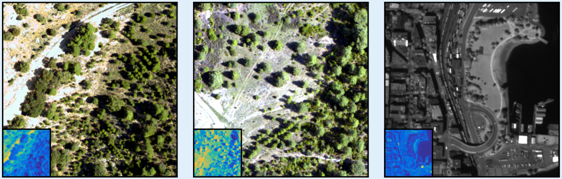
Examples: positron emission tomography (PET)



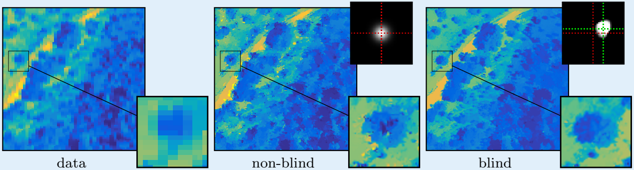


[1] Images by courtesy of Matthias J. Ehrhardt.

Hyperspectral imaging/remote sensing



Data f: hyperspectral & panchromatic image.

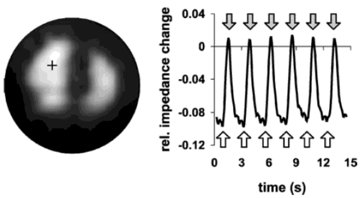


Reconstruction u and reconstructed kernel.

[1]Bungert et al., Blind Image Fusion for Hyperspectral Imaging with the Directional Total Variation, arXiv:1710.05705, (2017)

Examples: electrical impedance tomography (EIT)

EIT Setup



Reconstruction.

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n$$
$$u|_{\partial\Omega} = f$$

- $u : \Omega \to \mathbb{R}$ electric potential,
- $\sigma : \Omega \to \mathbb{R}$ conductivity,
- f voltage applied at the boundary $\partial\Omega$,

Calderón's problem: recover σ from boundary measurements $\Lambda_\sigma(f) = \sigma \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}$.

[1] Images taken from Wikipedia, CC BY 3.0, S. Heinrich, H. Schiffmann, A. Frerichs, A. Klockgether-Radke, and I. Frerichs.

Examples: motion estimation

Observed image $f(0,\cdot)$.

Observed image $f(1,\cdot)$.

Estimated velocities v.

- Given image sequence $f : [0,1] \times \Omega \subset \mathbb{R}^2 \to \mathbb{R}$,
- Recover velocity field $v : \Omega \to \mathbb{R}^2$ that satisfies optical flow equation

$$\partial_t f + \nabla f \cdot v = 0 \quad \text{in } [0,1] \times \Omega.$$

[1] Images taken from `http://vision.middlebury.edu/flow/data/`.

Examples: machine learning

Images from MNIST dataset.[1]

- Given training samples $\{x_i, y_i\}_{i=1}^n$ with
  - feature vectors $x_i \in \mathbb{R}^d$,
  - class labels $y_i \in \{0, 1, \ldots, 9\}$.
- Find classifier $f : \mathbb{R}^d \to \{0, 1, \ldots, 9\}$ minimising empirical risk

$$L(f) = \frac{1}{n} \sum_i c(f(x_i), y_i).$$

- $c : \{0, 1, \ldots, 9\}^2 \to [0, +\infty)$ is a cost/loss function.

[1] http://yann.lecun.com/exdb/mnist/

Lecture outline