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Inverse Problems in Imaging

Example sheet **3** Presentation **28 November 2018**, **1-2pm**, **MR14**.
Please submit after the lecture on **22 November 2018**.

Please submit Exercises **1 and 2**.

Exercise 1 (Total Variation - submit)
Consider total variation $\mathrm{TV}\colon L^1(\Omega) \to \bar{\mathbb{R}}$ as defined in the lecture.

(a) Show that TV is a seminorm on $L^1(\Omega)$.

(b) Show that for any $v \in L^1(\Omega)$ s.t. $\mathrm{TV}(v) = 0$ and any $u \in L^1(\Omega)$

$$\mathrm{TV}(u + v) = \mathrm{TV}(u).$$

(c) Find the convex conjugate of total variation $\mathrm{TV}^*\colon L^\infty(\Omega) \to \bar{\mathbb{R}}$. Using the fact that TV is absolute one-homogeneous, find its subdifferential at zero $\partial\,\mathrm{TV}(0)$.

Exercise 2 (Source Condition - submit)
Although in the lecture we derived the source condition only for absolute one-homogeneous regularisation functionals, it holds in a more general setting. What form does it take for Tikhonov regularisation (i.e. $\mathcal{J}(u) = \frac{1}{2}\|u\|_{\mathcal{U}}^2$, where $\mathcal{U}$ is a Hilbert space)? Provide convergence rates for Tikhonov regularisation under the source condition.

Exercise 3 (Source Condition for Total Variation)
Follow the lines of Example 5.2.5 in the lecture notes in the case when $\mathcal{C}$ has a smooth boundary. Why does the analysis of Example 5.2.5 not work anymore?
Hint: Consider a circle.

Exercise 4 (Bregman iteration)
Let $f \in \mathcal{V}, A \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $J : \mathcal{U} \to \bar{\mathbb{R}}$ be a functional. The iteration

$$\begin{aligned} u^{k+1} &\in \operatorname*{arg\,min}_{u \in \mathcal{U}} \left\{ \frac{1}{2}\|Au - f\|_{\mathcal{V}}^2 + D_J^{p^k}(u, u^k) \right\} \\ p^{k+1} &= p^k + A^*(f - Au^{k+1}) \end{aligned} \tag{1}$$

with $u^0 = p^0 = 0$ and $p^k \in \partial J(u^k)$ for all $k \in \mathbb{N}$ is known as *Bregman iteration*. In this exercise we will analyse properties of the Bregman iteration which can be seen as an iterative regularisation method.

(a) Show that the Bregman iterates monotonically decrease the data fidelity, i.e. they satisfy $\|Au^{k+1} - f\|_{\mathcal{V}} \le \|Au^k - f\|_{\mathcal{V}}$.

(b) Consider now the noisy case where u^k are the iterates for noisy data, i.e. f^δ replaces f in the definition of the iteration and $\|f - f^\delta\| \leq \delta$ for some $f \in \mathcal{R}(A)$. Let $u^* \in \mathcal{U}$ such that $Au^* = f$. Show that u^k progressively approximates u^* in a Bregman sense, i.e.

$$D_J^{p^{k+1}}(u^\dagger, u^{k+1}) \leq D_J^{p^k}(u^\dagger, u^k)\,,$$

as long as $\|Au^{k+1} - f^\delta\|_{\mathcal{V}} \geq \delta$. Thus the Bregman iteration with Morozov's discrepancy principle as a stopping criterion is a useful strategy to find J-minimising solutions.

Note that this also implies $D_J^{p^{k+1}}(u^\dagger, u^{k+1}) \leq D_J^{p^k}(u^\dagger, u^k)$ for all $k \in \mathbb{N}$ in case of $\delta = 0$.

(c) Consider again the noise-free case, i.e. $\delta = 0$. Show the estimate

$$D_J^{p^k}(u^\dagger, u^k) \leq \frac{\|w\|_{\mathcal{V}}^2}{2k}$$

for $k \geq 1$ on the speed of the convergence under a source condition $p^\dagger := A^*w \in \partial J(u^\dagger)$ for some $u^\dagger \in \mathcal{U}$ with $Au^\dagger = f$.

Exercise 5 (Dual of the ROF model)
Recall that the ROF model for image denoising is as follows (with the regularisation parameter α set 1, for simplicity)

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|u - f\|_{L^2}^2 + \mathrm{TV}(u).$$

Consider it in the finite-dimensional setting (i.e. after discretisation). Let $f \in \mathbb{R}^{n\times n}$ be a discrete noisy image. Then the ROF model can be re-written as follows

$$\min_{u \in \mathbb{R}^{n\times n}} \frac{1}{2}\|u - f\|_2^2 + \|Du\|_1,$$

where D denotes the (discrete) gradient (you can think of it being just a matrix). Derive the dual of the ROF problem following the lines of Section 5.1 and show that the solution of the primal problem can be easily derived from the solution of the dual problem.
Hint: Make use of the conjugate (transpose) matrix D^*.