UNIVERSITY OF CAMBRIDGE

Inverse Problems

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Organisation

- Lectures take place Tuesdays and Thursdays, 12pm–1pm, MR13.
- Course materials (lecture notes, example sheets, etc.) will be provided at

  www.damtp.cam.ac.uk/research/cia/teaching/201819michaelmasinvprob.html
- Three example sheets and example classes (details on the next page).
- Revision class: date TBA.
- Written exam: date TBA.
- For further questions email y.korolev@damtp.cam.ac.uk.

Example classes

- Example sheets will be discussed/presented on the following dates:
  - 31 October 2018 (Wednesday), 1–2pm, MR14.
  - 14 November 2018 (Wednesday), 1–2pm, MR14.
  - 28 November 2018 (Wednesday), 1–2pm, MR14.
- Hand in solutions to at least two questions which are specified in advance:
  - See course website.
  - Made available one week before deadline.
  - Hand in one day before respective example class (i.e. lecture on Tuesday 12pm).

What are inverse problems?

- Keller (1976):

  "We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former problem is called the direct problem, while the latter is called the inverse problem."

- Engl, Hanke, Neubauer (1996):

  "Inverse problems are concerned with determining causes for a desired or an observed effect."

- Direct problem:

  Cause (Parameter, Unkown, etc.) $\Rightarrow$ Effect (Data, Measurements, etc.)

- Inverse problem:

  Cause (Parameter, Unkown, etc.) $\Leftarrow$ Effect (Data, Measurements, etc.)

What are inverse problems?

- Inverse problems $\approx$ ill-posed/ill-conditioned problems.
- Well-posedness in the sense of Hadamard (1923):
  - Existence of a solution (for all admissible data),
  - Uniqueness of a solution,
  - Continuous dependence of the solution on the data.
- If any of these conditions is violated the problem is called ill-posed.

What are inverse problems?

- Given a (physical/mathematical) model $A : \mathcal{U} \to \mathcal{V}$,
- Given measurements $f \in \mathcal{V}$,
- Recover $u \in \mathcal{U}$ such that

$$Au = f.$$

- Main difficulty: A^{-1} does not exist or is not continuous.
- Applications in:
  - physics, biology, medicine,
  - engineering, finance, machine learning,
  - imaging (e.g., computed tomography, microscopy), computer vision, image processing,
  - and many more...

Examples: deblurring

Observed image f.

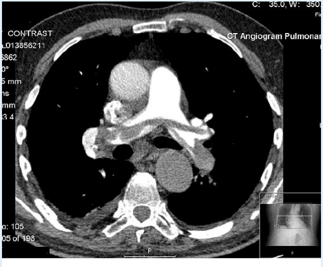
$$\xrightarrow{A^{-1}}$$

Original image u.

$$f(y) = (Au)(y) := \int_{\mathbb{R}^2} k(y-x)u(x)\, dx$$

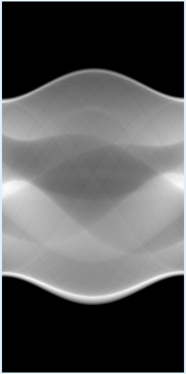
Examples: computerised tomography (CT)

CT scanner, Wikipedia, CC BY 2.0, by daveynin.



CT scan, Wikipedia, public domain.

Examples: computerised tomography (CT)



$\xrightarrow{A^{-1}}$

Measured data f.

Original image u.

$$f(\theta, s) = (Au)(\theta, s) := \int_{x\cdot\theta = s} u(x)\, dx$$

See videos at `http://www.siltanen-research.net/IPexamples/xray_tomography`.

Examples: motion estimation

Observed image $f(0,\cdot)$.

Observed image $f(1,\cdot)$.

Estimated velocities v.

- Given image sequence $f : [0,1] \times \Omega \subset \mathbb{R}^2 \to \mathbb{R}$,
- Recover velocity field $v : \Omega \to \mathbb{R}^2$ that satisfies optical flow equation

$$\partial_t f + \nabla f \cdot v = 0 \quad \text{in } [0,1] \times \Omega.$$

[1] Images taken from `http://vision.middlebury.edu/flow/data/`.

Examples: machine learning

Images from MNIST dataset.[1]

- Given training samples $\{x_i, y_i\}_{i=1}^n$ with
  - feature vectors $x_i \in \mathbb{R}^d$,
  - class labels $y_i \in \{0, 1, \dots, 9\}$.
- Find classifier $f : \mathbb{R}^d \to \{0, 1, \dots, 9\}$ minimising empirical risk

$$L(f) = \frac{1}{n} \sum_i c(f(x_i), y_i).$$

- $c : \{0, 1, \dots, 9\}^2 \to [0, +\infty)$ is a cost/loss function.

[1] `http://yann.lecun.com/exdb/mnist/`

Lecture outline

Lecture outline

To be continued on the blackboard...