Introduction to Nonlinear Spectral Analysis

Lent term 2021/2022

Leon Bungert and Yury Korolev

20 January 2022

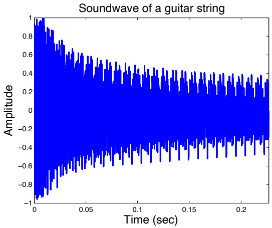
UNIVERSITY OF CAMBRIDGE

Layout

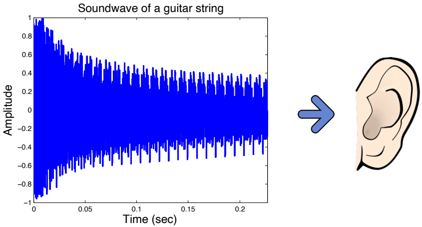
Interesting stuff

Boring stuff

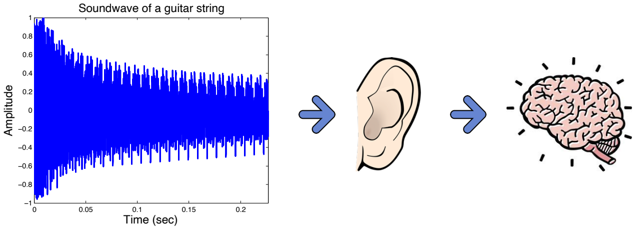
Linear spectral analysis



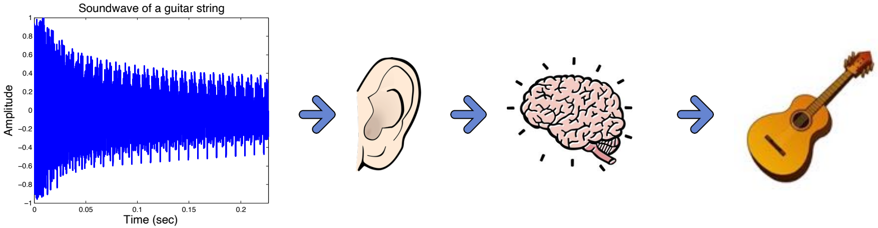
Linear spectral analysis



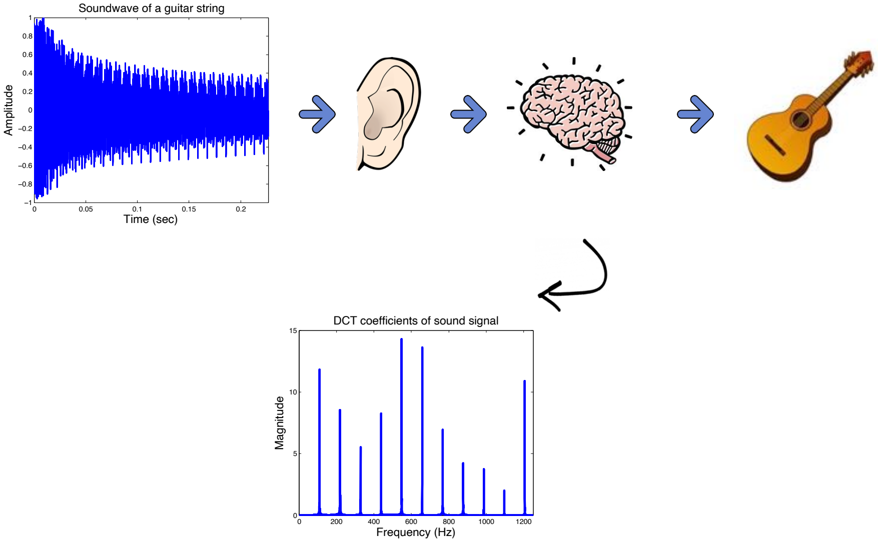
Linear spectral analysis



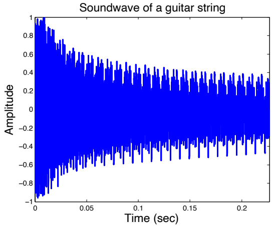
Linear spectral analysis

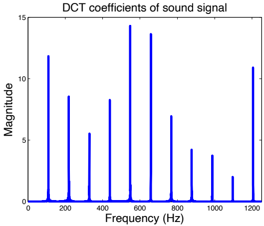


Linear spectral analysis



The Fourier transform



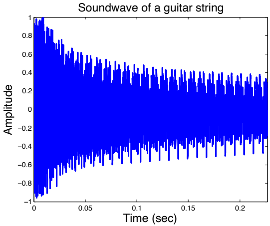


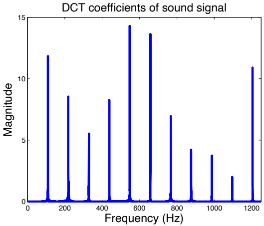
Any signal $f \in L^2(\mathbb{R}^n)$ can be written as

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\sigma) e^{-ix\cdot\sigma} \, \mathrm{d}\sigma,$$

where $\hat{f}$ are the Fourier coefficients and $e^{-ix\cdot\sigma}$ are eigenfunctions of $-\Delta$.

The Fourier transform





Any signal $f \in L^2(\mathbb{R}^n)$ can be written as

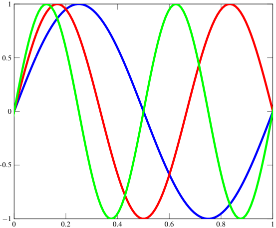
$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\sigma) e^{-ix\cdot\sigma} \, \mathrm{d}\sigma, \quad f_h(x) = \int_{\mathbb{R}^n} h(\sigma) \hat{f}(\sigma) e^{-ix\cdot\sigma} \, \mathrm{d}\sigma$$

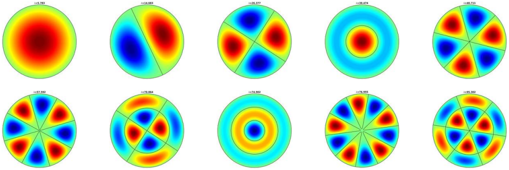
where $\hat{f}$ are the Fourier coefficients and $e^{-ix\cdot\sigma}$ are eigenfunctions of $-\Delta$.

Important *everyday application*: frequency filtering

Laplacian eigenfunctions

Laplacian eigenfunctions describe the *modes of vibration* of an object.





Applications: signal processing, clustering, PDEs, shape optimization, etc.

Linear vs. nonlinear eigenfunctions in applications

Laplacian eigenfunctions are of limited use for images and other data sources which are inherently *non-smooth*.

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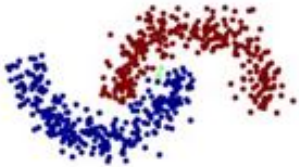

What do we expect from nonlinear spectral theory?

Wish list:

- To be able to *decompose* every datum f as
$$f = \int_0^\infty c(t)\zeta(t)\ \mathrm{d}t$$
$\zeta(t)$: eigenfunctions, $c(t)$: coefficients

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spectral filtering / denoising

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Applications:

spectral analysis / synthesis
spectral filtering / denoising

clustering,
segmentation
machine learning

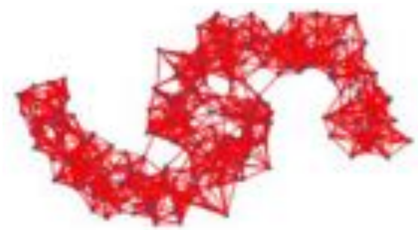
What are we prepared to give up?

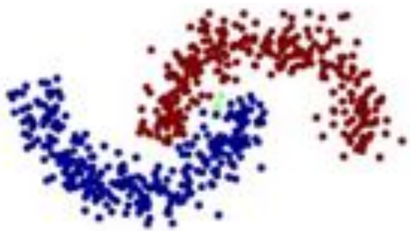
- Existence of an orthonormal basis of eigenfunctions;
- Eigenfunctions in the decomposition

$$f = \int_0^\infty c(t)\zeta(t)\,\mathrm{d}t$$

may become dependent on f. Such a decomposition may even not exist.

Example: nonlinear spectral graph clustering





Example: computing distance functions

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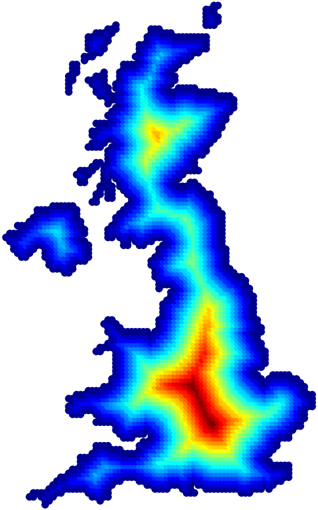
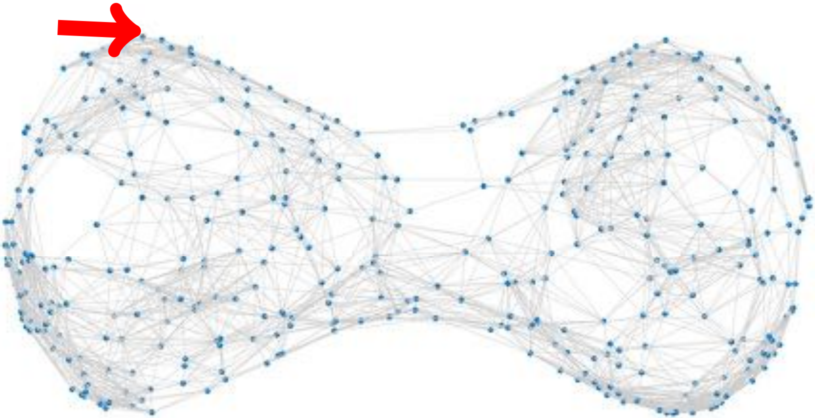


Example: discrete-to-continuum limits



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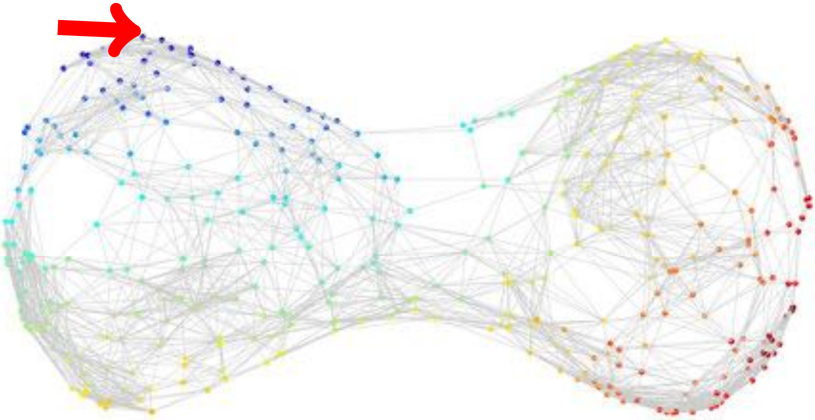

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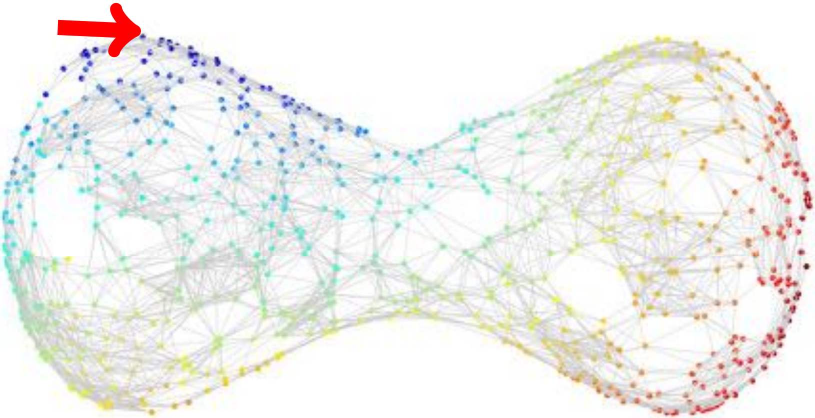

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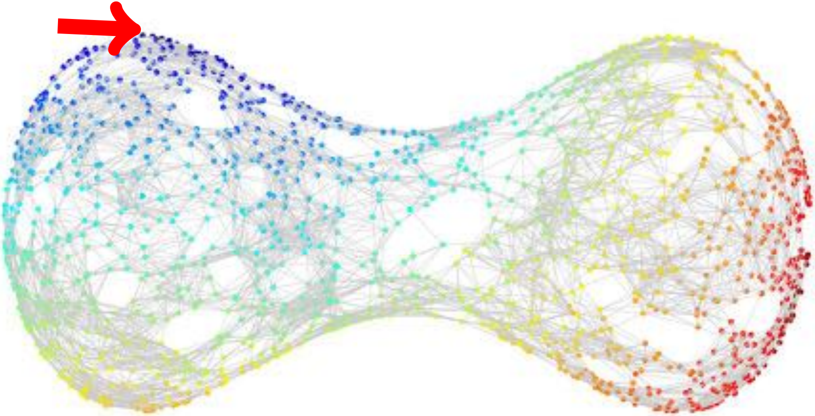

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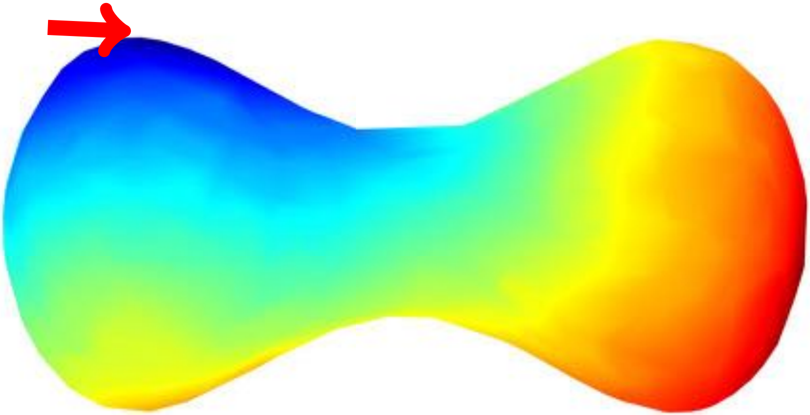
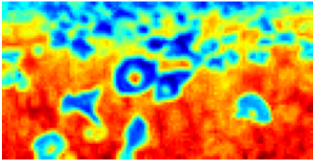


Example: colour maps



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Practicalities

- Lectures will take place on Tuesdays and Thursdays in a hybrid format (MR12 and zoom), 11am-12pm;
- Graduate course: no exam, example classes by arrangement if there is interest from the audience;
- Lectures will be recorded, recordings will be available on Moodle;
- Lecture notes will be made available on Moodle; they are still under development – please report any typos or factual errors you spot; please do not share the notes with 3rd parties;
- Nonlinear spectral analysis is not a complete theory yet; some questions are unanswered;
- The course web page is `https://www.damtp.cam.ac.uk/research/cia/introduction-nonlinear-spectral-analysis`
- Please email y.korolev@maths.cam.ac.uk with any questions.

Questions? Comments?